SHOCK STRUCTURE FOR PHYSICAL GASES

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Acknowledgements

Abstract

This report incorporates a systematic attempt to develop the theory of the structure of shock waves for real gases.

The simplest place to start is of course with one-dimensional steady flow. This is analysed in detail with reference to both exact solutions and general behaviour for shocks. The limits of vanishing diffusion and shock strength are also discussed.

This analysis is then built upon in two ways. Firstly, certain conclusions can be drawn from the previous analysis relating to shock structure in general fluid flow. These include a Reynolds' number analysis and a conjecture concerning the shock structure in the weak limit.

Secondly, the incorporation of nonequilibrium effects is discussed and a new system is presented as a possible alternative to the Navier-Stokes' equations. The relative merits and demerits are discussed along with several different types of analysis.

1. One-Dimensional Steady Advection Diffusion Systems

1.0 Introduction

This section begins with a formulation of the general system of equations in this context. Once this is achieved, algorithms are provided for attempting to determine the presence of diffusive shock waves. These algorithms are then tested and discussed.

Attention is then switched to existing exact solutions. A standard non-dimensionalisation procedure is employed to analyse the three main forms of diffusion (namely: viscosity, bulk relaxation and conduction) separately. A survey of other exact solutions is provided for completeness.

Finally, theory is developed for the general structure of these waves with limiting diffusion and/or limiting strength.

1.1 General Formulation with Examples

The general formulation will be taken to be the one-dimensional steady form of the general conservation system investigated by the author in [1] §2.2.2 - namely the set of equations

$$\frac{\partial u^i}{\partial t} + \sum_p \frac{\partial F^i_p}{\partial x_p} = S^i(\underline{u};\ \underline{x},t)\ , \qquad (1.1)$$

where S^i are the source terms and F^i_p are the diffusive fluxes defined by

$$F^i_p = f^i_p(\underline{u}) - \sum_j \sum_q \sum_m V^{ij}_{pqm}(\underline{u}) \frac{\partial u^j}{\partial x_q} d_m\ . \qquad (1.2)$$

where $\underline{u} = \underline{u}(\underline{x};\ t)$ is the vector of dependent variables; $f^i_p(\underline{u})$ are the non-diffusive fluxes; $V^{ij}_{pqm}(\underline{u})$ is the 'viscosity tensor'; and d_m are the scale values of the diffusion coefficients.

In the case of one-dimensional steady flow, the suffices p and q only take the value 1 so they may be omitted without confusion. Also the first term on the left-hand side of equation (1.1) vanishes, as does the t-dependence of the right-hand side. Instead of $\underline{x}$, we write x. This gives us the system

$$\frac{\partial F^i}{\partial x} = S^i(\underline{u};\ x) \tag{1.3}$$

$$F^i = f^i(\underline{u}) - \sum_j \sum_m V^{ij}_m(\underline{u}) \frac{\partial u^j}{\partial x} d_m \ . \tag{1.4}$$

Finally, it is notationally convenient to collapse the viscosity tensor with the diffusion scale values as the latter are merely constants. This may be achieved by introducing $W^{ij}(\underline{u})$ and ϵ such that

$$d = \left[\sum_m d_m^2\right]^{1/2} \tag{1.5}$$

$$dW^{ij}(\underline{u}) = \sum_m V^{ij}_m(\underline{u}) d_m \ . \tag{1.6}$$

These, together with equations (1.3) and (1.4), and noticing the derivative is now total rather than partial, give us

$$\frac{d}{dx}\left\{f^i(\underline{u}) - \sum_j dW^{ij}(\underline{u}) \frac{du^j}{dx}\right\} = S^i(\underline{u};\ x) \ , \tag{1.7}$$

where $\underline{u} = \underline{u}(x)$.

Two examples are presented below:

Example 1.1

The viscous Burgers' equation:

$$u_t + uu_x = \epsilon u_{xx} , \tag{1.8}$$

with

$$u = u(x-Ut) . \tag{1.9}$$

This gives the steady equation

$$(u-U) \frac{du}{d\xi} = \epsilon \frac{d^2u}{d\xi^2} , \tag{1.10}$$

where

$$\xi = x - Ut . \tag{1.11}$$

Fitting this to the structure of equation (1.7), we note:

i) i and j only take the value 1;

ii) $$\epsilon \frac{d^2u}{d\xi^2} = \frac{d}{d\xi} \left[\epsilon \frac{du}{d\xi}\right] . \tag{1.12}$$

Hence we can obtain a fit by setting

$$\left.\begin{aligned} d &= \epsilon \\ x &= \xi \end{aligned}\right\} , \tag{1.13}$$

$$f(u) = u - U , \tag{1.14}$$

$$W(u) = 1 , \tag{1.15}$$

and

$$S(u; \xi) \equiv 0 . \tag{1.16}$$

□

Example 1.2

The one-dimensional steady Navier-Stokes' equations with constant viscosity and thermal conductivity:

$$\left.\begin{aligned} \frac{d}{dx}\left\{\rho(u-U)\right\} &= 0 \\ \frac{d}{dx}\left\{p + \rho u(u-U) - \tfrac{4}{3}\,\mu\,\frac{du}{dx}\right\} &= 0 \\ \frac{d}{dx}\left\{\rho E(u-U) + pu - \tfrac{4}{3}\,\mu\, u\,\frac{du}{dx} - \kappa\,\frac{dT}{dx}\right\} &= 0 \end{aligned}\right\} . \tag{1.17}$$

with

$$p = R\rho T \tag{1.18}$$

$$E = \frac{p}{\gamma-1} + \tfrac{1}{2}\, u^2 . \tag{1.19}$$

In order to avoid confusion, the dependent variable vector shall be changed from $\underline{u}$ to $\underline{v}$. We set

$$\underline{v} = (v_1, v_2, v_3)^T = (\rho, u, p)^T . \tag{1.20}$$

Equation (1.5) leads us to

$$d = \sqrt{(\mu^2 + \kappa^2)} , \tag{1.21}$$

assuming equations (1.17) are not premultiplied. This may be achieved by also setting

$$\underline{f}(\underline{v}) = \begin{bmatrix} v_1(v_2-U) \\ Rv_1v_3 + v_1v_2(v_2-U) \\ \left[\frac{R}{\gamma-1}\, v_1v_3 + \tfrac{1}{2}\, v_2^2\right](v_2-U) + Rv_1v_3v_2 \end{bmatrix} . \tag{1.22}$$

$$W(\underline{v}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{4\mu}{3\sqrt{(\mu^2+\kappa^2)}} & 0 \\ 0 & \dfrac{4\mu v_2}{3\sqrt{(\mu^2+\kappa^2)}} & \dfrac{\kappa}{\sqrt{(\mu^2+\kappa^2)}} \end{bmatrix} \qquad (1.23)$$

$$\underline{S}(\underline{v};\ x) = \underline{0}\ . \qquad (1.24)$$

1.2 Shock Control Domains and Intervals

1.2.1 Motivation

For simplicity, we shall here consider general solutions to the single equation version of (1.7), i.e.

$$\left.\begin{aligned} u &= u(x;\ \epsilon,\ u_{-\infty},\ u_{\infty})\ , \\ u_{-\infty} &= u(-\ \infty)\ , \\ u_{\infty} &= u(\infty) \end{aligned}\right\} . \qquad (1.25)$$

where

Our aim here is to determine the position of diffusive shock waves. Obviously, as these are not discontinuities, this will require the specification of a representative interval for the given diffusive shock. This will be called the shock control interval. It turns out that it is also helpful to define a domain over which it is feasible to look for the shock control interval. Such an outer domain will be called the shock control domain.

These definitions have both constructive and non-constructive applications. Constructively, they are intended to provide an improved theoretical foundation to the process of shock detection, once a solution has been obtained. It may be that the artificial viscosity, be it implicit or explicit, that is incorporated into the numerical scheme, will in some sense correspond to the coefficient ϵ in this formulation.

Non-constructively, the procedures provided will be used later in this section as a basis to weak limit theory.

1.2.2 Formulation

From geometrical arguments, it is clear that most diffusive shock waves contain an inflection point. This condition is therefore imposed and used in the following definition.

Definition 1.1

The shock control domain about the inflection point x_0 of u is called $D(x_0)$ and is defined to be the largest simply connected open interval satisfying

i) $\{x \in D(x_0) : u''(x) = 0\} = \{x_0\}$. (1.26)

and

ii) $\{x \in D(x_0) : u'(x) = 0\} = \emptyset$. (1.27)

If we write

$$D(x_0) = (X_L, X_R) \tag{1.28}$$

(this is clearly well-defined as $D(x_0)$ is simply connected and open), then an easy corollary of definition 1.1 is

$$\left.\begin{array}{l} (X_L = -\infty \text{ or } u'(X_L) = 0 \text{ or } u''(X_L) = 0) \\ \text{and} \\ (X_R = \infty \text{ or } u'(X_R) = 0 \text{ or } u''(X_R) = 0) \end{array}\right\} . \tag{1.29}$$

We now turn our attention to the definition of the shock control interval. To this end, let us consider an arbitrary interval $(x_L, x_R) \subseteq D(x_0)$. We define

$$[x] = x_R - x_L \tag{1.30}$$

$$\left.\begin{aligned} [u] &= u(x_R) - u(x_L) \\ u_0 &= u(x_0) \\ \bar{u} &= \tfrac{1}{2}(u(x_L) + u(x_R)) \end{aligned}\right\} \tag{1.31}$$

$$\left.\begin{aligned} [u'] &= u'(x_R) - u'(x_L) \\ u_0' &= u'(x_0) \\ \bar{u}' &= \tfrac{1}{2}(u'(x_L) + u'(x_R)) \end{aligned}\right\} \tag{1.32}$$

etc.

