OPTIMISATION OF STEAM INJECTION INTO AN OIL RESERVOIR

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ABSTRACT

A simple model is developed for steam injection into an idealised oil reservoir. The development of the steam zone is governed by a weakly singular integral equation which in turn becomes the equation of state in a related control problem. The control problem arises from optimisation of suitable quantities involved in the injection process and development of the resulting steam zone. Finally an elementary profit functional is formed and an economic limit imposed on the model.

ACKNOWLEDGEMENTS

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I also wish to thank Mr. P. Briggs and Mr. P. Clyne of BP Petroleum Development Limited for their help and ideas over the last eight months.

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INTRODUCTION

The problem under consideration is the optimization of heavy oil recovery by cyclic steam injection. This involves injecting steam into the reservoir for a certain amount of time, then allowing a soak period followed by a production period, this cycle is then repeated successively until it becomes uneconomic to do so.

The eventual aim of the work is to maximise the overall profit taking into account such factors as running costs of the well, the cost of injecting steam and the cost of extracting the oil and water produced.

This involves determining the optimal placings of the switch points from one cycle to the next and determining the switch points from injection to soak and from soak to production within each cycle. The total number of cycles also has to be found along with quantities such as optimal steam injection rates.

The first step in this work is to look at one period of steam injection and to investigate how certain quantities related to this process can be optimized. This is the subject of the present report.

In chapter one the actual model that is to be used for the injection process is presented together with the exact solution for a specific case. In chapter two we go on to look at maximizing the area penetrated by the steam and minimizing the amount of steam injected. Finally, in chapter three, we briefly look at imposing economic limits on the model.

CHAPTER ONE

THE INJECTION MODEL

SECTION A - THE MODEL

A very simple model of the injection period is described in [1] and [2]. This model estimates thermal invasion rates and cumulative heated area for an idealized reservoir making full allowances for non-productive reservoir heat losses. The underlying assumptions of the model are detailed below. This model is valid only up to some critical time t_c which shall be discussed later.

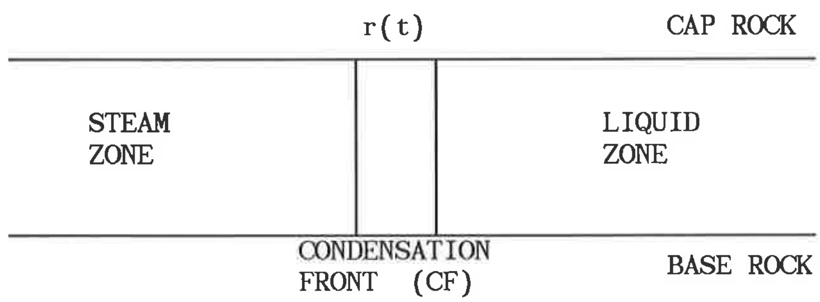


FIG. 1

Assumptions and Conditions

A.1 Disregard changes in the shape of the condensation front, see fig. 1, and assume that it is perpendicular to the reservoir boundaries.

A.2 Assume that the reservoir has a constant thickness and is homogeneous and isotropic.

A.3 The very small changes in C_w and C_s, the specific heat per unit mass of water and solid respectively, with temperature may be safely neglected - i.e. take C_w and C_s to be constant.

A.4 The specific heat C_0 of oil is treated as a function of temperature T, i.e. $C_0 = C_0(T)$.

A.5 At any point the components of the fluid/solid system are in thermostatic equilibrium, i.e. fluids and solids have the same local temperature.

A.6 Temperatures and saturations are constant over any cross-section perpendicular to the flow direction.

A.7 The temperature in the steam zone is a constant and is independent of position and time.

A.8 All temperatures will be measured from the reservoir temperature T_{RI}, which serves as zero level.

A.9 The steam and liquid densities p_{st}, p_w and p_0 will be treated as functions of the temperature.

In order to develop the model expressions for various quantities such as the heat contained in the steam and liquid zones, the heat flux through a cross-section in each of the zones and the rate of heat loss to the surrounding rock, must be found. The assumption of no heat flow across the condensation front must also be formulated.

