NON-LINEAR OPTIMAL CONTROL OF TIDAL POWER SCHEMES IN LONG ESTUARIES.

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ABSTRACT

Numerical techiques are applied to solve an optimal control problem occuring in tidal power generation. The problem considered is that of maximising the average revenue from a tidal barrage subject to the hydrodynamic equations of flow in an estuary. The flows in an estuary are modelled by the non-linear channel equations, and the head-flow characteristics of the barrage are given as a set of non-linear functions for the turbines and sluices. A set of necessary conditions for optimality of the controls are given, and the gradient of the revenue functional is derived. A numerical algorithm which couples a gradient optimisation technique with a finite difference solution of the flow and associated adjoint equations is described. Two numerical examples are provided, taking data for a proposed Severn estuary tidal power scheme. The first example shows the operation of the algorithm to maximise total energy output over a single tide, and the second uses the algorithm to compare the maximum energy solution with the maximum revenue solution, taking a typical winter tariff function over a complete spring-neap-spring cycle. It is seen that it is possible to increase revenue by 4% over the maximum energy solution using the tidal barrage in ebb generation mode.

1. INTRODUCTION

A recent study [11] has shown that the use of tidal energy to generate electrical power is economically viable. In order to evaluate any tidal power scheme, it is important to know that the plant is operating in the most efficient way and several studies have been made [1], [9], [15], using a variety of mathematical techniques and models in order to assess the best operating strategies. One technique which has been shown to be particularly effective is that of applying the mathematical theory of optimal control. In previous studies [2], [3], [4], [5], [6], [8], optimal control methods are applied to the problem of maximising the average power or revenue functional modelling the output of a tidal power station, subject to the satisfaction of the equations of flow in a tidal estuary. The resulting optimal control problem is then solved approximately using numerical techniques. It has been found that this approach is both computationally feasible and flexible in being able to treat ebb or two way schemes, non-linear head-flow relationships, and variable estuarine geometries. Initial studies concentrated on establishing the computational feasibility of the approach and were centred around solving problems where the estuarine dynamics were described by linear ordinary and partial differential equation models. A more recent study [7], has extended the work to problems where the estuarine dynamics are described by a non-linear ordinary differential equation where effects such as the drying out of sand bars can more reallistically be modelled. In Britain, the interest in tidal power has centred around large estuaries such as the Severn. For such estuaries it is known that accurate mathematical modelling of the fluid dynamics is only possible using more sophisticated models [12], since the ordinary differential equation model takes no account of the different phases of the tides at different points along the estuary.

In this report we present the application of optimal control techniques to the problem of maximising the output from a tidal power plant where the estuarine dynamics are modelled by a set of non-linear partial differential equations based on the one-dimensional channel flow equations [14].

In the next section the mathematical model of a tidal power scheme is described and the corresponding optimal control problem is formulated. Necessary conditions for optimality are derived. In Section 3 a numerical method for determining the optimal control strategy is developed and a computational algorithm is presented. Results are given in Section 4 using data approximating that for a scheme in the Severn estuary. Conclusions are presented in Section 5.

2. THE MATHEMATICAL MODEL

2.1 The Equations of flow

The fluid dynamics in the estuary are modelled by the one-dimensional, non-linear channel flow equations [14] :

$$\begin{aligned} b(\eta,x)\eta_t &= -(A(\eta,x)u)_x \\ u_t &= \{-uu_x\}-g\eta_x-gn^2u|u|/r^{4/3}(\eta,x) \end{aligned} \quad x \in [-l_1,+l_2], \qquad (1)$$

where $b(\eta,x) > 0$, $A(\eta,x) > 0$, $r(\eta,x) > 0$ are the breadth, the vertical cross-section and the hydraulic radius of the channel, the surface of which is at a level $\eta(x,t)$ above a given datum at a distance x from the tidal barrier and at time t. The hydraulic radius r is defined to be the ratio of the cross-section A to the wetted perimeter of the channel and for a wide shallow estuary such as the Severn is approximated by the formula $r = A/b$. The average flow velocity through a given cross-section is $u(x,t)$, and the constants $g > 0$ and $n > 0$ are the acceleration due to gravity and Manning's constant.

The tidal basin is taken to lie upstream of the tidal barrier, located at $x = 0$, as shown in Figure 1. At the seaward end of the estuary ($x = -l_1$) the tidal elevation, assumed periodic with period T, is imposed, and at the upstream end of the basin ($x = l_2$) zero flow is assumed, giving boundary conditions

$$\eta(-l_1,t) = f(t) \ , \quad u(l_2,t) = 0 \ , \qquad (2)$$

where f(t) is the imposed tidal elevation assumed periodic with period T. Across the barrier the continuity condition

$$Q(0,t) = A(\eta^+,0^+)u(0^+,t) = A(\eta^-,0^-)u(0^-,t) \qquad (3)$$

where $Q(0,t)$ is the volumetric flow rate of water through the barrier. The functions η, u are required to be periodic in time with period T, such that

$$\eta(x,0) = \eta(x,T), \quad u(x,0) = u(x,T). \qquad (4)$$

In many cases the term uu_x is negligible in equations (1) and may be dropped. If this is so then in the following work all terms inside the braces { } should also be dropped. The barrier is assumed to contain two types of device, namely turbines and sluices, which can both be controlled. The discharge of water through each turbine and sluice is denoted by $q_1(t)$, $q_2(t)$, respectively, and the relationships between discharge and head-difference $H(t)$ are described by

$$q_1(t) = P(H(t)), \quad q_2(t) = R(H(t))$$

where P and R are differentiable functions with derivatives $P' > 0$, $R' > 0$, and $H(t)$ is defined by

$$H(t) = \eta(0^-,t) - \eta(0^+,t). \tag{5}$$

The total influx of fluid $Q(0,t)$ from the estuary to the basin across the barrier is then given by

$$Q(0,t) = K_1\alpha_1(t)P(H(t)) + K_2\alpha_2(t)R(H(t)) \tag{6}$$

where the control vector $\alpha = [\alpha_1,\alpha_2]^T$ gives the proportional discharge across the turbines and sluices, respectively, and K_1, K_2 are positive constants representing the maximum number of turbines and sluices available for operation. The controls are thus bounded such that

$$0 \leq \alpha_1(t),\ \alpha_2(t) \leq 1\ . \tag{7}$$

We assume that, when operating, each turbine gives rise to F(H) Watts (electricity) for a given head difference H (metres), where F is a differentiable function. The tariff associated with electrical production is denoted by $C(t) \geq 0$, and thus the average total revenue $\bar{P}$ derivable from operating the tidal power scheme is given by :

$$\bar{P} = \frac{1}{T}\int_0^T K_1 C(t)\alpha_1(t)F(H(t))\,dt. \tag{8}$$

2.2 The Optimal Control Problem.

The optimisation problem is then to determine the control functions $\alpha_1(t)$, $\alpha_2(t)$ in order to maximise $\bar{P}$ subject to equations (1),(2),(3),(4), and (6), and constraints (7) being satisfied.

2.3 Necessary Conditions.

Necessary conditions for the solution of the optimal control problem are derived using the Lagrangian formulation of the problem. This approach provides the basis for the numerical procedure described in the next section

The Lagrangian functional $L(\alpha)$ associated with the optimal control problem is defined by

$$L(\alpha) = \int_0^T C\alpha_1 F(H)+\gamma_1 (A(\eta^-,0^-)u^- - \alpha_1 K_1 P(H) - \alpha_2 K_2 R(H)) + \gamma_2 (A(\eta^-,0^-)u^- - A(\eta^+,0^+)u^+)\, dt \; +$$

$$⨍_{l_2}^{l_1} \int_0^T \lambda(-A_t - Q_x) + \mu(-u_t - uu_x - g\eta_x - gn^2 u|u|/r^{4/3}(\eta,x)) + \nu(A - a(\eta,x))\, dx\, dt \tag{9}$$

where $Q = Au$, $H = \eta(0^-,t)-\eta(0^+,t)$, $u^{\pm} = u(0\pm,t)$, $\eta^{\pm} = \eta(0\pm,t)$, etc., $A = a(\eta,x)$, and

$$⨍_{l_1}^{l_2} z(x)\, dx \triangleq \int_{l_1}^{0^-} z(x)\, dx + \int_{0^+}^{l_2} z(x)\, dx.$$