From figure 1, it can clearly be seen that the following two qualities are desirable for this interval:

i) $\left|\frac{[u]}{[x]}\right| << |u_0'|$; (1.33)

ii) $|\bar{u}'| << \left|\frac{[u]}{[x]}\right|$. (1.34)

The following example shows that these two qualities are not in themselves sufficient to derive a 'good' shock interval in all circumstances.

Example 1.3

We again consider the steady viscous Burgers' equation. It can be shown that this has solution

$$u = u_L + U\left\{1 + \tanh\left[\frac{U(x+c)}{2\epsilon}\right]\right\} . \tag{1.35}$$

Without loss of generality, we may normalise equation (1.35) to:

$$u = u_0 + a \tanh \frac{ax}{2\epsilon} . \tag{1.36}$$

In equation (1.36), we have placed the inflection point at the origin. As a tanh curve has no turning points and only a single inflection point, we have

$$D(0) = (-\infty, \infty) . \tag{1.37}$$

It seems reasonable that we should only consider symmetrical intervals as tanh is a symmetrical function. Let the interval therefore be $(-X, X)$. We then have

$$[x] = 2X \tag{1.38}$$

$$[u] = 2a \tanh \frac{aX}{2\epsilon} \tag{1.39}$$

$$u_0' = \frac{a^2}{2\epsilon} \tag{1.40}$$

$$\overline{u}' = \frac{a^2}{2\epsilon} \operatorname{sech}^2 \frac{aX}{2\epsilon} . \tag{1.41}$$

These equations imply

$$\left|\frac{[u]/[x]}{u_0'}\right| = \frac{2\epsilon}{aX} \tanh \frac{aX}{2\epsilon} , \tag{1.42}$$

and

$$\left|\frac{\overline{u}'}{[u]/[x]}\right| = \frac{aX}{2\epsilon} \operatorname{sech}^2 \frac{aX}{2\epsilon} \coth \frac{aX}{2\epsilon} . \tag{1.43}$$

Now, if we consider the limit $X \to \infty$, we obtain

$$\left|\frac{[u]/[x]}{u_0'}\right| \sim \frac{2\epsilon}{aX} \tag{1.44}$$

$$\left|\frac{\bar{u}'}{[u]/[x]}\right| \sim \frac{2aX}{\epsilon}\, \rho^{-aX/\epsilon} \; . \tag{1.45}$$

Clearly both these functions tend to zero as $X \to \infty$ (for $a, \epsilon > 0$). This implies that the 'best' shock control interval is the whole of $D(x_0)$, i.e. $(-\infty, \infty)$. This is clearly undesirable.

□

The problem with the example of the tanh curve is the exponential decay of the asymptotes. In practice this could be the case (or something similar). To make our definition sufficient we shall need the following extra quality:

iii) $[x] \to 0$ as the shock stiffens up into a discontinuity (i.e., as $\epsilon \to 0$).

We may combine these conditions in the following definition:

<u>Definition 1.2</u>

For an arbitrary interval $(x_L, x_R) \in D(x_0)$, we define

$$\theta(x_L, x_R;\ \alpha,\beta,\gamma) = \max\left\{\alpha\left|\frac{[u]/[x]}{u_0'}\right| ,\ \beta\left|\frac{\bar{u}'}{[u[/[x]}\right| ,\ \frac{\gamma[x]}{L}\right\} , \tag{1.46}$$

where $\alpha,\beta,\gamma > 0$ and L is a global length scale in x for u.

This function measures the quality of a given interval. It is now simple to define the best interval in a formal sense.

Definition 1.3

The shock control interval $I(\alpha,\beta,\gamma) = (x_L^*, x_R^*)$ is the interval in $D(x_0)$ that minimises $\theta(x_L, x_R; \alpha,\beta,\gamma)$ for the given values of α,β and γ. We may therefore write

$$\left.\begin{aligned} \theta^*(\alpha,\beta,\gamma) &= \theta(x_L^*, x_R^*; \alpha,\beta,\gamma) \\ &= \min_{(x_L,x_R) \subseteq D(x_0)} \{\theta(x_L, x_R; \alpha,\beta,\gamma)\} \end{aligned}\right\} . \tag{1.47}$$

We now see how this definition works for the function in our previous example.

Example 1.4

We again have the function given by equation (1.36). As before, we only consider symmetric intervals. As there is no known global length scale for u, we set $L = 1$.

We also need to assume that condition iii) holds, so we are at liberty to assume X is small. However, this does not necessarily imply that $\frac{aX}{2\epsilon}$ is small, as we are concerned with the limit $\epsilon \to 0$. Let

$$Z = \frac{aX}{2\epsilon} . \tag{1.48}$$

We shall assume Z is large. Equations (1.44) and (1.45) imply

$$\left|\frac{[u]/[x]}{u_0'}\right| \sim \frac{1}{Z} \tag{1.49}$$

$$\left|\frac{\bar{u}'}{[u]/[x]}\right| \sim 4Ze^{-2Z} . \qquad (1.50)$$

We assume Z increases in scale as $\epsilon \to 0$ so, for suitably small ϵ,

$$\max\left\{\alpha\left|\frac{[u]/[x]}{u_0'}\right| , \beta\left|\frac{\bar{u}'}{[u]/[x]}\right|\right\} \sim \frac{\alpha}{Z} . \qquad (1.51)$$

Clearly $\frac{\alpha}{Z}$ increases as $[x]$ decreases. Hence θ does have a minimum, at which point

$$\frac{\alpha}{Z} \sim \gamma\cdot 2X , \qquad (1.52)$$

which implies

$$X^* \sim \sqrt{\left[\frac{\alpha}{\gamma a}\epsilon\right]} , \qquad (1.53)$$

where

$$(x_L^*, x_R^*) = (-X^*, X^*) . \qquad (1.54)$$

This is consistent with the assumptions made previously.

□

1.2.3 Normalisation

The normalisation provided here will be used in the limit theory of §1.4. We recall the function $u(x;\ \epsilon,\ u_{-\infty},\ u_\infty)$ introduced in equation (1.25). Suppose it does have a turning point at x_0. Let us define

$$\left.\begin{aligned} v(\xi;\ \epsilon, v_{-\infty}, v_\infty) &= u(x_0+\xi;\ \epsilon, u_{-\infty}, u_\infty) - u_0 , \\ \text{where}\quad v_{-\infty} &= v(-\infty) = u_{-\infty} - u_0 \\ v_\infty &= v(\infty) = u_\infty - u_0 \end{aligned}\right\} . \qquad (1.55)$$

This normalises the position and the value about the inflection point. We have a corresponding definition for the shock control interval

$$\left.\begin{aligned} J(\alpha,\ \beta,\ \gamma) &= (\xi_L^*,\ \xi_R^*)\ , \\ \text{where} \qquad \xi_L^* &= x_L^* - x_0 < 0 \\ \xi_R^* &= x_R^* - x_0 > 0 \end{aligned}\right\} . \tag{1.56}$$

1.3 Exact Solutions

1.3.1 Nondimensionalisation Procedure

In this subsection, we shall be concerned with exact solutions to approximations to the Navier-Stokes' equations for one-dimensional steady flow. The non-dimensionalisation procedure is based upon constants derived from integrating the equations of motion.

The one-dimensional steady Navier-Stokes' equations can be written

$$\frac{d}{dx}\left(\underline{f} - \mu \underline{f}_v - \kappa \underline{f}_c\right) = 0\ , \tag{1.57}$$

where

$$\underline{f} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho u\left(U + \frac{p}{\rho} + \frac{1}{2} u^2\right) \end{bmatrix} \tag{1.58}$$

$$\underline{f}_v = \begin{bmatrix} u \\ \frac{4}{3} u' \\ \frac{4}{3} u u' \end{bmatrix} \tag{1.59}$$

$$\underline{f}_c = \begin{bmatrix} 0 \\ 0 \\ T' \end{bmatrix} . \tag{1.60}$$

where

$$p = p(\rho,\ T) \tag{1.61}$$

is the equation of state and U is the internal energy, which may have a non-equilibrium form. μ and κ need not be constants.