<u>Heat Content</u> (see [2])

For H_1 and H_2, the heat content per unit volume of the fluid-solid mixture in the steam and liquid zones respectively, we obtain, when taking the heat content at original reservoir temperature as reference level,

$$H_1 = p_1C_1T_1 + \phi S_{st}p_{st}(T_1)L_v \qquad (1.1)$$

where

$$p_1C_1 = p_sC_s(1-\phi) + p_w(T_1)C_w\phi S_{w1}$$
$$+ p_0(T_1)C_0(T_1)\phi S_{01} + p_{st}(T_1)C_w\phi S_{st}$$

$$H_2 = p_2C_2T_2 \qquad (1.2)$$

where

$$p_2C_2 \equiv p_sC_s(1-\phi) + p_w(T_2)C_w\phi S_{w2} \ .$$

Here p represents density, C specific heat, T temperature, S saturation and ϕ porosity.

<u>Heat Flux and the Neglect of Heat Flow Across the Condensation Front (see [2])</u>

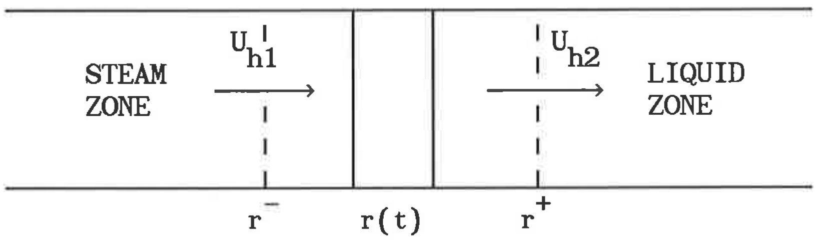


FIG. 2

Let U_{h1}, U_{h2} be the heat fluxes through the fixed cross-sections (see fig. 2). From a heat balance for the volume enclosed by the two fixed cross-sections

$$U_{h1} - U_{h2} = v(t)\left[H_1(r^-, t) - H_2(r^+, t)\right] \tag{1.3}$$

where

$$v(t) = \frac{dr}{dt} .$$

In the present model the assumption is also made that all heat arriving at the C.F. is consumed there while heating the matrix and residual oil from original reservoir temperature to steam temperature. This means that the heat flux $U_{h2} - vH_2$ is neglected. Using this assumption in equation (1.3) results in

$$U_{h1} = v(t)H_1(r^-, t) . \tag{1.4}$$

The heat flux U_{h1} is obtained by subtracting $\dot{Q}_{st}(t)$, the rate at which the whole steam zone loses heat to the surrounding rock layers, from the rate of heat injection, and by taking into account changes in the heat content in the steam zone that are associated with saturation changes, namely,

$$\begin{aligned} U_{h1} = {} & W_{st}(t)\left[C_w T_1 + L_v\right] + W_w(t)C_w T_1 - \dot{Q}_{st}(t) \\ & - \phi p_{st}(T_1)\left[C_w T_1 + L_v\right] \int_0^{r(t)} \frac{\partial}{\partial t} S_{st} dx \\ & - \phi p_w(T_1)C_w T_1 \int_0^{r(t)} \frac{\partial}{\partial t} S_{w1} dx \\ & - \phi p_o(T_1)C_O(T_1)T_1 \int_0^{r(t)} \frac{\partial}{\partial t} S_{01} dx \end{aligned}$$

where W_{st} and W_w represent the mass injection rates per unit cross-sectional area of steam and hot water respectively and L_v is the latent heat of steam.

We can introduce the instantaneous average saturations from the steam zone

$$\bar{S} = \frac{1}{r(t)} \int_0^{r^-(t)} S(x,t)dx \tag{1.5}$$

and thus obtain from substituting equations (1.5) and (1.1) into (1.4)

$$\begin{aligned}&\left[W_{st}(t) + W_w(t)\right]C_w T_1 + W_{st}(t)L_v - \dot{Q}_{st}(t)\\ &= v(t)\phi p_{st}(T_1)L_v\bar{S}_{st} + v(t)T_1\overline{p_1C_1}\\ &\quad + \phi r(t)\left[p_{st}(T_1)L_v + T_1C_w(p_{st}(T_1) - p_w(T_1))\right]\frac{d}{dt}\bar{S}_{st}\end{aligned}$$

where

$$\begin{aligned}\overline{p_1C_1} &= p_sC_s(1-\phi) + p_w(T_1)C_w\phi\bar{S}_{w1}\\ &\quad + p_O(T_1)C_O(T_1)\phi\bar{S}_{O1} + p_{st}(T_1)C_w\phi\bar{S}_{st}\end{aligned}$$

and we have made the assumption

$$\frac{d}{dt}\bar{S}_{O1} = 0 \qquad \frac{d}{dt}\bar{S}_{w1} = -\frac{d}{dt}\bar{S}_{st} .$$