The functions $\gamma_1(t)$, $\gamma_2(t)$, ν, $\lambda(x,t)$, and $\mu(x,t)$ are Lagrange multipliers and we note that if η, u satisfy all the constraining equations (1)-(4) and (6) then $\bar{P}(\alpha) = K_1 L(\alpha)/T$. For $\alpha(t)$ to be optimal, it is necessary that the first variation $\delta L(\alpha, \delta\alpha)$ of the functional L is negative, where δL is defined to be linear with respect to $\delta\alpha = \beta - \alpha$ and such that

$$L(\beta) - L(\alpha) = \delta L(\alpha,\delta\alpha) + o(\|\beta - \alpha\|_2)$$

for all (smooth) admissible controls β. To simplify the notation we denote the difference between the responses of the system (1)-(6) to the controls β and α by $\delta\eta(x,t)$ and $\delta u(x,t)$, and let $\delta\alpha = \beta-\alpha$. Now taking variations and integrating by parts, we find that the first variation of the Lagrangian can be written in the form

$$
\begin{aligned}
\delta L = \int_0^T & C\delta\alpha_1 F(H) + C\alpha_1 F'\delta H + \gamma_1 (\delta Q_0 - [\alpha_1 K_1 P' + \alpha_2 K_2 R']\delta H - \delta\alpha_1 K_1 P - \delta\alpha_2 K_2 R) + \\
& \gamma_2 (b_0^- u_0^- \delta\eta_0^- + A_0^- \delta u_0^- - b_0^+ \delta\eta_0^+ u_0^+ - A_0^+ \delta u_0^+)\, dt + \\
& \int_{l_1}^{l_2} [-\lambda\delta A - \mu\delta u]_0^T \, dx + \\
& \int_0^T [-\lambda\delta Q - \{\mu u \delta u\} - g\mu\delta\eta]_{-l_1}^{0^-} + [-\lambda\delta Q - \{\mu u \delta u\} - g\mu\delta\eta]_{0^+}^{l_2} \, dt + \\
\int_{l_1}^{l_2} \int_0^T & \lambda_t \delta A + \lambda_x \delta Q + \mu_t \delta u + \{\mu_x u \delta u\} + g\mu_x \delta\eta - \\
& \mu g n^2 (2|u|\delta u / r^{4/3} - 4u|u| r_\eta \delta\eta / 3r^{7/3}) + \nu(\delta A - b\delta\eta)\, dx\, dt \\
& + o\|\delta\alpha\|,
\end{aligned}
\tag{10}
$$

where we have used the fact that $\frac{\partial a}{\partial \eta} = b$. We now use the fact that

$\delta H = \delta\eta_0^- - \delta\eta_0^+$, and $\delta Q = A\delta u + bu\delta\eta$, and simplify the expression (10) by imposing the following conditions on the Lagrange multipliers:

$$
\begin{aligned}
b(\lambda_t + u\lambda_x) + g\mu_x + \frac{4\mu g n^2 u|u| r_\eta}{3r^{7/3}} &= 0 \\
\mu_t + \{u\mu_x\} + A\lambda_x - \frac{2\mu g n^2 |u|}{r^{4/3}} &= 0
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
A\lambda + \{\mu u\} = 0 \quad & \text{at} \quad x = -l_1 \\
\mu = 0 \quad & \text{at} \quad x = +l_2
\end{aligned}
\tag{12a}
$$

$$
\mu(0^+, t)(g - \{b_0^+ u_0^{+2} / A_0^+\}) + \gamma_1 (\alpha_1 K_1 P' + \alpha_2 K_2 R') = C\alpha_1 F'
\tag{12b}
$$

[illegible]

$$
\begin{aligned}
\lambda(x,0) &= \lambda(x,T) \\
\mu(x,0) &= \mu(x,T)
\end{aligned}
\tag{13}
$$

where $\gamma_1(t) = \lambda_0^- - \lambda_0^+ + \{\mu_0^- u_0^- / A_0^- - \mu_0^+ u_0^+ / A_0^+\}$,

$\gamma_2(t) = - \lambda_0^+ - \{\mu_0^+ u_0^+ / A_0^+\}$

It is seen that equations (11) form a linear hyperbolic system of partial differential equations with wave-speeds $u \pm \sqrt{gA/b}$. Equations (12) and (13) provide boundary and initial conditions respectively, so that (11)-(13) form a well posed initial value problem in reverse time [3] which we will call the adjoint flow problem. When equations (1)-(6) and (11)-(13) are simultaneously satisfied then it is seen that (10) simplifies to

$$\delta L(\alpha,\delta\alpha) = \int_0^T \nabla E(\alpha)(t).\delta\alpha(t)\,dt, \qquad \text{where}$$

$$\nabla E(\alpha)(t) = \left\{ \begin{matrix} C(t)F(H)-\gamma_1(t)K_1P(H) \\ -\gamma_1(t)K_2R(H) \end{matrix} \right\}. \tag{14}$$

We may now write the necessary condition for optimality of the control as

$$\langle \nabla E(\alpha), \beta - \alpha \rangle \leq 0 \tag{15}$$

for each (smooth) admissible control α, where the inner product $\langle\, .\,,\, .\,\rangle$ is defined by

$$\langle p,q \rangle = \int_0^T p(t).q(t)\,dt\ ,$$

and where $\nabla E(\alpha)(t)$ is proportional to the function space gradient of $\bar{P}$ with respect to the controls [3] ($\nabla\bar{P} = K_1 \nabla E/T$). For a given control the gradient vector $\nabla E(\alpha)(t)$ may be evaluated from equation (14) after first solving the flow problem (1)-(6) with the control α, and then solving the adjoint flow problem (11)-(13). The inequality (15) may then be easily tested since the controls take on values in the closed interval [0,1] [3]. The ability to calculate the gradient of the functional $\bar{P}$ also means that we can apply gradient optimisation methods to the optimal control problem [3], [10]. In order to apply gradient techniques we therefore require to solve a pair of hyperbolic initial value problems, and this may be done approximately using numerical integration techniques. The numerical procedure described in the next section uses a gradient technique to generate a sequence of approximations to the optimal control problem.

3. THE NUMERICAL METHOD

The computational method which we use to solve the optimal control problem consists of a constrained optimisation technique for iteratively determining the optimal control function, together with a numerical procedure for solving the estuarine flow problem and the adjoint flow problem with a given control.

3.1 Conditional Gradient Method.

Many optimisation techniques are described in the literature [10]. We describe here a conditional gradient method with a step size selection procedure based on finding an approximation to the coefficients in a Taylor expansion of the functional $\bar{P}$, up to second order terms. This method generates a sequence of piecewise continuous controls $\alpha^k(t)$, $k = 1,2,\ldots$ approximating the optimal control $\alpha(t)$. Since the set of admissible controls U is convex, there exists a maximal displacement $\delta\alpha^k$ in the direction of the gradient $\nabla E(\alpha^k)$ such that $\alpha^k + \delta\alpha^k$ lies in U. The conditional gradient method generates the controls such that $\alpha^{k+1} = \alpha^k + \theta\delta\alpha^k$ where $\theta \in (0,1]$, and such that either $\bar{P}(\alpha^{k+1}) > \bar{P}(\alpha^k)$ or α^{k+1} satisfies the necessary conditions (15). In practice the iteration is terminated when the measure $M(\alpha^k)$ is less than a given positive tolerance, where $M(\alpha)$ is given by

$$M(\alpha) = \max_{\beta \in U} \langle \nabla E(\alpha), \beta - \alpha \rangle , \tag{16}$$

and α^k is then accepted as a good solution to the optimal control problem. The solutions of the flow problem (1)-(6), and adjoint problem (11)-(13) are approximated using an explicit integration method based on the Leap-Frog scheme [13], the flow problem being integrated forward in time and the adjoint problem being integrated backwards in time. In order to find the step length θ at each step of the iterative method we consider an approximation of the functional L by a Taylor series about the current control α^k .

$$L(\alpha^k + \theta(\beta^k-\alpha^k)) = L(\alpha^k) + \theta D_1 + \theta^2 D_2/2 + O(\theta^3\|\beta-\alpha\|^3) \qquad (17)$$

where D_1 and D_2 are the first and second directional derivatives of L in the direction of $\beta^k - \alpha^k$ respectively. Here the control β is taken using the formula

$$D_1 = \langle\nabla E(\alpha^k),\beta^k-\alpha^k\rangle = \max_{\beta\in U}\langle\nabla E(\alpha^k),\beta-\alpha^k\rangle , \qquad (18)$$

and then θ is taken so as to maximise the quadratic expression

$$L(\alpha^k) + \theta D_1 + 1/2\ \theta^2 D_2$$

subject to the constraint that $\theta \in [0,1]$, and hence that the new control $\alpha^{k+1} = \alpha^k + \theta\delta\alpha^k$ is admissible whenever α is. The second derivative D_2 is approximated using

$$D_2 \approx \frac{\langle\nabla E(\alpha^k + h(\beta^k-\alpha^k)),\beta^k - \alpha^k\rangle - D_1}{h} \qquad (19)$$

for small h.

The numerical optimisation algorithm is then obtained by replacing all integrations in the following algorithm by discrete approximations.

Algorithm

STEP 0 : Choose $\alpha^0 \in U$ $(\alpha^0(t) \not\equiv 0)$

Choose $\theta \in (0,1]$

$\bar{P} := 0$

$k := 0$

$\alpha := \alpha^k$

STEP 1: Solve the flow problem (1)-(6) and evaluate $\bar{P}$ given by (8).

STEP 2: Solve the adjoint problem (11)-(13).

STEP 3: Evaluate $\nabla E(\alpha)(t)$ given by (14).

STEP 4: Evaluate D_1, find β^k given in (18) and use (19) to approximate D_2 .