Despite all this possible freedom, equation (1.57) may still be integrated to give

$$\underline{f} - \mu\underline{f}_v - \kappa\underline{f}_c = \begin{bmatrix} Q \\ P \\ QH \end{bmatrix} . \qquad (1.62)$$

Q is called the flow rate, P the flow stress and H the entralpy.

Now we turn to the non-dimensionalisation process. We introduce scale values:

$$u_0 = \frac{P}{Q} \qquad (1.63)$$

and x_o, t_0 may take any values obeying

$$\frac{x_0}{t_0} = u_0 . \qquad (1.64)$$

Also, define

$$\lambda = \frac{H}{u_0^2} . \qquad (1.65)$$

The scale values for the other variables are given as

$$\left. \begin{aligned} \rho_0 &= \frac{Q^2}{P} \\ P_0 &= P \\ T_0 &= \frac{u_0^2}{R} \\ U_0 &= u_0^2 \end{aligned} \right\} . \qquad (1.66)$$

Substituting into equations (1.57) to (1.60) eventually yields

$$\left.\begin{aligned} \tilde{\rho}\tilde{u} &= 1 \\ \tilde{P} + \tilde{\rho}\tilde{u}^2 - \tfrac{4}{3}\,\tilde{u}\,\frac{d\tilde{u}}{d\tilde{x}} &= 1 \\ \tilde{U} + \frac{\tilde{p}}{\tilde{\rho}} + \tfrac{1}{2}\,\tilde{u}^2 - \tfrac{4}{3}\,\tilde{u}\tilde{u}\,\frac{d\tilde{u}}{d\tilde{x}} - \tilde{K}\,\frac{d\tilde{T}}{d\tilde{x}} &= \lambda \end{aligned}\right\} , \qquad (1.67)$$

where, $\tilde{u} = \dfrac{u}{u_0}$, etc. ....

In particular,

$$\tilde{u} = \frac{u}{u_0}, \quad \text{where} \quad \mu_0 = Pt_0 , \qquad (1.68)$$

and

$$\tilde{K} = \frac{K}{K_0}, \quad \text{where} \quad K_0 = PRt_0 . \qquad (1.69)$$

Equations (1.67) are the non-dimensionalised equations.

1.3.2 Taylor Shock

The idea now is to analyse the system of equations (1.57) into the three dominant physical diffusion processes. The processes are viscosity, bulk relaxation and conduction.

Taking the first process, we restrict equations (1.57) with the assumptions

$$\left.\begin{aligned} K &= 0 \\ \mu &= \text{const} \\ P &= \rho RT \\ U &= \frac{p}{(\gamma-1)\rho} \end{aligned}\right\} . \qquad (1.70)$$

Substituting these values into equations (1.67) and removing the '~' signs yields:

$$\left.\begin{aligned} \rho &= 1/u \\ \frac{p}{\rho} &= u - u^2 + \tfrac{4}{3}\mu u \frac{du}{dx} \end{aligned}\right\} . \qquad (1.71)$$

Substituting for U and $\frac{p}{\rho}$ in the third equation of (1.70) gives, eventually

$$\frac{4}{3(\gamma-1)}\mu u \frac{du}{dx} = \frac{\gamma+1}{2(\gamma-1)} u^2 - \frac{\gamma}{\gamma-1} u + \lambda . \qquad (1.72)$$

But we know $\frac{du}{dx} = 0$ when $u = u_{-\infty}$ and $u = u_{\infty}$. Hence equation (1.72) must be

$$\frac{4}{3(\gamma-1)}\mu u \frac{du}{dx} = -\frac{\gamma+1}{2(\gamma-1)} (u_{-\infty} - u)(u - u_{\infty}) \qquad (1.73)$$

as $u_{-\infty} > u > u_{\infty}$ for $-\infty < x < \infty$. This in turn implies

$$x = -\frac{8\mu}{3(\gamma+1)} \int_{u(0)}^{u(x)} \frac{vdv}{(u_{-\infty}-v)(v-u_{\infty})} . \qquad (1.74)$$

This may be integrated analytically. The resulting solution is known as a Taylor shock after G.I. Taylor.

1.3.3 Relaxing Flow Shock

The second dominating physical diffusion process is that of relaxation, where some of the energy of the gas is modelled as lagging behind its equilibrium value. The case we shall investigate here is that of the internal energy corresponding to the rotational degrees of freedom lagging behind those degrees of freedom corresponding to translational motion.

In general, let the gas have α translational and β rotational degrees of freedom. For a diatomic gas, $\alpha = 3$ and $\beta = 2$. We shall merely assume α and β are constant. Let V be the translational part of the internal energy and B the rotational part. Thus

$$V = B + V . \tag{1.71}$$

We shall also assume the translational part takes the form as described previously:

$$V = \frac{\alpha}{2}\frac{p}{\rho} . \tag{1.72}$$

However, the non-equilibrium form of the rotational energy will be given by

$$\left.\begin{aligned} \frac{DB}{Dt} &= -\frac{1}{\tau}(B-B_0) , \\ \text{where} \qquad B_0 &= \frac{\beta}{2}\frac{p}{\rho} \end{aligned}\right\} . \tag{1.73}$$

This is the Landau-Teller form (see [2] p.204). τ is the lag time for the rotational energy. It has been found, theoretically and experimentally, to be a certain function of pressure and temperature. We shall assume it is a constant (an acceptable approximation for weak shocks).

As we are analysing this process, we shall also assume

$$\left.\begin{array}{l} K = 0 \\ \mu = 0 \end{array}\right\} . \qquad (1.74)$$

The ideal gas law of equation (1.70) will also be assumed. Eventually, again assuming the same conditions on the stationary points of u, we obtain

$$u\left\{u - \frac{(2+\beta)}{2(1+\beta)}\right\}u' = \left[\frac{1+\alpha+\beta}{2(1+\beta)}\right](u_{-\infty} - u)(u - u_{\infty}) . \qquad (1.75)$$

This system is also analytically solvable for $x = x(u)$. However, there is a problem with the potential sign of the left-hand side. This gives a critical speed of

$$u_{crit} = \frac{2+\beta}{2(1+\beta)} . \qquad (1.76)$$

Writing in dimensionalised variables, this gives

$$\left.\begin{array}{ll} & u^2_{crit} = \dfrac{\gamma_r p}{\rho} , \\ \text{where} & \gamma_r = \dfrac{2+\beta}{\beta} \end{array}\right\} . \qquad (1.77)$$

This gives a critical mach number of

$$M_{crit} = \sqrt{\frac{\gamma_r}{\gamma}} . \qquad (1.78)$$

It turns out that when $u_{-\infty} < u_{crit}$, the system is well-posed, but when $u_{-\infty} > u_{crit}$, the solution curve is double-valued and never reaches the left-hand side. These cases are shown in figure 2.

This situation also occurs in traffic flow equations (see [3], p76]. In this case, Whitham argues that when the solution becomes double-valued, a fundamental process (namely, some sort of diffusion) is being ignored. A discontinuity could be fitted to the solution, but it is not known where.

1.3.4 Conductive Shock

This analysis is provided in order to complete the section rather than model a physically isolated diffusion process. In fact, strange things (such as loss of regularity in shock fronts) can happen in some circumstances when heat conduction is allowed to dominate viscosity (see [4]).

We consider a very simple case here, with the ideal gas law and internal energy relations as in §1.3.2, along with the assumptions:

$$\left.\begin{aligned} \mu &= 0 \\ K &= \text{const} \end{aligned}\right\} . \tag{1.79}$$

Putting these values into equations (1.67) eventually gives us

$$\frac{4(\gamma-1)}{\gamma+1}\tilde{K}(u - \tfrac{1}{2})u' = -(u_{-\infty} - u)(u - u_{\infty}) . \tag{1.80}$$

This equation may also be integrated analytically to give $x = x(u)$. However, as in §1.3.3, we again have problems with the sign of the left-hand side. This gives us a critical speed of

$$u^2_{crit} = \frac{p}{\rho} \tag{1.81}$$

in re-dimensionalised variables, and a critical mach number of

$$M_{crit} = \frac{1}{\sqrt{\gamma}} . \tag{1.82}$$

It turns out that, in this case, we have the opposite problem to before; i.e. that the system is only well-posed for $u_\infty > u_{crit}$, otherwise some other diffusion process needs to be incorporated.

1.3.5 Survey of Other Solutions

The set of solutions provided so far, with their severe simplifications, do not display the limits of analytical methods for the one-dimensional steady Navier-Stokes' equations.

Lighthill ([5]) has provided a model for advection-diffusion which incorporates all three of the processes described above into a single diffusion coefficient.

Becker ([6]) has solved the equations when viscosity and thermal conductivity are constant and related by having a constant Prandtl number of ¾ (which is close to the value for air).

Finally, Pike ([7]) has extended Becker's solution to the case of varying viscosity (still keeping the Prandtl number at ¾).

Obviously, other exact solutions do exist, but it is hoped that the survey provided is fairly complete.