If we make the added assumption that $\frac{d\bar{S}_{st}}{dt} = 0$ then we obtain

$$\begin{aligned}&\left[W_{st}(t) + W_w(t)\right]C_w T_1 + W_{st}(t)L_v - \dot{Q}_{st}(t)\\ &= v(t)\left[\phi p_{st}(T_1)L_v\bar{S}_{st} + T_1\overline{p_1C_1}\right] .\end{aligned} \tag{1.6}$$

<u>Rate of Heat Loss, $\dot{Q}_{st}(t)$ (see [2])</u>

Assuming vertical heat flow in the surrounding rock layers, the rate of heat loss $\dot{Q}_{st}(t)$ can be expressed as a functional of v as follows.

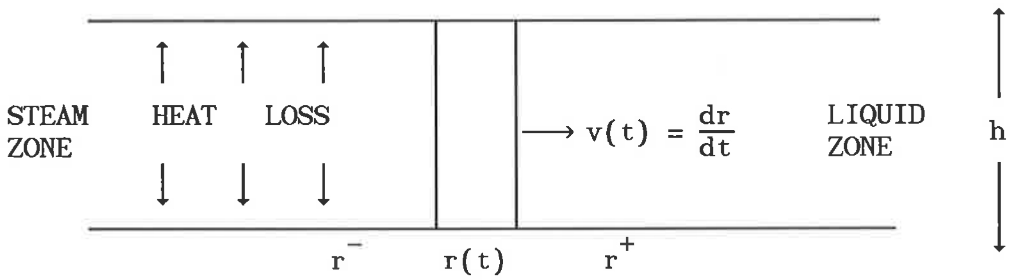


FIG. 3

The local heat flux from the steam zone at x to the overlying and underlying rock is known [3] to be

$$\delta\dot{Q}_{st}(x, t) = \frac{2T_1\sqrt{K_{hf}p_f C_f}}{\sqrt{\pi(t-s)}} \tag{1.7}$$

where K_{hf}, p_f and C_f refer to the thermal conductivity, density and specific heat respectively of the cap and base rock. Here s stands for the instant at which the originally cold boundary at x first became exposed to steam temperature T_1 and hence $(t-s)$ represents the time the cross-section at x has been effected by steam, i.e. $r(s) = x$.

Integrating (1.7) over the total steam zone and referring the heat loss to unit reservoir cross-section we obtain, on substituting into equation (1.6), the following expression

$$\left[W_{st}(t) + W_w(t)\right]C_w T_1 + W_{st}(t)L_v \tag{1.8}$$

$$= \left[\phi p_{st}(T_1)L_v\bar{S}_{st} + T_1\overline{p_1 C_1}\right]v(t)$$

$$+ \frac{2T_1\sqrt{K_{hf}p_f C_f}}{h\sqrt{\pi}} \int_0^t \frac{v(s)}{\sqrt{t-s}}\,ds \quad .$$

We have combined expressions for heat flux, the heat content of the steam zone and the rate of heat loss to the surrounding rock in the form of a heat balance equation with the rate of heat being injected into the reservoir, to finally obtain equation (1.8). This equation determines the position of the condensation front, remembering that $v(t) = \frac{dr}{dt}$.

Constraint On Model

Mandl-Volek [2] claim that this simple model is only valid up to some critical time. This is because steam injection must supply the content of latent heat stored in the expanding steam zone which is not possible after some critical time. This information, which is not contained in equation (1.6) can be formulated as the constraint

$$W_{st}(t)L_v - \dot{Q}_{st}(t) \geq \phi p_{st}L_v\bar{S}_{st}v(t)$$

having made the assumption $\frac{d\bar{S}_{st}}{dt} = 0$. The above inequality can also be written in the form

$$W_{st}(t)L_v \geq \phi p_{st}L_v\bar{S}_{st}\frac{dr}{dt} + \frac{2T_1\sqrt{K_{hf}p_f C_f}}{h\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-s}}\frac{dr}{ds}\,ds \quad . \tag{1.9}$$

The next stage is to non-dimensionalize equations (1.8) and (1.9).