STEP 5: If $D_2 \geq 0$ then GO TO STEP 7

STEP 6: $\theta = -D_1/D_2$, if $\theta > 1$ then $\theta = 1$.

STEP 7: If $D_1 < tol$ then STOP.

STEP 8: $\alpha := \alpha^k + \theta(\beta^k - \alpha^k)$

$\bar{P} := \bar{P}(\alpha^k)$.

STEP 9: $k := k+1$

$\alpha^k := \alpha$

GO TO STEP 1.

Details of the numerical integration schemes are given in the Appendix.

4. RESULTS

4.1 Half day cycles.

Numerical results are described for a problem which approximates a Severn estuary tidal scheme located at the position of the 1981 preferred scheme [11]. For this problem we take a repeating tide of period 12.4 hours. The estuarine geometry is numerically approximated by taking a linear interpolation between low water and high water breadths (Figures 2,3) to give $b(\eta,x)$. The turbine and sluice characteristics are modelled by (ebb-scheme) :

$$P(y) = \begin{cases} 390(1+\tanh(10(y-2.27)) & y \leq 0 \\ 0 & y > 0 \end{cases} \qquad \text{(turbine flow function)}$$

$$R(y) = 520\sqrt{(2gy)}\,\mathrm{sgn}(y) \qquad \text{(sluice flow function)}$$

(Figure 4). The power output characteristics $F(y)$ for each turbine are given by a piecewise cubic polynomial approximation (Figure 5). The rest of the data is as follows :

$$T = 44714 \text{ s} \qquad \text{(tidal period)}$$
$$C(t) = 1 \qquad \text{(unit tariff)}$$
$$f(t) = F_0 \cos(2\pi t/T) + 0.15 \qquad \text{(tidal elevation at seaward boundary)}$$

where F_0 is the tidal amplitude in metres. It is noted that P,R, and F are not differentiable at $y = 0$. From a mathematical point of view we can always replace these functions by smooth approximations near the origin. This is in fact what is done in practice.

Table 1 shows the result of applying the numerical algorithm described in section 3 to this test problem. The initial control is taken to be the constant control vector $(1.00,0.1)^T$. We see that the average power output from the tidal power model rises rapidly as the iteration procedes. Figure 6 shows the main flow parameters for the best computed control strategy (iteration 8).

4.2 Half lunar cycles.

The second numerical example we give is a Severn estuary model, where we seek to maximise a revenue functional over a 14 day spring-neap-spring cycle. We take turbine and sluice data as given in §4.1 and consider the tidal power plant to be working in ebb-generation mode. The tidal elevation f(t) is given by

$$f(t) = \{2.25\cos^2(\pi t/T) + 2.00\}\cos(54\pi t/T)$$

where T is the half lunar period. We take two tariff functions C(t). The first tariff is based on winter rate electricity prices as follows

Weekdays	0030 - 0730	1.37 p/kW hr
	0730 - 2000	6.05 p/kW hr
	2000 - 0030	2.43 p/kW hr
Weekends		2.43 p/kW hr ,

where it is assumed that the first high spring tide occurs at 7 a.m. on a Sunday. The second tariff function $C(t) \equiv 1$, so that we are merely seeking to maximise average power rather than revenue. Tables 2,3 show the results of the numerical iteration for the winter-rate and unit tariff respectively and Figures (7a,7b), (8a,8b) show the main flow parameters for the winter-rate and unit tariffs respectively. As a comparison, Table 4 gives the average power output in GW, and the total revenue in M£ for the best computed schemes with C(t) = 1 and the winter tariff. As expected an increase in revenue over the maximum output scheme is achieved by a reduction in the average power, although this is only of the order of 4%. It is also seen that the maximal revenue scheme produces an extra 4% profit over the maximum output scheme.

5. CONCLUSIONS

In this report we examine a non-linear channel flow model suitable for the accurate description of a tidal power scheme in a large estuary such as the Severn. We describe an optimal control technique for determining the maximum revenue derivable from the model. A numerical optimisation procedure which is an extension of methods previously developed is presented, and, for an example based on data derived from the Severn estuary, is shown to be highly efficient.

We conclude that the application of optimal control techniques to the problem of optimising tidal power scheme output is feasible even in the case where the mathematical model is highly non-linear. It is also concluded that, using an ebb generation scheme, it is possible to re-schedule power output to increase revenue over a maximum energy output policy.

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APPENDIX

NUMERICAL SOLUTION OF FLOW AND ADJOINT EQUATIONS

The numerical solution of the periodic initial value problem (1)-(6) is obtained by using a Leap-Frog approximation to the differential equations, followed by repeated numerical integration over the time interval [0,T] until the initial values η_j^0, u_j^0, and final values η_j^M, u_j^M where $\Delta t = T/M$, agree to within some small tolerance. The difference approximation is given by

$$\eta_j^{n+1} - \eta_j^{n-1} = -\nu[Q_{j+1}^n - Q_{j-1}^n]/b_j^n,$$

$$(\ Q_j^n = A_j^n u_j^n\ )\ ,$$

$$u_j^{n+1} = [u_j^{n-1}/d_j^{n-1} - \nu(\{u_{j+1}^n - u_{j-1}^n\}u_j^n + g(\eta_{j+1}^n - \eta_{j-1}^n))/d_j^n$$

followed by a post processing step

$$\eta_j^n := (\eta_j^{n+1} + 2\eta_j^n + \eta_j^{n-1})/4$$

where $\nu = \Delta x/\Delta t$, $d_j^n = 1 + (gn^2|u_j^n|\Delta t)/(r_j^n)^{4/3}$. The numerical boundary conditions are based on the problem boundary conditions plus an implicit, stable boundary condition at the barrier and two extra boundary conditions derived using considerations of symmetry. These are given by :

$\eta_0^n = f(t)$ (imposed tidal curve)

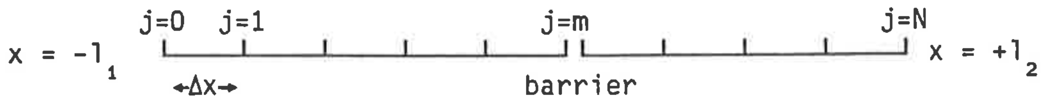
$Q_N^n = 0$ (zero flow at upstream end)

$Q_{N-1}^n = -Q_{N+1}^n$ (symmetry condition)

$f(t) - \eta_{-1}^n = \eta_{+1}^n - f(t)$ (symmetry condition)

$$H^n = (\eta_{m-1}^{n+1} + \eta_{m-1}^{n-1} - \eta_{m+1}^{n+1} - \eta_{m+1}^{n-1})/2.$$

The last condition leads to a pair of non-linear algebraic equations to be solved at each timestep. It is convenient to use a damped Newton method for the numerical solution of these algebraic equations since both P' and R' are available. The finite difference mesh is shown in the figure following .



The adjoint problem (11)-(13) is approximated in the reverse time direction by an analogous scheme, where $\lambda(x,t),\mu(x,t)$ are the adjoint elevation and flow velocity respectively. Both the flow problem (1)-(6) and the adjoint problem (11)-(13) are solved on the same finite difference mesh.

Non-linear P.D.E problem

```
Number of space intervals,N1...=    20
Number of space intervals,N2...=    15
Number of time  intervals,M....=   250
Maximum number of iterations...=    10
Length of outer estuary........= 70000.    Metres
Length of estuary basin........= 50000.    Metres
Mannings n.....................=      .0250
Tidal period T.................= 44714.    Seconds
Number of turbines available...=   140.
Number of sluices available....=   187.
Tidal amplitude Fo.............=     4.25 Metres
```

```
              *** Ebb-generation scheme ***

For U=0, stability check as follows..
              Minimum Courant number=     .4642
              Maximum Courant number=     .9038
At  7.0m o.d.,Maximum Courant number=     .9981
```

```
I.............................is the iteration number
THETA.....................is the gradient step-length
P..........................is the average power (GW)
E................is the weighted average power (GW)
MAX(DE) is an upper bound on the 1st variation of E
DEDK1.......is the rate of change of E with respect
              to the number of turbines (MW/Turbine).
DEDK2.......is the rate of change of E with respect
               to the number of sluices  (MW/Sluice).
```

I	THETA	P (GW)	E	MAX(DE)	DEDK1 (MW)	DEDK2 (MW)
1	1.000000	.276021	.276021	3.695330	1.212040	.471796
2	1.000000	2.433201	2.433201	.043648	8.938183	.715937
3	.087014	2.437560	2.437560	.026599	9.010824	.690572
4	.290393	2.444120	2.444120	.009818	9.072761	.692672

Table 1.

Non-linear P.D.E problem

Number of space intervals,N1..= 20
Number of space intervals,N2..= 15
Number of time intervals,M...= 8000
Maximum number of iterations..= 20
Length of outer estuary.....= 70000. Metres
Length of estuary basin.......= 50000. Metres
Mannings n....................= .0250
Tidal period T................= 1209600. Seconds
Number of turbines available..= 140.
Number of sluices available...= 187.