1.4 The Non-diffusive and Weak Limits

1.4.1 The Order of the Limits

In [8], it was shown that the α-fraction width of the first order solution to Lighthill's model is

$$\lambda_{\alpha}^{(1)} = \frac{4\delta}{(\gamma+1)[u]} \ln\left[\frac{1-\alpha}{\alpha}\right] \tag{1.83}$$

where

$$[u] = u_{-\infty} - u_{\infty} , \tag{1.84}$$

and δ is the 'diffusivity of sound' (the single diffusion coefficient mentioned above). The structure of equation (1.83) is typical of all the models described so far in as much as the width is directly proportional to the diffusion coefficient and inversely proportional to the shock strength.

This leads us to the conjecture in the limit $d \to 0$ and $[u] \to 0$,

$$\lambda \propto \frac{d}{[u]} , \tag{1.85}$$

where d is the scale value of the diffusion coefficient and λ is a measure of the shock width. From this conjecture, we obtain the simple results

$$\left.\begin{array}{l} \lim\limits_{d\to 0^+} \lim\limits_{[u]\to 0^+} \lambda = \infty \\ \lim\limits_{[u]\to 0^+} \lim\limits_{d\to 0^+} \lambda = 0 \end{array}\right\} . \tag{1.86}$$

These results have direct relevance to the structure of the shock tip in two-dimensional steady flow in the non-diffusive limit. The first equation implies that the shock width at the tip will become infinite whenever there is any diffusion present. The second equation implies the shock tip is a discontinuity in the non-diffusive system.

Figures 2 and 3 are three-dimensional surface plots for the leading order approximation of u against x and $[u]^2$ in the cases $d > 0$,

$d = 0$ respectively. The axis $[u]^2$ has been chosen instead of $[u]$ in order to relate it to the two-dimensional steady case (see [9], §2.3). The infinite gradient of u along the axis $[u] = 0$ is not expected to be a physical phenomenon however.

1.4.2 Asymptotic Expansions

The aim here is to show the leading order structure of the solution to a single general conservation law is indeed a tanh curve.

We therefore begin with the single equation version of (1.7):

$$\frac{d}{dx}\left\{f(u) - dW(u)\frac{du}{dx}\right\} = s(u;\ x) \ . \tag{1.87}$$

For simplicity, we shall also assume there is no source term. This leads us immediately, by simple integration to

$$\frac{du}{dx} = \frac{f(u) - A}{dW(u)} \ , \tag{1.88}$$

for some constant A. As in §1.3, we know that $\frac{du}{dx} = 0$ for $u = u_{-\infty}$ and $u = u_{\infty}$. This implies

$$f(u_{-\infty}) = f(u_{\infty}) = A \ , \tag{1.89}$$

assuming $W(u)$ is bounded. As in §1.3.3, the case $f(u) = A$ for some $u \in (u_{\infty},\ u_{-\infty})$ is discounted as it leads to a badly-posed problem. Therefore we may assume f is either convex or concave in u over the interval $(u_{\infty},\ u_{-\infty})$. This in turn implies there exists a unique value, u_0 say, such that

$$\left.\begin{aligned} & u_0 \in (u_\infty, u_{-\infty}) \\ & \frac{df}{du}(u_0) = 0 \end{aligned}\right\} \qquad (1.90)$$

We now introduce the following normalisations (c.f. §1.2.3)

$$\left.\begin{aligned} v &= u - u_0 \\ g(v) &= f(u_0 + v) - A = f(u) - A \\ L(v) &= W(u_0 + v) = W(u) \\ v_{-\infty} &= u_{-\infty} - u_0 \ (> 0) \\ v_\infty &= u_\infty - u_0 \ \ (< 0) \end{aligned}\right\} \qquad (1.91)$$

Thus,

$$g(0) = f(u_0) - A . \qquad (1.92)$$

Without loss of generality, we assume $f(u_0) - A > 0$, so we may write

$$g(0) = \lambda^2 . \qquad (1.93)$$

for some $\lambda \in \mathbb{R}$.

We also have

$$g'(0) = 0 , \qquad (1.94)$$

and

$$\left.\begin{aligned} g(v_{-\infty}) &= 0 \\ g(v_\infty) &= 0 \end{aligned}\right\} \qquad (1.95)$$

Now, equation (1.88) transforms to

$$\frac{dv}{dx} = \frac{g(v)}{dL(v)} , \qquad (1.96)$$

which implies

$$\frac{x}{d} = \int_0^V \frac{L(w)}{g(w)} dw , \tag{1.97}$$

where we have normalised x so that $v(0) = 0$. We now seek a Taylor expansion of $g(w)$ about $w = 0$. Equations (1.93) and (1.94) imply

$$g(w) = \lambda^2 + \frac{g''(0)}{2!} w^2 + \frac{1}{2!} \int_0^w (w-z)^2 g'''(z) dZ . \tag{1.98}$$

Assuming $g'''(v)$ is bounded, we see that for suitably small values of $v_{-\infty}$ and v_{∞}, $g''(0)$ must be negative in order to satisfy equations (1.95). Let us therefore assume

$$\frac{g''(0)}{2} = - \mu^2 . \tag{1.99}$$

So equation (1.97) becomes

$$\frac{x}{d} = \int_0^V \frac{L(w)dw}{\lambda^2 - \mu^2 w^2 + ½ \int_0^w (w-z)^2 g'''(z) dZ} . \tag{1.100}$$

Unfortunately, we are not able to expand the denominator any further as $\lambda'' - \mu^2 w^2$ becomes small for w at the extreme values. We may only conclude that the leading order truncated equation

$$\frac{x}{d} = \int_0^V \frac{L(0)dw}{\lambda^2 - \mu^2 w^2} \tag{1.101}$$

does indeed have a tanh-like solution. It is in fact

$$u(x) = u_0 - \frac{\lambda}{\mu} \tanh\left[\frac{\lambda\mu}{dL(0)} x\right] \qquad (1.102)$$

with

$$\frac{\lambda}{\mu} \sim \frac{[u]}{2} \quad \text{as} \quad [u] \to 0 \ . \qquad (1.103)$$

From equation (1.99), it is clear that μ remains constant in the limit $[u] \to 0$. So we do indeed recover the relationship (1.85).

Our conclusion here is that even limitingly weak shocks for a single equation in one dimension may contain a non-trivial structure.

1.4.3 Theory and Conjectures

We now switch our attention to the recovery of shock discontinuities and their related jump conditions in the nondiffusive limit.

For one-dimensional steady flow, the appropriate form of the Rankine-Hugoniot jump conditions is the simple relations

$$[f^i(\underline{u})] = 0 \ . \qquad (1.104)$$

Our purposes may be achieved here if we are able to construct a functional of diffusive jump functions $\{\cdot\}_d$, where

$$\{\phi\}_d = \phi(x_R(d)) - \phi(x_L(d)) \qquad (1.105)$$

$$x_L(d) < x_0(d) < x_R(d) \qquad (1.106)$$

$$\left.\begin{aligned} \lim_{d\to 0^+} x_L(d) &= x_0(0)^- \\ \lim_{d\to 0^+} x_R(d) &= x_0(0)^+ \end{aligned}\right\} \tag{1.107}$$

$$\lim_{d\to 0} \{\cdot\}_d \equiv [\cdot] , \tag{1.108}$$

where [ ] measures a jump discontinuity at $x_0(0)$. x_L, x_0 and x_R are defined as in §1.2.1 and are considered continuous functions of d for all $d < d_0$.

Of course, a natural choice of $\{\ \}_d$ is to use the interval $I(\alpha, \beta, \gamma; d)$ of §1.2.2, for some α, β, γ. Equation (1.46) may be rewritten to give

$$\theta^*(\alpha, \beta, \gamma; d) = \max\left\{\alpha \left|\frac{\{u\}_d/\{x\}_d}{u_0'}\right| , \beta \left|\frac{\bar{u}_d'}{\{u\}_d/\{x\}_d}\right| , \gamma \frac{\{x\}_d}{L}\right\} , \tag{1.109}$$

from which we may infer

$$\left|\frac{\{u\}_d/\{x\}_d}{u_0'}\right| \leq \frac{\theta^*(\alpha,\beta,\gamma;\ d)}{\alpha} , \tag{1.110}$$

$$\left|\frac{\bar{u}_0'}{\{u\}_d/\{x\}_d}\right| \leq \frac{\theta^*(\alpha,\beta,\gamma;\ d)}{\beta} , \tag{1.111}$$

and

$$\frac{\{x\}_d}{L} \leq \frac{\theta^*(\alpha,\beta,\gamma;\ d)}{\gamma} . \tag{1.112}$$

Equation (1.87) integrates to give

$$\{f(u)\}_d = \int_{I(\alpha,\beta,\gamma;\ d)} \left[dW(u)\ \frac{du}{dx} + S(u;\ x)\right]dx\ . \tag{1.113}$$

But

$$\int_{I(\alpha,\beta,\gamma;\ d)} W(u)\ \frac{du}{dx}\ dx = \left\{\int W(u)du\right\}_d\ . \tag{1.114}$$

and, assuming $S(u;\ x)$ is bounded,

$$\exists\ k > 0 \ \text{ st } \ \forall\ d < d_0\ ,$$

$$\left| \int_{I(\alpha,\beta,\gamma;\ d)} S(u;\ x)dx \right| < K\{x\}_d\ . \tag{1.115}$$

Thus,

$$\left|\{f(u)\}_d\right| < d/\left\{\int W(u)du\right\}_d\left| + K\{x\}_d\right.\ . \tag{1.116}$$

So, provided $\theta^*(\alpha,\ \beta,\ \gamma;\ d) \to 0$ as $d \to 0$, equations (1.112) and (1.116) imply that

$$\{f(u)\}_d \to 0 \ \text{ as } \ d \to 0\ . \tag{1.117}$$

Equations (1.110) and (1.11?) also suggest that the interval $I(\alpha,\ \beta,\ \delta;\ d)$ is of the right form for equation (1.108) to hold.