Non-Dimensionalization

Put $t = \frac{L}{V_i} t_D$ and $r = Lr_D$ where t_D and r_D are both dimensionless quantities (see [2]).

$$L = \text{length of reservoir}$$

$$V_i \equiv v(0) = \frac{dr}{dt}(0) .$$

Then

1. $$v(t) = \frac{dr}{dt} = L\frac{dr_D}{dt} = L\frac{dr_D}{dt_D}\frac{V_i}{L} = V_i\frac{dr_D}{dt_D}$$

2. $$\int_0^t \frac{v(s)}{\sqrt{t-s}}\,ds = \int_0^t \frac{1}{\sqrt{t-s}}\frac{dr}{ds}\,ds$$

$$= \sqrt{LV_i}\int_0^{t_D}\frac{1}{\sqrt{t_D-s_D}}\frac{dr_D}{ds_D}\,ds_D$$

3. $$W_{st}(t) = W(t_D)W_{st}(0)$$

$$W_{st}(t)L_v + \left[W_{st}(t) + W_w(t)\right]C_wT_1$$

$$= F(t_D)\left\{W_{st}(0)L_v + \left[W_{st}(0) + W_w(0)\right]C_wT_1\right\}$$

where the quantities $W(t_D)$ and $F(t_D)$ are dimensionless.

Using the relations 1-3 the equations (1.8) and (1.9) can now be re-written in the form

$$\left\{W_{st}(0)L_v + \left[W_{st}(0) + W_w(0)\right]C_wT_1\right\} \boxed{F(t_D)}$$

$$- \frac{2T_1\sqrt{K_{hf}P_fC_fLV_i}}{h\sqrt{\pi}} \boxed{\int_0^{t_D}\frac{1}{\sqrt{t_D-s_D}}\frac{dr_D}{ds_D}\,ds_D} . \tag{1.8a}$$

$$= \left[\phi p_{st} L_v \bar{S}_{st} + \overline{p_1 C_1 T_1}\right] V_i \boxed{\frac{dr_D}{dt_D}}$$

$$W_{st}(0) L_v \boxed{W(t_D)} \geq \phi p_{st} L_v \bar{S}_{st} V_i \boxed{\frac{dr_D}{dt_D}}$$

$$+ \frac{2T_1 \sqrt{K_{hf} p_f C_f} L V_i}{h \sqrt{\pi}} \boxed{\int_0^{t_D} \frac{1}{\sqrt{t_D - s_D}} \frac{dr_D}{ds_D} ds_D} \qquad (1.9a)$$

where the boxed terms are dimensionless. Putting $t = 0$ in equation (1.6) gives the relationship

$$W_{st}(0) L_v + \left[W_{st}(0) + W_w(0)\right] C_w T_1 = \left[\phi p_{st} L_v \bar{S}_{st} + p_1 C_1 T_1\right] V_i \, . \qquad (1.10)$$

Using (1.10) to eliminate V_i in equation (1.8a) and dividing (1.8a) through by the LHS of (1.10) we eventually obtain

$$\frac{dr_D}{dt_D} = F(t_D) - \sqrt{\frac{\sigma}{\pi}} \int_0^{t_D} \frac{1}{\sqrt{t_D - s_D}} \frac{dr_D}{ds_D} ds_D \qquad (1.11)$$

where

$$\sigma = \frac{4 T_1{}^2 K_{hf} p_f C_f L}{h^2 \left\{W_{st}(0) L_v + \left[W_{st}(0) + W_w(0)\right] C_w T_1\right\} \left\{\phi p_{st} L_v \bar{S}_{st} + \overline{p_1 C_1 T_1}\right\}}$$

and

$$F(t_D) = \frac{W_{st}(t_D) L_v + \left[W_{st}(t_D) + W_w(t_D)\right] C_w T_1}{W_{st}(0) L_v + \left[W_{st}(0) + W_w(0)\right] C_w T_1} \, .$$

If we take the origin to be the time at which injection actually starts then the denominator of $F(t_D)$ will not be zero.

Again using (1.10) to eliminate V_i in equation (1.9a) and dividing through by $W_{st}(0)L_v$ we eventually obtain

$$w(t_D) \geq \alpha \frac{dr_D}{dt_D} + \sqrt{\frac{\beta}{\pi}} \int_0^{t_D} \frac{1}{\sqrt{t_D - s_D}} \frac{dr_D}{ds_D} ds_D \qquad (1.12)$$

where

$$\alpha = \frac{1 + [1 + W_w(0)/W_{st}(0)][C_w T_1/L_v]}{1 + \overline{p_1 C_1} T_1/\phi p_{st} L_v \bar{S}_{st}}$$

$$\beta = \frac{4T_1{}^2 K_{hf} p_f C_f L \left\{1 + [1 + W_w(0)/W_{st}(0)][C_w T_1/L_v]\right\}}{h^2 W_{st}(0) L_v \left[\phi p_{st} L_v \bar{S}_{st} + \overline{p_1 C_1} T_1\right]} . \qquad (1.13)$$

Note

The form of $F(t_D)$ automatically gives us the initial condition that $F(0) = 1$.