*** Ebb-generation scheme ***

For U=0, stability check as follows..
Minimum Courant number= .3924
Maximum Courant number= .7641
At 7.0m o.d.,Maximum Courant number= .8438

I.............................is the iteration number
THETA...................is the gradient step-length
P.........................is the average power (GW)
E..................is the weighted average power (10^4 £/hour)
MAX(DE) is an upper bound on the 1st variation of E
DEDK1.......is the rate of change of E with respect to the number of turbines (£/hour/Turbine)
DEDK2.......is the rate of change of E with respect to the number of sluices (£/hour/sluice)

I	THETA	P (GW)	E	MAX(DE)	DEDK1	DEDK2
1	1.000000	.213729	.758387	6.825834	4.307169	.755756
2	1.000000	1.206439	4.402184	3.360253	2.801880	2.499796
3	.124131	1.273718	4.679106	1.274518	6.957050	2.445662
4	.206793	1.319601	4.880753	.542338	11.150896	2.122245
5	.678052	1.344498	5.069303	.356483	14.959934	1.514917
6	.498349	1.365714	5.142847	.203838	14.506371	1.508367
7	1.000000	1.334106	5.137614	.500521	15.615578	1.369551
8	.253066	1.361284	5.208131	.197642	16.259735	1.398727
9	.339230	1.370665	5.234169	.086228	15.392380	1.438005
10	.221709	1.374507	5.246522	.055172	16.067478	1.415052
11	.267066	1.376418	5.251740	.045368	15.524546	1.434107

Table 2.

Non-linear P.D.E problem

```
Number of space intervals,N1..=      20
Number of space intervals,N2..=      15
Number of time  intervals,M...=    8000
Maximum number of iterations..=      20
Length of outer estuary.......=   70000. Metres
Length of estuary basin.......=   50000. Metres
Mannings n....................=       .0250
Tidal period T................=1209600. Seconds
Number of turbines available..=     140.
Number of sluices available...=     187.
```

```
          *** Ebb-generation scheme ***

For U=0, stability check as follows..
              Minimum Courant number=      .3924
              Maximum Courant number=      .7641
At  7.0m o.d.,Maximum Courant number=      .8438
```

I...........................is the iteration number
THETA.....................is the gradient step-length
P..........................is the average power (GW)
E...............is the weighted average power (10^4 £/hour)
MAX(DE) is an upper bound on the 1st variation of E
DEDK1.......is the rate of change of E with respect to the number of turbines (£/hour/Turbine)
DEDK2.......is the rate of change of E with respect to the number of sluices (£/hour/sluice)

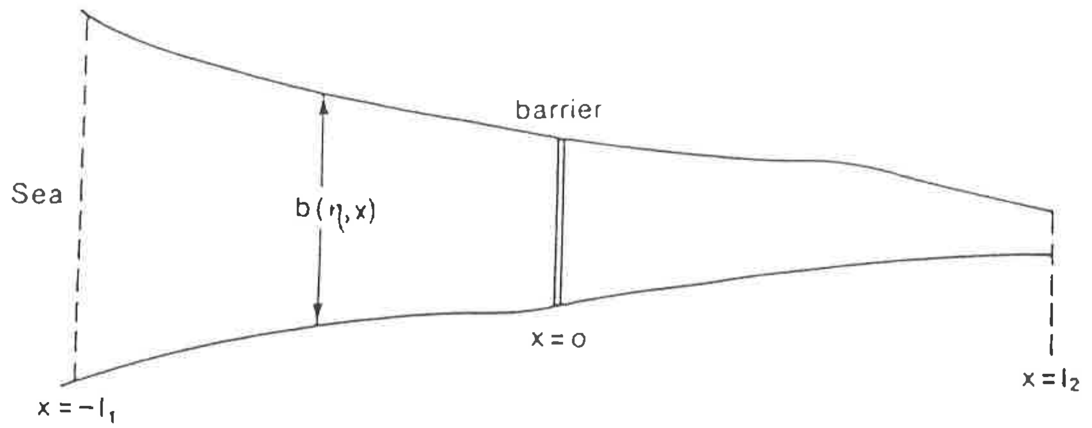
I	THETA	P (GW)	E	MAX(DE)	DEDK1	DEDK2
1	1.000000	.213729	.213729	1.916060	1.215065	.210469
2	1.000000	1.206401	1.206401	.934885	.468807	.712733
3	.092902	1.266634	1.266634	.362592	1.502947	.674002
4	.214890	1.328319	1.328319	.144351	2.798630	.584197
5	.366676	1.367231	1.367231	.065132	3.684248	.440182
6	1.000000	1.374870	1.374870	.122586	3.670606	.337606
7	.358198	1.398210	1.398210	.042006	4.134104	.342669
8	.345718	1.406879	1.406879	.020512	4.165015	.339810
9	.447278	1.409972	1.409972	.028797	3.955593	.352600
10	.191736	1.414122	1.414122	.015900	4.129676	.347591
11	.275419	1.417030	1.417030	.013047	4.185672	.344431

Table 2

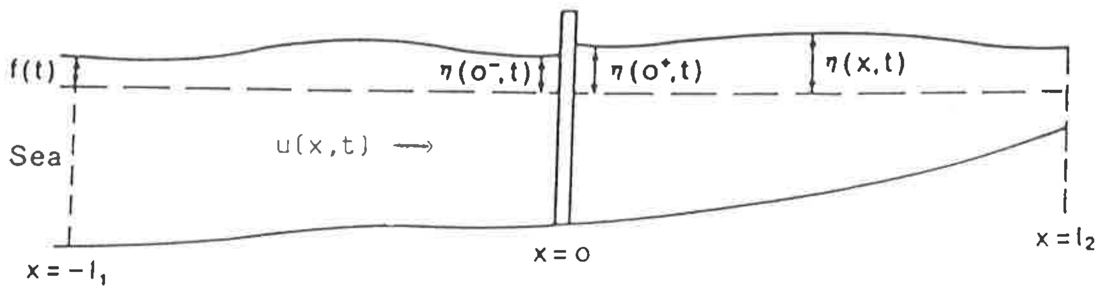
Spring-Neap-Spring cycle.		
	Average power (GW)	Total revenue over a 14 day period (M£)
C(t)≡1	1.42	17.0
C based [illegible] winter tariff.	[illegible]	[illegible]

Table 4.

Plan of model estuary



Section of estuary



Cross section of estuary

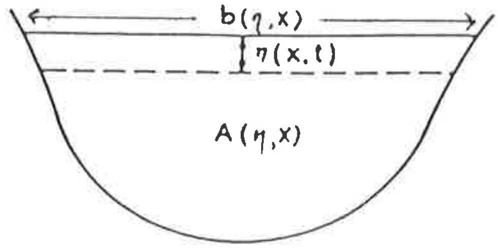
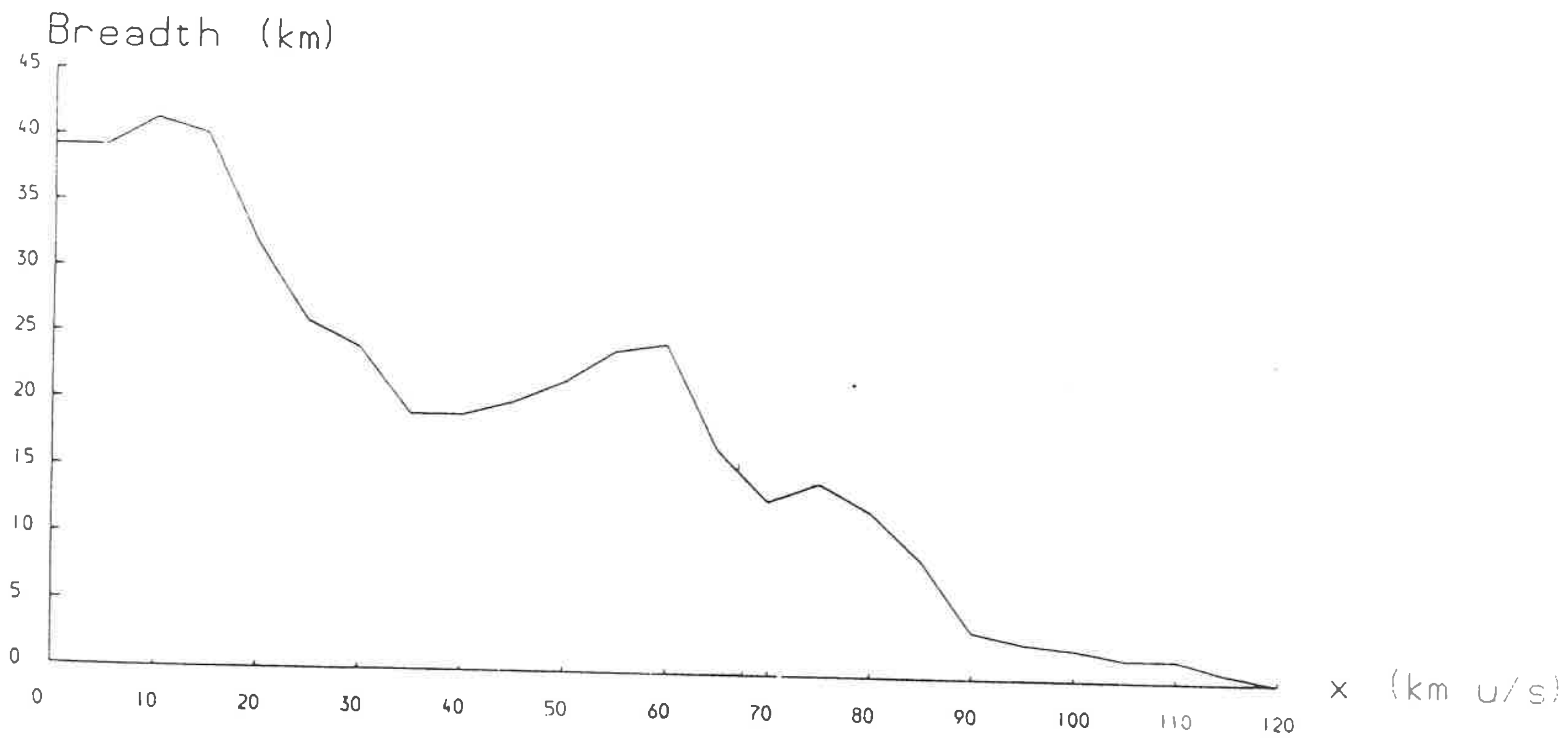