The argument provided is not exact, nor does it say anything about whether a discontinuity will form in $u(x;\ u_{-\infty},\ u_{\infty},\ d)$ in the limit $d \to 0$, only that if it does, then we are likely to pick it up. (It can be shown that if u is a constant function with a step at x_0, then $I = (x_0^-,\ x_0^+)$ and $\theta^* = 0 \quad \forall\ \alpha,\ \beta,\ \gamma$).

We finish this section by stating a conjecture which is intended to be intermediate between our initial equations and a strong notion of the formation of discontinuities. We shall require the appropriate simplification of the norm introduced in [9], viz

$$\|(x, v(x))\|_{L,v} = \sqrt{\left[\left[\frac{x}{L}\right]^2 + \left[\frac{v(x)}{U}\right]^2\right]} \; . \tag{1.118}$$

We shall also require the normalisations introduced in §1.2.3; given by equations (1.55) and (1.56). We shall drop the dependence of v on $v_{-\infty}$ as these values are taken to be constant here.

Putting this altogether gives us the hypothesis

$\forall$ L, U > 0 $\forall \epsilon > 0$ $\exists$ $d(\epsilon) < d_0$ such that $\forall$ $d_0 d_2 \in (0, d(\epsilon))$

$\forall \xi_1 \in J_{max}(\quad d_1) \cap J_{max}(\quad d_2)$

$\exists \xi_2 \in J_{max}(\quad d_1) \cap J_{max}(\quad d_2)$ such that

$$\|(\xi_1, v(\xi_1; d_1)) - (\xi_2, v(\xi_2; d_2))\|_{L,v} < \epsilon \tag{1.119}$$

where

$$J_{max}(\quad d) = (X_L(d) - x_{dd}), \; X_R(d) - x_0(d)) \; . \tag{1.120}$$

2. Stability Theory

2.0 Introduction

In this chapter, we shall discuss two different stability theories used in describing the behaviour of non-linear first order differential equations with source terms.

The first approach involves transforming the system into a single perturbation equation and then comparing the wavespeeds of the leading order operator and the preceding lower order operators.

The second approach involves imposing the standard plane wave form of perturbation to the system. It turns out that only one form of perturbation is possible and that the method is equivalent to taking the Fourier transform of the system.

The author has not been able to find references to the theory of these methods, only to applications of them, apart from the wavespeed comparisons for the first method.

2.1 Whitham's Method

2.1.1 Initial System

We have named this method after the process used by Whitham in [10]. Our description of the argument proceeds as follows. Let us initially consider the system of equations:

$$L\underline{u}(\underline{x}) = \underline{0} , \tag{2.1}$$

where $\underline{x}$ is the vector of m independent variables, $\underline{u}$ is the vector of n dependent variables and L is a matrix of linear first order partial differential operators with respect of $\underline{x}$. We may therefore write

$$L \equiv \sum_{p=1}^{m} L^{(p)} \partial/\partial x_p , \tag{2.2}$$

for some constant matrices $L^{(1)},\ldots,L^{(m)}$.

Initially, let us consider the case $m = 2$. Let us use the simplified notation

$$\left.\begin{array}{lll} \underline{x} & = & (x,\ y) \\ L^{(1)} & = & A \\ L^{(2)} & = & B \end{array}\right\} . \tag{2.3}$$

Here equation (3.1) becomes

$$A\,\frac{\partial \underline{u}}{\partial x} + B\,\frac{\partial \underline{u}}{\partial y} \;=\; 0 \;. \tag{2.4}$$

By premultiplying by A^{-1} and transforming variables $\underline{u} = M\underline{v}$, for a constant matrix M, it is possible to transform equation (2.4) into

$$\frac{\partial \underline{v}}{\partial x} + \Lambda\,\frac{\partial \underline{v}}{\partial y} \;=\; 0 \;, \tag{2.5}$$

where

$$\Lambda \;=\; \mathrm{diag}\{\lambda_1,\ldots,\lambda_n\} \;, \tag{2.6}$$

for certain constants $\lambda_1,\ldots,\lambda_n$ - the eigenvalues of $A^{-1}B$.

Now, let us consider the function

$$\phi \;=\; \sum_{i=1}^{n} C_i v_i \;, \tag{2.7}$$

for arbitrary constants $C_1,\ldots,C_n$. Equation (2.5) gives us

$$\left[\frac{\partial}{\partial x} + \lambda_i\,\frac{\partial}{\partial y}\right] v_i \;=\; 0 \qquad \forall\ i \;. \tag{2.8}$$

From equations (2.7) and (2.8) we therefore infer

$$\left\{ \prod_{i=1}^{n} \left[\frac{\partial}{\partial x} + \lambda_i \frac{\partial}{\partial y} \right] \right\} \phi = 0 . \tag{2.9}$$

The coefficients λ_i represent the wavespeeds for the arbitrary perturbation ϕ.

Now, let us return to the case of general m. We now introduce the following notation

$$(\text{adj } L)_{ij} \equiv (-1)^{i+j} \det L^{(j,i)} , \tag{2.10}$$

where

$$L^{(i,j)}_{p,q} \equiv L_{p+I(p\geq i),q+I(q\geq j)} \tag{2.11}$$

where $I(W)$ is the indicator function for the equation W such that

$$\left. \begin{array}{l} I(W) = 1 \text{ when } W \text{ is true} \\ I(W) = 0 \text{ when } W \text{ is false} \end{array} \right\} \tag{2.12}$$

The det operator is defined as in normal linear algebra, i.e.

$$\det L \equiv \sum_{\rho \in S_n} \epsilon_\rho \prod_{i=1}^{n} L_{i,\rho(i)} , \tag{2.13}$$

where S_n is the symmetry group of size n and ϵ is the permutation function. The operator is well-defined because the operators $L_{i,j}$ are commutative, associative and distributive (although they are not invertible). This implies that the following relation holds analogously to normal linear algebra.

$$(\text{adj } L)L \equiv (\det L)I \ . \tag{2.14}$$

So, premultiplying equation (2.1) by adj L gives

$$(\det L)\underline{u} = \underline{0} \ . \tag{2.15}$$

So if we now use

$$\phi = \sum_{i=1}^{n} c_i u_i \ , \tag{2.16}$$

we obtain

$$(\det L)\phi = 0 \ , \tag{2.17}$$

which degenerates to equation (2.9) in the case $m = 2$, as the constants c_i are arbitrary.

2.1.1 Additional Source Term

Let us now consider a slightly more complex initial equation:

$$L\underline{u} = A(\underline{u} - \underline{u}_0) \ , \tag{2.18}$$

for some constant matrix A and vector $\underline{u}_0$. Introducing

$$\underline{v} = \underline{u} - \underline{u}_0 \ , \tag{2.19}$$

equation (2.18) transforms to

$$L\underline{v} = A\underline{v} \ ,$$

i.e.

$$(L - A)\underline{v} = \underline{0} \ . \tag{2.20}$$

Now, even though L is an operator and A is a matrix, they are still both commutative, associative and distributive with respect to multiplication and addition. This is true even though the operation of partial differentiation on the constant terms of A will result in zero. We may also derive the following result analogously to become:

$$(\mathrm{adj}(L - A))(L - A) \equiv \det(L - A)I \ , \qquad (2.21)$$

whose adj and det are defined in the same way as before.