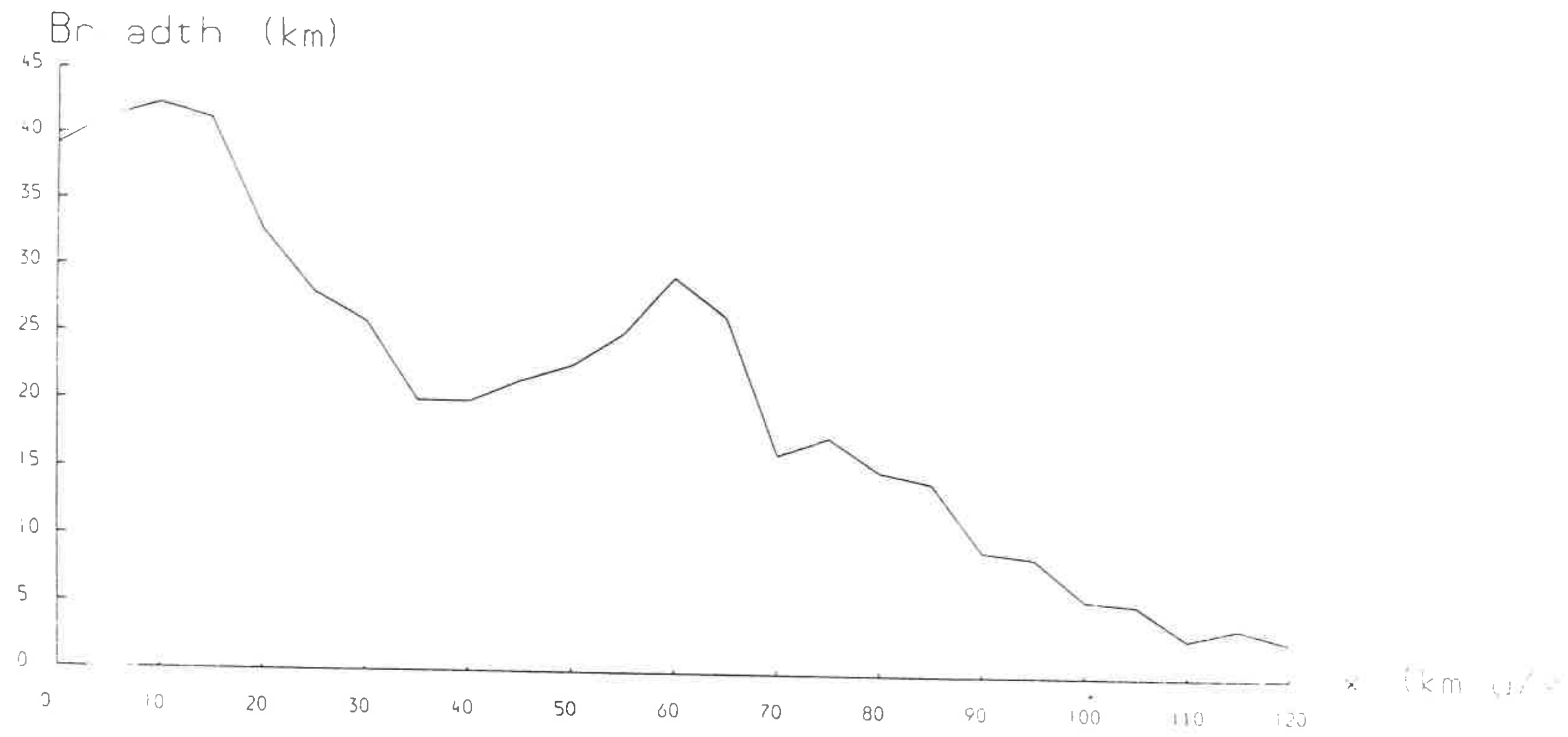
Fig. 1

Geometry of model estuary

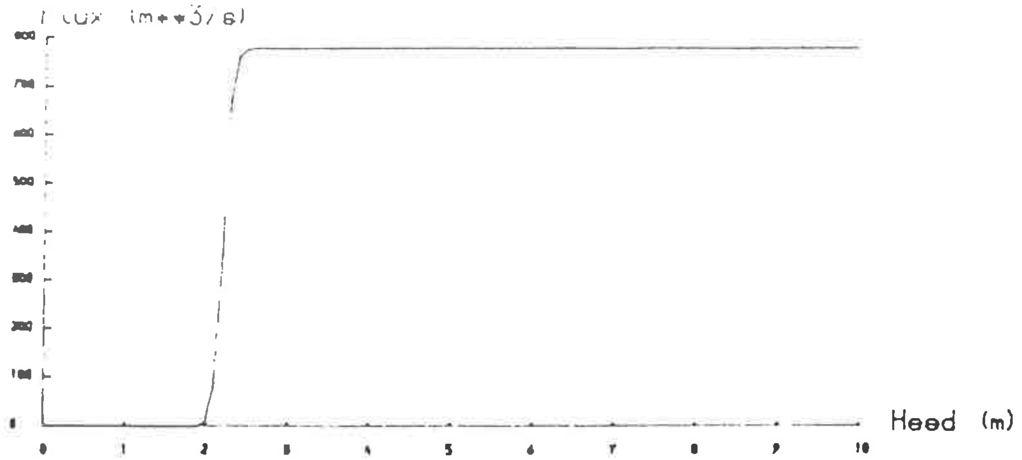
Low Water Breadths (-6m OD) u/s Ilfracombe.



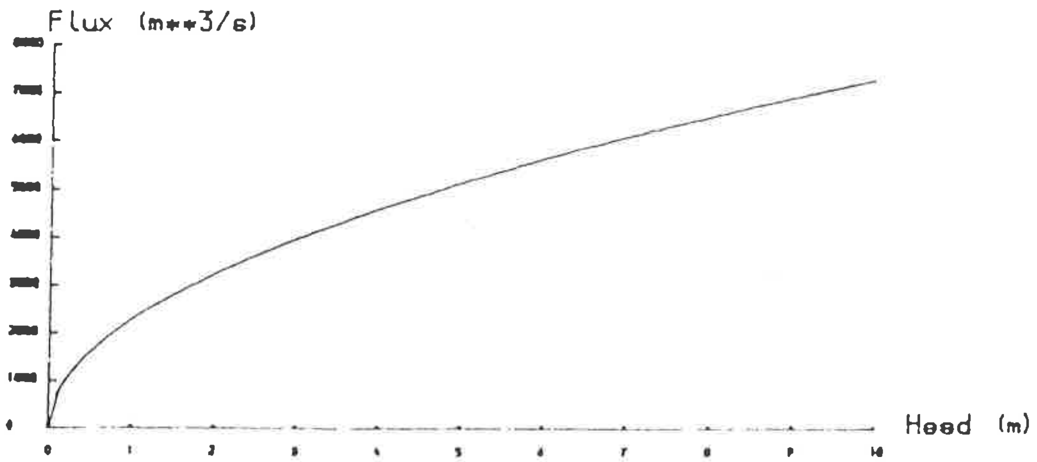
High Water Breadths (+8m OD) u/s Ilfracombe.



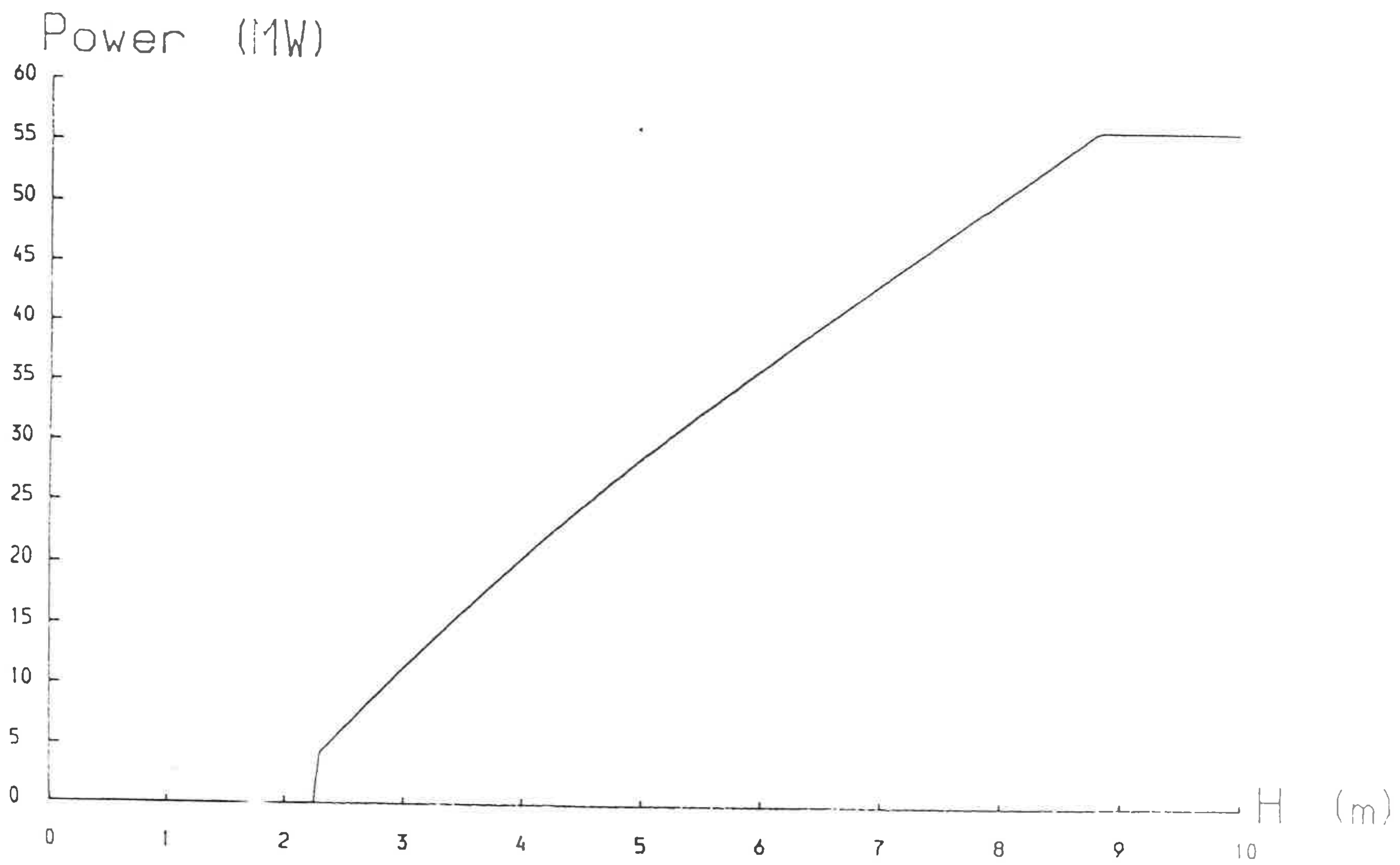
Turbine Characteristics



Sluice Characteristics.



Turbine Performance vs Head.



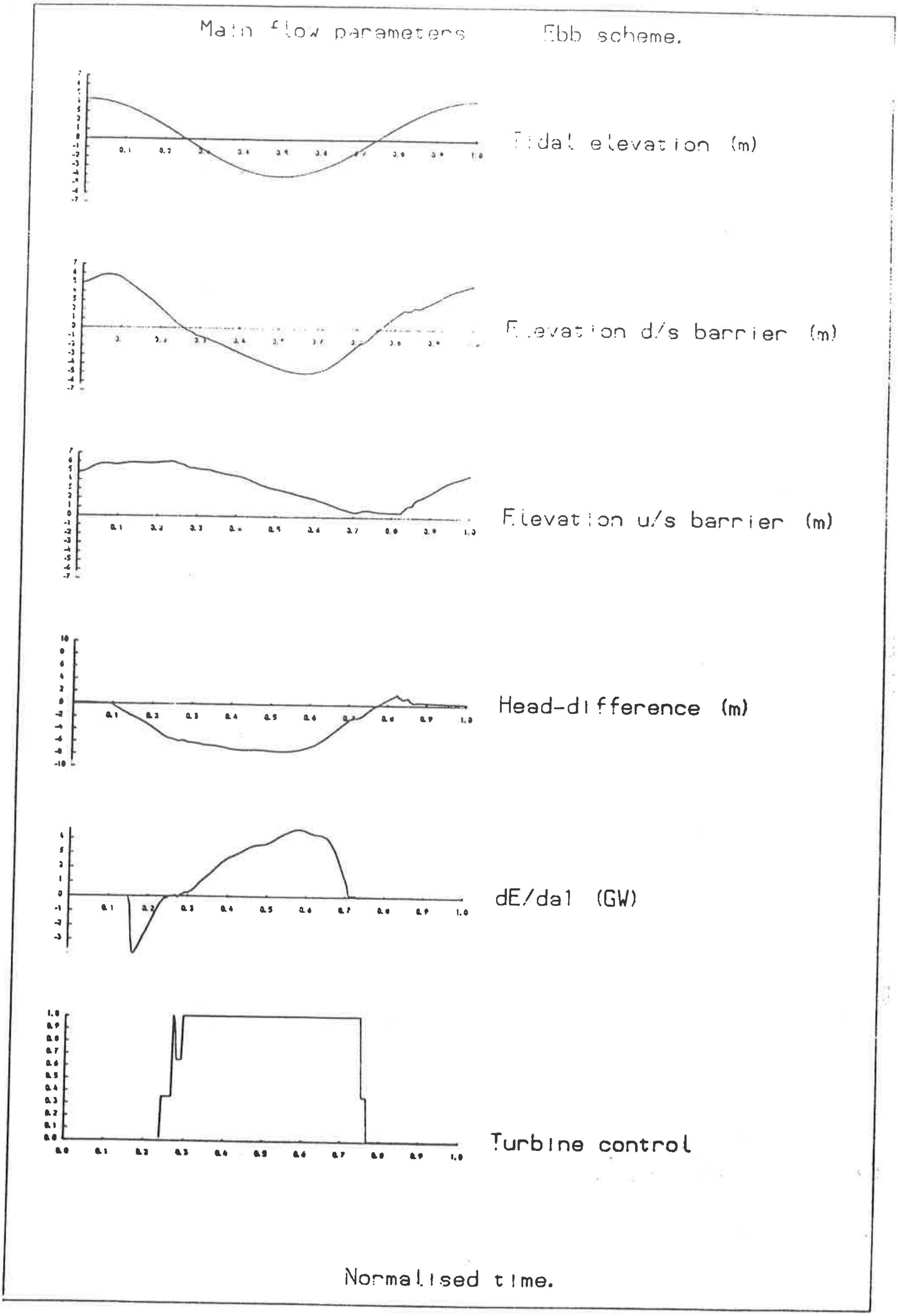
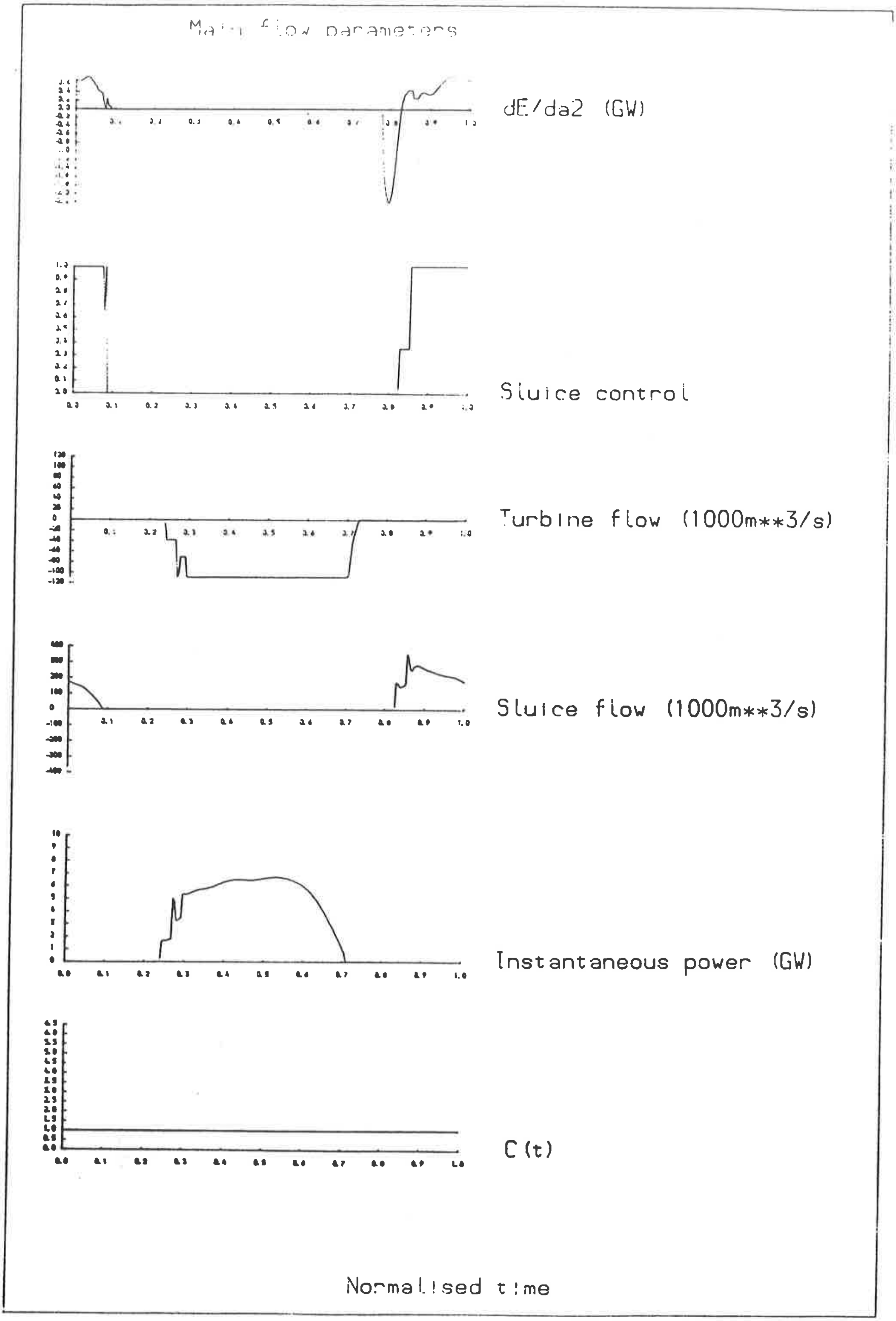