We now require the following lemma:

<u>Lemma 2.1</u>

$$\det(A-B) = \det A - \mathrm{tr}(\mathrm{adj}\ A\ B) + \ldots + (-1)\mathrm{tr}(A\ \mathrm{adj}\ B) + (-1)^n \det B \ . \qquad (2.22)$$

Proof

$$\det(A-B) = \sum_{\rho \in S_n} \epsilon_\rho \prod_{i-1}^{n} (A_{i,\rho(i)} - B_{i,\rho(i)}) \ , \qquad (2.23)$$

from equation (2.13). Now

$$\prod_{i=1}^{n} (A_{i,\rho(i)} - B_{i,\rho(i)}) = \prod_{i=1}^{n} A_{i,\rho(i)} - \sum_{j=1}^{n} B_{j,\rho(j)} \prod_{i \neq j} A_{i,\rho(i)} + \ldots$$

$$\ldots + (-1)^{n-1} \sum_{j=1}^{n} A_{j,\rho(j)} \prod_{i \neq j} B_{i,\rho(i)} + (-1)^n \prod_{i=1}^{n} B_{i,\rho(i)} \ . \qquad (2.24)$$

Taking the second term of the right-hand side of equation (2.24) and applying the summation of equation (2.23) gives

$$\sum_{\rho\in S_n} \epsilon_\rho \sum_{j=1}^{n} B_{j,\rho(j)} \prod_{i\neq j} A_{i,\rho(i)} = \sum_{j,k=1}^{n} B_{j,k} \sum_{\substack{\rho\in S_n \text{ st} \\ \rho(j)=k}} \epsilon_\rho \prod_{i\neq j} A_{i,\rho(i)} \quad . \qquad (2.25)$$

We assert that

$$\sum_{\substack{\rho\in S_n \text{ st} \\ \rho(j)=k}} \epsilon_\rho \prod_{i\neq j} A_{i,p(i)} = (-1)^{j+k} \sum_{\tau\in S_{n-1}} \epsilon_\tau \prod_{i=1}^{n-1} A_{i+I(i\geq j),\tau(i)+I(\tau(i)\geq k)}$$

$$= (-1)^{j+k} \sum_{\tau\in S_{n-1}} \epsilon_\tau \prod_{i=1}^{n-1} A^{(j,k)}_{i,\tau(i)}$$

$$= (-1)^{j+k} \det A^{(j,k)}$$

$$= (\mathrm{adj}A)_{k,j} \quad . \qquad (2.26)$$

Substituting into equation (2.25) gives

$$\sum_{\rho\in S_n} \epsilon_\rho \sum_{j=1}^{n} B_{j,\rho(j)} \prod_{i\neq j} A_{i,\rho(i)} = \sum_{j,k=1}^{n} B_{j,k}(\mathrm{adj}A)_{k,j} \quad . \qquad (2.27)$$

We clearly have the dual result:

$$\sum_{\rho\in S_n} \epsilon_\rho \sum_{j=1}^{n} A_{j,\rho(i)} \prod_{i\neq j} B_{i,\rho(i)} = \sum_{j,k=1}^{n} A_{j,k}(\mathrm{adj}A)_{k,j} \qquad (2.28)$$

Equations (2.23), (2.24), (2.27) and (2.28) therefore give the result. □

Lemma 2.1 implies that we may expand $\det(L-A)$ in the same way to give

$$\det(L-A) \equiv \det L - \operatorname{tr}(A \operatorname{adj} L) + \ldots , \tag{2.29}$$

where the neglected terms are linear operators of degree less than $(n-1)$.

Equations (2.20), (2.21) and (2.29) imply

$$\{\det L - \operatorname{tr}(A \operatorname{adj} L) + \ldots\}\underline{v} = \underline{0} . \tag{2.30}$$

So, setting

$$\phi = \sum_{i=1}^{n} c_i v_i , \tag{2.31}$$

we obtain

$$\{\det L - \operatorname{tr}(A \operatorname{adj} L) + \ldots\}\phi = 0 , \tag{2.32}$$

for arbitrary constants $c_1,\ldots,c_n$.

Now, only taking terms with derivatives of order n or $(n-1)$ leads to the truncated system:

$$\{\det L - \operatorname{tr}(A \operatorname{adj} L)\}\phi = 0 . \tag{2.33}$$

The term $(\det L)\phi$ is the same as that encountered in §2.1.1. In the case $m = 2$ it gives the product of wavespeeds. The other term, $-\operatorname{tr}(A \operatorname{adj} L)\phi$, gives the wave equation for the reduced set of equations. In the case $m = 2$ it will also be expandable in the form:

$$\operatorname{tr}(A \operatorname{adj} L) \equiv \alpha \prod_{i=1}^{n-1} \left[\frac{\partial}{\partial x} + \mu_i \frac{\partial}{\partial y}\right] , \tag{2.34}$$

for some constants $\mu_1,\ldots,\mu_{n-1}$. Whereas previously the constant α could be removed, here, because equation (2.33) has the two terms, it will remain. So the system (2.33) converts to one of the form

$$\left\{\prod_{i=1}^{n} \left[\frac{\partial}{\partial x} + \lambda_i \frac{\partial}{\partial y}\right] + \beta \prod_{i=1}^{n-1} \left[\frac{\partial}{\partial x} + \mu_i \frac{\partial}{\partial y}\right]\right\}\phi = 0 . \tag{2.35}$$

Further terms on this expansion could be found by expanding det(L−A) directly and then regrouping terms of the same order. This will lead to a full system of the form

$$\left\{\sum_{k=0}^{n} \alpha_k \prod_{i=1}^{k} \left[\frac{\partial}{\partial x} + \lambda_i^{(k)} \frac{\partial}{\partial y}\right]\right\}\phi = 0 , \tag{2.36}$$

for some constants $\alpha_0,\ldots,\alpha_n$, $\lambda_i^{(k)}$: $k \le n$, $i \le k$. Without loss of generality, we may impose $\alpha_n = 1$. This theory is applied to some extent in the next section.

2.1.3 First Order Nonlinear System

We now consider the system

$$L(\underline{u})\underline{u} = \underline{f}(\underline{u}) , \tag{2.37}$$

where

$$L(\underline{u}) \equiv \sum_{p=1}^{m} L^{(p)}(\underline{u}) \frac{\partial}{\partial x_p} , \tag{2.38}$$

and $\underline{f}$ is a general nonlinear function of $\underline{u}$.

The first order approximation to equation (2.37) is found by imposing the perturbation equation (2.19). This gives us

$$L(\underline{u}_0)\underline{v} = \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)\underline{v} . \tag{2.39}$$

This is because we must have

$$\underline{f}(\underline{u}_0) = \underline{0} \tag{2.40}$$

in order to balance both sides of equation (2.37) to leading order.

But equation (2.39) is identical in structure to equation (2.20). We therefore infer the highest order truncation equation corresponding to equation (2.33) to be

$$\left\{\det L(\underline{u}_0) - \mathrm{tr}\left[\frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)\mathrm{adj}(L(\underline{u}_0))\right]\right\}\phi = 0 . \tag{2.41}$$

In the case $m = 2$, we obtain a wave equation corresponding to equation (2.35), except the coefficients $\lambda_1,\ldots,\lambda_n$, β, $\mu_1,\ldots,\mu_{n-1}$ are now functions of $\underline{u}_0$. In particular, the size of the coefficient β indicates whether the operator of order n or the operator of order $(n-1)$ is dominating the equation.

This argument may be extended to the case of the full sequence of operators given by equation (2.36). This is a remarkable result because, not only does it tell us what the appropriate wavespeeds of the reduced systems should be, it also tells us when to make the appropriate approximations - namely: according to the relative sizes of the coefficients $\alpha_0(\underline{u}_0),\ldots,\alpha_{n-1}(\underline{u}_0)$.

A simple example provided in [10] is now given.

Example 2.1 - **Flood Waves**

$$\left.\begin{aligned} \underline{u} &= (h,\ u)^T \\ \underline{x} &= (t,\ x)^T \end{aligned}\right\} \qquad (2.42)$$

$$\left.\begin{aligned} L(\underline{u}) &\equiv A(\underline{u})\frac{\partial}{\partial t} + B(\underline{u})\frac{\partial}{\partial x} \\ A(\underline{u}) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ B(\underline{u}) &= \begin{bmatrix} u & h \\ g & u \end{bmatrix} \\ \underline{f}(\underline{u}) &= \begin{bmatrix} 0 \\ gS - gu^2/c^2h \end{bmatrix} \end{aligned}\right\} . \qquad (2.43)$$

We impose $\underline{f}(\underline{u}_0) = \underline{0}$ and the leading order expansion equation (2.41). In fact because $n = 2$ and $\underline{f}(\underline{u})$ has a zero, there are no other terms present.

We obtain the system

$$\left\{\left[\frac{\partial}{\partial t} + c_1\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x}\right] + \alpha\left[\frac{\partial}{\partial t} + a_1\frac{\partial}{\partial x}\right]\right\}\phi = 0 , \qquad (2.45)$$

where

$$\left.\begin{aligned} c_1 &= u_0 - \sqrt{gh_0} , \\ c_2 &= u_0 - \sqrt{gh_0} , \\ a_1 &= \tfrac{3}{2}u_0 , \\ \alpha &= \frac{2gS}{u_0} \end{aligned}\right\} . \qquad (2.45)$$

and

□

The observation made in the last example is generalised in the following lemma.

Lemma 2.2

When $\underline{f}(\underline{u})$ has k constant terms, the lowest order operator present in the first order expansion wave equation will be of order at least k.