Main flow parameters Ebb scheme.
Tidal elevation (m)
Elevation d/s barrier (m)
Elevation u/s barrier (m)
Head-difference (m)
dE/dal (GW)
Turbine control
Normalised time.

Figure 6 a.


Main flow parameters
dE/da2 (GW)
Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Instantaneous power (GW)
C(t)
Normalised time

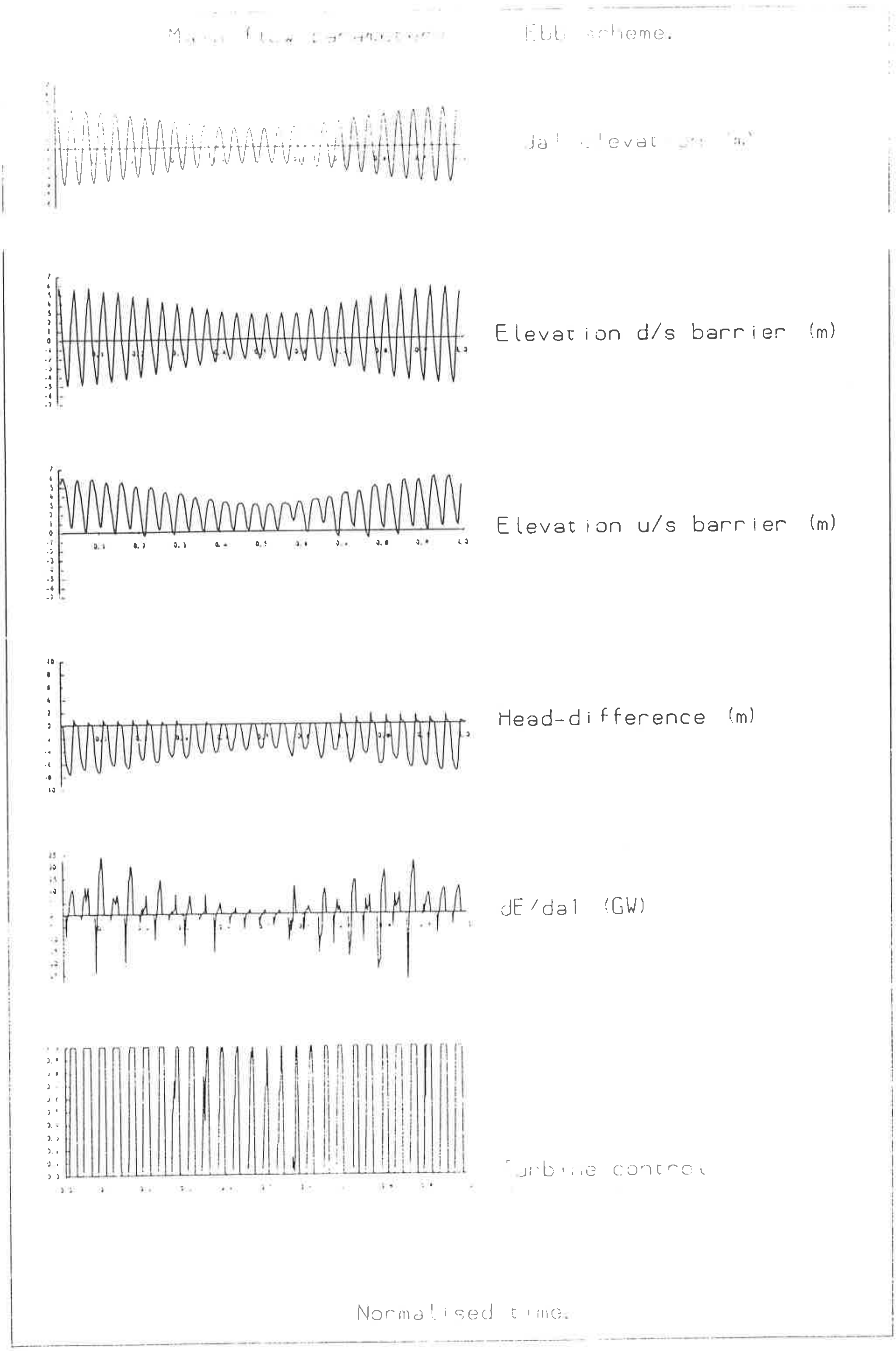

Ebb scheme.
Elevation d/s barrier (m)
Elevation u/s barrier (m)
Head-difference (m)
dE/dal (GW)
Normalised time.

Figure 7a.

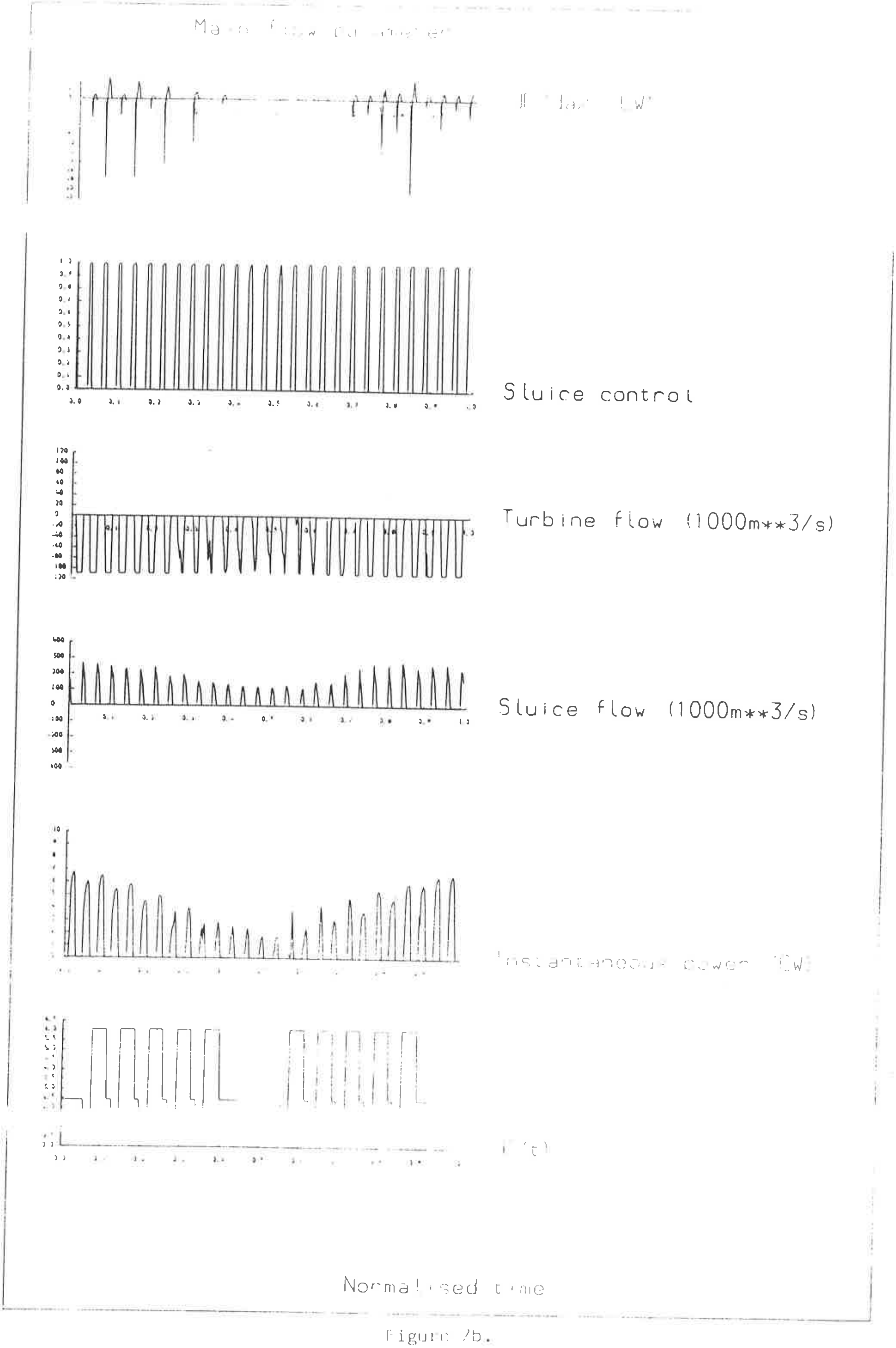

Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Normalised time

Figure 2b.