Proof

If $\underline{f}(\underline{u})$ has k constant terms (which must all be zeros, as $\underline{f}(\underline{u}_0) = \underline{0}$), then $\frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)$ will have k rows of zeros. Consider the expansion of $\det(L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0))$. If we take less than k products from $L(\underline{u}_0)$ in any term, we will have to take more than (n-k) products from $\frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)$. However, by the structure of the determinant function, these must all be taken from different rows. Therefore one of them must be taken from a row containing all zeros.

■

2.1.4 Second Order Nonlinear System

Again, we start with equation (2.37). We now expand it up to order $|\underline{v}|^3$. We obtain

$$L(\underline{u}_0)\underline{v} + \left[\underline{v}\cdot\frac{\partial L}{\partial \underline{u}}(\underline{u}_0)\right]\underline{v} = \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)\underline{v} + \left[\frac{\partial^2 \underline{f}}{\partial \underline{u}^2}(\underline{u}_0)\underline{v}\right]\underline{v} . \tag{2.46}$$

This may be rewritten in the form

$$\left[L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0) + \underline{v}\cdot\frac{\partial L}{\partial \underline{u}}(\underline{u}_0) - \frac{\partial^2 \underline{f}}{\partial \underline{u}^2}(\underline{u}_0)\underline{v}\right]\underline{v} = \underline{0} . \tag{2.47}$$

So, by the same argument to before, we obtain the equation

$$\det\left[L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0) + \underline{v}\cdot\frac{\partial L}{\partial \underline{u}}(\underline{u}_0) - \frac{\partial^2 \underline{f}}{\partial \underline{u}^2}(\underline{u}_0)\underline{v}\right]\phi = 0 \qquad (2.48)$$

for an arbitrary perturbation ϕ. We need to neglect terms of order $|\underline{v}|^2$ in equation (2.48). This gives us

$$\left\{\det(L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)) + \mathrm{tr}\left[\left[\underline{v}\cdot\frac{\partial L}{\partial \underline{u}}(\underline{u}_0) - \frac{\partial^2 \underline{f}}{\partial \underline{u}^2}(\underline{u}_0)\underline{v}\right] \mathrm{adj}\left[L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)\right]\right]\right\}\phi = 0 \qquad (2.49)$$

by use of Lemma 2.1.

Let us use the symbol $\bar{\psi}^k$ to represent taking the differential operators in ψ of order k only.

Clearly, as already shown,

$$\left.\begin{aligned} \overline{\det(L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0))}^{\,n} &= \det(L(\underline{u}_0)) \\ \overline{\det(L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0))}^{\,n-1} &= -\,\mathrm{tr}\left[\frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)\mathrm{adj}(L(\underline{u}_0))\right] \end{aligned}\right\} . \qquad (2.50)$$

Also, it may easily be shown that

$$\overline{\mathrm{adj}(L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0))}^{\,n-1} = \mathrm{adj}(L(\underline{u}_0))\Big] . \qquad (2.51)$$

Combining equations (2.49) to (2.51) shows us that the operators on ϕ of order n and $(n-1)$ up to order $|\underline{v}|^2$ are:

$$\Bigg\{\Bigg[\det(L(\underline{u}_0)) + \mathrm{tr}\left[\underline{v}\cdot\frac{\partial L}{\partial \underline{u}}(\underline{u}_0)\mathrm{adj}(L(\underline{u}_0))\right]\Bigg]$$

$$- \left[\mathrm{tr}\left[\frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0)\mathrm{adj}(L(\underline{u}_0))\right] + \mathrm{tr}\left[\frac{\partial^2 \underline{f}}{\partial \underline{u}^2}(\underline{u}_0)\underline{v}\ \mathrm{adj}(L(\underline{u}_0))\right]\right.$$

$$\left. - \ \mathrm{tr}\left[\underline{v}\cdot\frac{\partial L}{\partial \underline{u}}(\underline{u}_0)\ \overline{\mathrm{adj}(L(\underline{u}_0) - \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{u}_0))}^{\,n-2}\ \right]\right]\Bigg\}\phi = 0 . \tag{2.52}$$

We thus obtain correction terms to the differential operators acting on ϕ, which will in turn lead to correction terms to the wavespeeds and hence the critical wavespeeds (the constants α_k in equation (2.36) will also be corrected). The process may of course be extended to the inclusion of higher order terms in $\underline{v}$ and lower order differential operators. The whole method is analogous to the perturbation approximation to the canonical form of a nonlinear system devised in §1.4 of [9].

2.2 Theory of Stability Conditions on the Wavespeeds

In his concise paper, Wu ([11]) gives stability conditions on the wavespeed for systems in the form

$$\left\{\prod_{i=1}^{n}\left[\frac{\partial}{\partial t} + c_i\frac{\partial}{\partial x}\right] + \alpha\prod_{j=1}^{m}\left[\frac{\partial}{\partial t} + a_j\frac{\partial}{\partial x}\right]\right\}\phi = 0 \tag{2.53}$$

where $\alpha \in \mathbb{R}$ and with solutions for ϕ of the form

$$\phi(x,t) = e^{iK(x-Yt)} , \tag{2.54}$$

where $k \in \mathbb{R}$, $K > 0$.

The system given by equation (2.53) will clearly be stable for all time when

$$\psi_m(Y) \leq 0 . \tag{2.55}$$

The sufficient conditions for stability are stated by Wu as follows:

Theorem 2.3

Equation (2.53) has a stable solution of the form of equation (2.54) if and only if equation (2.55) holds and either

i)
$$\left.\begin{array}{l} n - m = 1 , \\ \alpha \geq 0 , \\ \text{and} \quad c_1 > a_1 > c_2 > a_2 > \ldots > a_m > c_{m+1} ; \end{array}\right\} \tag{2.56}$$

or

ii)
$$\left.\begin{array}{l} n - m = 2 \\ \alpha \geq 0 \\ (c_{i+1}, a_i) \text{ occur in pairs, for } 1 \leq i \leq n, \\ \text{with their relative positions unimportant.} \end{array}\right\} \tag{2.57}$$

Proof

See [11].

■

Although the conditions given are sufficient, Wu gives no argument to show that they are necessary. So, for example, we could consider the case $n-m = 3$. The structure of the theorem seems to suggest the following hypothesis:

For a system of the form of equation (2.36), ϕ remains stable when

$$\left.\begin{aligned} &\alpha_k \geq 0 \qquad \forall\, k\,, \\ &\text{and,}\quad \forall\, k < n\,, \\ &\lambda_1^{(k+1)} > \lambda_1^{(k)} > \lambda_2^{(k+1)} > \lambda_2^{(k)} > \ldots > \lambda_k^{(k)} > \lambda_{k+1}^{(k+1)} \end{aligned}\right\} . \qquad (2.58)$$

This hypothesis reduces to equation (2.56) in the case $\alpha_{n-2} = 0,\ldots,\alpha_1 = 0$ and to equation (2.57) in the case $\alpha_{n-1} = 0$, $\alpha_{n-3} = 0,\ldots,\alpha_1 = 0$ (as the pairing condition is equivalent to the condition on the wavespeeds $\lambda_i^{(k)}$ and $\lambda_i^{(k+1)}$ in the cases $k = n-2$ and $k = n-1$).

In the case $n = 1$, we obtain the relation

$$\zeta m(Y) = -\frac{x_0}{K}\,, \qquad (2.59)$$

which is always satisfied.

In the case $n = 2$, the following condition may be obtained:

$$\zeta m(Y) \leq 0 \Leftrightarrow \left[\lambda_2^{(2)} - \lambda_1^{(1)}\right]\left[\lambda_1^{(1)} - \lambda_1^{(2)}\right] \geq \frac{x_0}{4K^2}\,. \qquad (2.60)$$

This is clearly unsatisfiable for suitably small k. Hence, unless $\alpha_0 = 0$, the system is unconditionally unstable (the case $\alpha_0 = 0$ corresponds to case i) of theorem 3 with $n = 2$).

Hence the hypothesis given by equation (2.58) is false. It therefore needs to be weakened in some way. The two obvious weakenings are

i) Impose the condition $\alpha_0 = 0$.

ii) Impose the condition that equation (2.36) is in the reduced form of equation (2.53).

Examples of both these weakened hypotheses require $n \geq 3$ in order not to be cases of theorem 3. This leads to rather intractable calculations for Y as it will be the solution of a polynomial equation of degree ≥ 3.

Of the two conjectures, the latter seems to be the weaker and more likely one.

Wu gives two examples for the case $n-m = 2$:

i) The linearized Korteweg-DeVries equation

$$\phi_t + a\phi_x + v\phi_{xxx} = 0 , \tag{2.61}$$

which is unconditionally unstable,

and

ii) The linearized Boussinesq equation

$$\phi_{tt} - c^2\phi_{xx} = \upsilon\phi_{xxtt} , \tag{2.62}$$

which is unconditionally stable for $\upsilon > 0$.

2.3 Roe's Analysis

The analysis used by Roe in [12] can be generalised to the following argument.

As in §2.1, we initially consider equation (2.1). However, let us now consider a multi-dimensional Fourier transform

$$\underline{v}(\underline{\xi}) = (2\pi)^{-m/2} \int_{\underline{\xi}\in\mathbb{R}^m} \underline{u}(\underline{x}) e^{i\underline{x}\cdot\underline{\xi}} \, \underline{d^m x} \; . \tag{2.63}$$

By the Fourier inversion theorem, we have the inverse relation

$$\underline{u}(\underline{x}) = (2\pi)^{-m/2} \int_{\underline{\xi}\in\mathbb{R}^m} \underline{v}(\underline{\xi}) e^{-i\underline{x}\cdot\underline{\xi}} \, \underline{d^m \xi} \; . \tag{2.64}$$

Hence,

$$L\underline{u} = \sum_{p=1}^{m} L^{(p)} \frac{\partial \underline{u}}{\partial x_p}$$

$$= (2\pi)^{-m/2} \int_{\underline{\xi}\in\mathbb{R}^m} - i\sum_p \xi_p L^{(p)} \underline{v}(\underline{\xi}) e^{-i\underline{x}\cdot\underline{\xi}} \, \underline{d^m \xi} \; . \tag{2.65}$$

A sufficient condition for equations (2.1) and (2.65) to be satisfied is

$$\sum_p \xi_p L^{(p)} \underline{v}(\underline{\xi}) = \underline{0} \; . \tag{2.66}$$

Finally, pre-multiplying equation (2.66) by $\mathrm{adj}\left[\sum_p \xi_p L^{(p)}\right]$ gives

$$\det\left[\sum_p \xi_p L^{(p)}\right] \underline{v} = 0 \; . \tag{2.67}$$

A sufficient condition for equation (2.67) to be satisfied is

$$\det\left[\sum_p \xi_p L^{(p)}\right] = 0 . \qquad (2.68)$$

It seems certain that this condition is also necessary for equation (2.67), as the latter is a vector equation for an arbitrary vector $\underline{v}(\underline{\xi})$.

In the case $m = 2$, let us put $\xi_1 = \omega$, $\xi_2 = \xi$ and $x_1 = t$, $x_2 = x$. Without loss of generality (assuming $L^{(1)}$ is invertible) we may assume that $L^{(1)} = I$ and $L^{(2)} = A$ for some constant matrix A. Equation (2.68) then becomes

$$\det(\omega I + \xi A) = 0 . \qquad (2.69)$$

This gives a relationship between the transformed independent variables.

It is also useful to think of the Fourier transform on terms of a Fourier series, i.e., let us rewrite equation (2.64) as

$$\underline{u}(\underline{x}) = (2\pi)^{-m/2} \sum_{\underline{\xi}\in\underline{W}} \underline{V}_{\underline{\xi}} e^{-i\underline{x}\cdot\underline{\xi}} , \qquad (2.70)$$

for some space $\underline{W}$ and some coefficients $\underline{V}_{\underline{\xi}}$ (i.e. they are independent of $\underline{x}$).

We therefore see that the analysis given is equivalent to considering a single Fourier component, $e^{-i\underline{x}\cdot\underline{\xi}}$, of $\underline{u}(\underline{x})$. Each component must then obey equation (2.68).

This analysis may easily be extended to the system with an additional source term given in §2.1.2. It can be shown that this gives the equation

$$\det\left[i \sum_p \xi_p L^{(p)} + A\right] = 0 . \qquad (2.71)$$

Therefore, the first order nonlinear system given in §2.1.3 will obey the equation

$$\det\left[i \sum_p \xi_p L^{(p)}(\underline{u}_0) + \frac{\partial \underline{f}}{\partial \underline{u}} (\underline{u}_0)\right] = 0 . \qquad (2.72)$$

In these two cases, the Fourier transform is applied to the correction of $\underline{u}$ from the steady state, $(\underline{u}_0)$ (see equation (2.19)).

It is envisaged that it will be possible to obtain a correction to equation (2.72) corresponding to the second order nonlinear system of §2.1.4. The analysis is not presented here.

Let us now consider the case $m = 2$ again. Using the notation introduced previously in this subsection, equation (2.72) becomes

$$\det\left\{i\left[\omega I + \xi A(\underline{u}_0)\right] + \frac{\partial \underline{f}}{\partial \underline{u}} (\underline{u}_0)\right\} = 0 . \qquad (2.73)$$

(Note: it is possible to write down a determinant equation of the form (2.72) even when $L^{(1)}$ is not invertible).

Equation (2.73) corresponds to the Fourier term $e^{-i(\omega t+\xi x)}$. Now, assume that equation (2.1) is a normal initial value problem solved over the domain $\{(x,t): x \in \mathbb{R},\ t \geq 0\}$. The perturbation will remain stable only when

$$\xi \in \mathbb{R} \quad \text{and} \quad \mathcal{I}m(\omega) \leq 0 \tag{2.74}$$

(c.f. the conditions on K and Y in §2.2).

So the stability condition is only met when equation (2.73) implies equation (2.74).

Example 2.2 - Nonequilibrium Flow (see [12]).

$$\left.\begin{aligned} \underline{u} &= (\rho,\ u,\ p,\ B)^T \\ \underline{x} &= (t,\ x)^T \end{aligned}\right\} \tag{2.75}$$

$$\underline{u}_t + A(\underline{u})\underline{u}_x = \underline{f}(\underline{u}) \tag{2.76}$$

$$A = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & 1/\rho & 0 \\ 0 & \gamma_r p & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \tag{2.77}$$

$$\underline{f}(\underline{u}) = \begin{bmatrix} 0 \\ 0 \\ \frac{2\rho}{\beta\tau}(B - B_0) \\ -1/\tau\ (B - B_0) \end{bmatrix}, \tag{2.78}$$

where B is the internal energy corresponding to the rotational degrees of freedom of the gas and

$$B_0 = \frac{\beta}{2}\frac{P}{\rho} \tag{2.79}$$

(c.f. §1.3.3).

Roe uses a slightly different framework with the Fourier perturbation written $e^{i(\omega t - \xi x)}$ and the initial equations not written in the form of equation (2.76). He plots

$$Z = \frac{\omega}{a_f \xi}, \tag{2.80}$$

against $a = \frac{a_e}{a_f}$ and $\tau' = \frac{\tau\xi\alpha a p}{\alpha+\beta}$, deriving the equation

$$Z(Z^2 - 1) + \frac{i}{\tau'}(Z^2 - a^2) = 0 \tag{2.81}$$

(a_e and a_f are the equilibrium and frozen sound speeds respectively). Without loss of generality, he makes the assumption $\xi > 0$ (although this is not explicitly stated). An original diagram plot of $Z(\tau'; a)$ shows that the wavespeed has undesired behaviour for $a < 1/\sqrt{5}$. However, this condition is unphysical as it corresponds to

$$\beta = \frac{2\alpha(1+\alpha)}{1 - 2\alpha}, \tag{2.82}$$

implying $\alpha < ½$ for $\beta > 0$.

□

It is hoped that the method used by Roe can be generalised to other domains and higher values of m. It is anticipated that equation (2.73) may be expressible in the form

$$\psi(Z;\ \Pi_1, \ldots, \Pi_N) = 0\ , \tag{2.83}$$

for some function ψ, where

$$Z \propto \frac{\omega}{|\underline{\xi}|}\ , \tag{2.84}$$

such that Z is non-dimensionalised, where the independent variables are now $(t,\ \underline{x})$ and their Fourier transform is $(\omega,\ \underline{\xi})$.

Also, $\Pi_1, \ldots, \Pi_N$ are independent non-dimensional parameters which are functions of $\underline{u}_0$ and $\underline{\xi}$. We conjecture that $N = n-1$ (in Roe's example, one of the components of $\underline{a}_0$ is set to zero, reducing the number of non-dimensional products from 3 to 2).

The general method will involve solving equation (2.83) for Z or plotting Z with respect to $\Pi_1, \ldots, \Pi_N$ as parametric variables. The locus of Z will then need to obey certain criteria corresponding to acceptable physical behaviour.

The process of determining the non-dimensional products $\Pi_1, \ldots, \Pi_N$ is described in [13].

3. Towards a Theory of the Nondiffusive Limit for General Flow

3.0 Introduction

In this section, we shall be considering systems of the general form given in §2.2.2 of [1] with governing equations as in (1.1) and (1.2). We shall consider only a fixed space domain, Δ. The diffusion scale coefficients $d_1, \ldots, d_\alpha$ will be treated as variables independent of the system.

Our purpose here is to attempt to describe the behaviour of shockwaves in these systems as $\underline{d} \to \underline{0}$. The first subsection gives a Reynold's number-type analysis of the general system. This gives us insight into the relative strengths in advection and diffusion within viscous shock regions of varying strength.

In the second and final subsections, a different course is taken. First of all, the concept of a shock interval developed in §1.2 is generalised to this multi-dimensional unsteady context. After this, a framework for asserting the convergence of viscous shockwaves is devised.

It is intended that such a framework will lead to a generalisation of the Rankine Hugoniot jump conditions for diffusive flow.

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