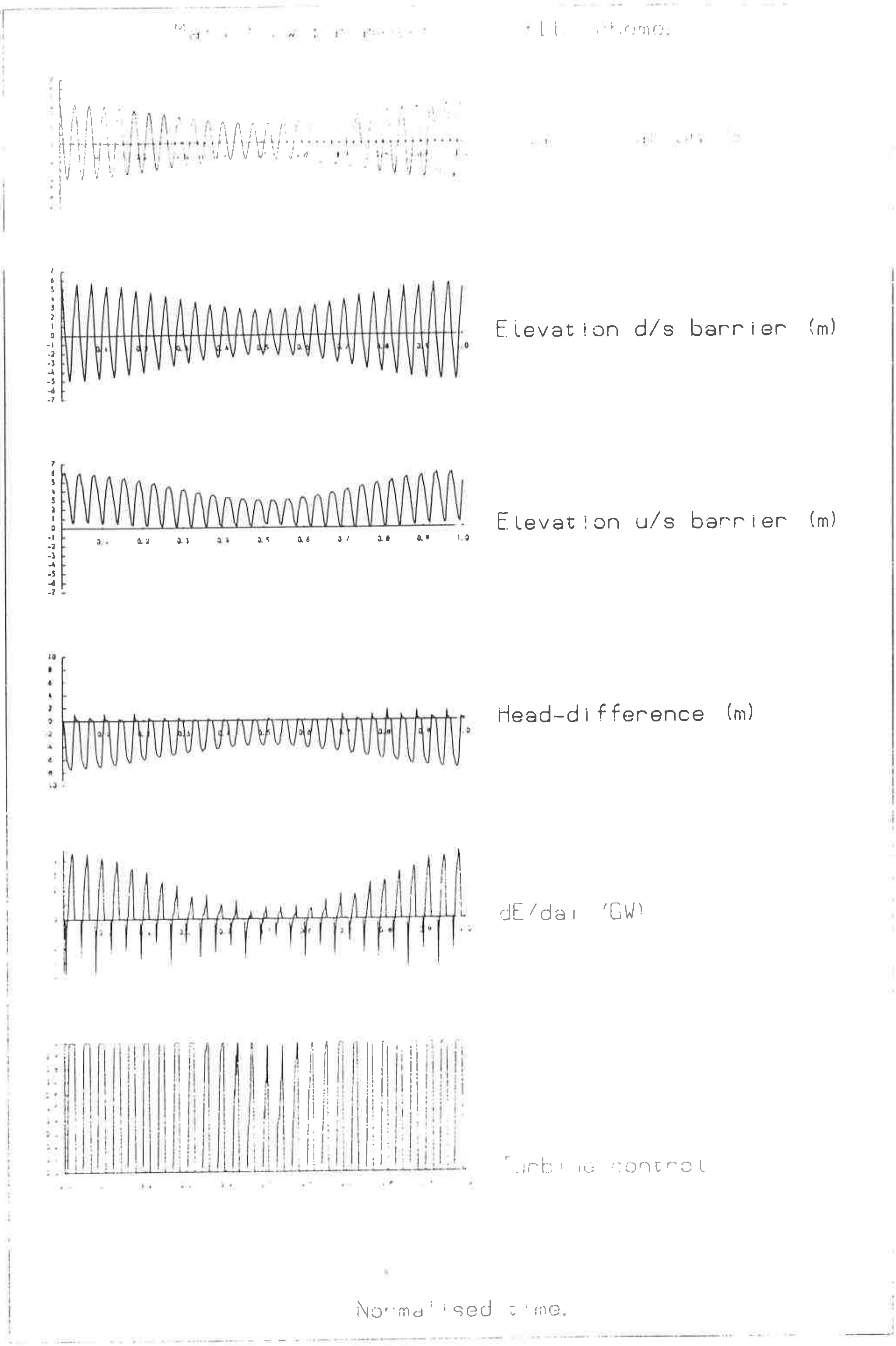

Elevation d/s barrier (m)
Elevation u/s barrier (m)
Head-difference (m)

Figure 8a.

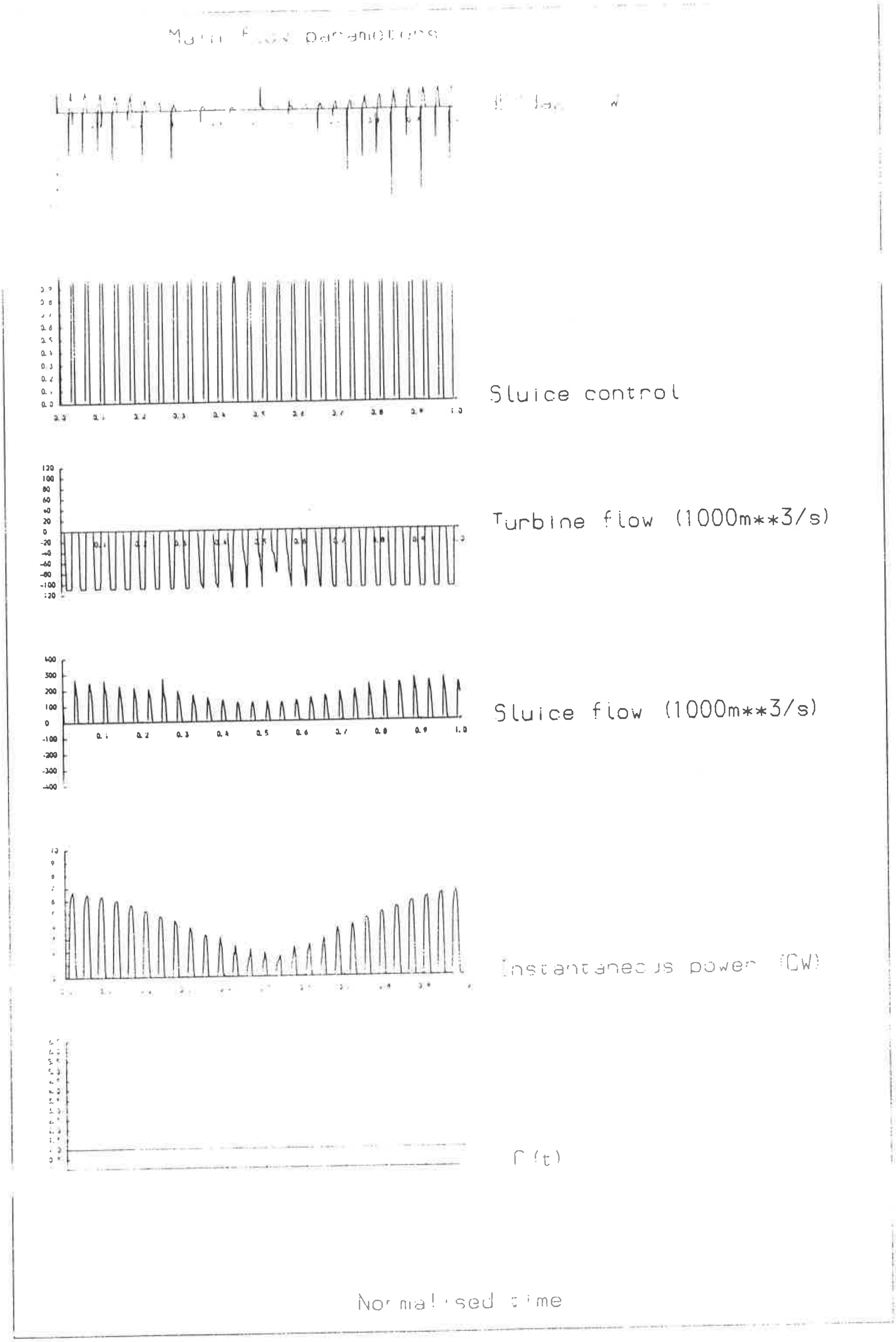

Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Instantaneous power (GW)
Normalised time

Figure 8b.