Summarising we can describe the movement of the steam front by the equation

$$\frac{dr_D}{dt_D} = F(t_D) - \sqrt{\frac{\sigma}{\pi}} \int_0^{t_D} \frac{1}{\sqrt{t_D - s_D}} \frac{dr_D}{ds_D} ds_D \qquad (1.11)$$

as long as the inequality

$$\frac{dr_D}{dt_D} \le \frac{1}{\alpha} W(t_D) - \sqrt{\frac{\beta}{\alpha^2\pi}} \int_0^{t_D} \frac{1}{\sqrt{t_D - s_D}} \frac{dr_D}{ds_D} ds_D \qquad (1.12a)$$

is satisfied.

There is an exact solution to equation (1.11) for the case where $F(t_D)$ is piecewise constant. This will be described in the next section.

SECTION B - THE EXACT SOLUTION OF THE MODEL EQUATION

Our model equation (1.11) is

$$\frac{dr}{dt} = F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} \frac{dr}{ds} ds ,$$

dropping subscripts. Let us first consider the case where $F(t) = F$, a constant. We can now write down the exact solution of equation (1.14) using the following argument due to D. Porter [4].

Define the operator K by

$$(Kf)(t) = \int_0^t \frac{f(s)}{\sqrt{t-s}} ds . \qquad (1.15)$$

It can readily be shown that

$$(K^2f(t) = \pi \int_0^t f(s)ds . \qquad (1.16)$$

Substituting (1.15) into (1.14) gives

$$\frac{dr}{dt} = F(t) - \sqrt{\frac{\sigma}{\pi}} K \frac{dr}{dt}$$

$$\left[I + \sqrt{\frac{\sigma}{\pi}} K\right] \frac{dr}{dt} = F(t) \quad . \tag{1.17}$$

We can now apply the operator $\left[I - \sqrt{\frac{\sigma}{\pi}} K\right]$ to both sides of (1.17), remembering that $F(t) = F$, a constant, to obtain

$$\left[I - \frac{\sigma}{\pi} K^2\right] \frac{dr}{dt} = F\left[1 - 2\sqrt{\frac{\sigma}{\pi}\, t}\,\right] \quad .$$

Using (1.16) yields

$$\frac{dr}{dt} - \sigma \int_0^t \frac{dr}{ds}\, ds = F\left[1 - 2\sqrt{\frac{\sigma}{\pi}\, t}\right] \quad .$$

Now we know that $r(0) = 0$ and so

$$\frac{dr}{dt} - \sigma r(t) = F\left[1 - 2\sqrt{\frac{\sigma t}{\pi}}\right] \quad . \tag{1.18}$$

Multiplying (1.18) by the integrating factor e^{-rt} and integrating gives

$$r(t) = e^{\sigma t} \int_0^t Fe^{-\sigma s}\left[1 - 2\sqrt{\frac{\sigma s}{\pi}}\right] ds \quad .$$

Finally, making the substitution $x = \sqrt{\sigma t}$

$$r(t) = \frac{F}{\sigma}\left[e^{x^2} \operatorname{erfc} x + \frac{2x}{\sqrt{\pi}} - 1\right] \tag{1.19}$$

where

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-s^2} ds$$

Differentiating (1.19) gives

$$\frac{dr}{dt} = F\, e^{x^2} \operatorname{erfc} x \quad . \tag{1.20}$$

and so we can write equation (1.12a) in the form

$$\frac{dr}{dt} \leq \frac{W}{\alpha} e^{x^2} \operatorname{erfc} x \quad \text{where} \quad x = \sqrt{\frac{\beta t}{\alpha^2}} \quad . \tag{1.21}$$

Using the data in table 1, page 42, and assuming a constant rate of steam injection the parameters in the above equations take the following values,

$F = 1$

$W = 1$

$\sigma = 0{\cdot}0009$

$\alpha = 0{\cdot}00003$

$\beta = 0{\cdot}0018$.

Substituting these values into equations (1.19), (1.20) and (1.21) we obtain

$$r(t) = 1111\left[e^{x^2} \operatorname{erfc} x + \frac{2x}{\sqrt{\pi}} - 1\right] \quad \text{where} \quad x = 0{\cdot}03\sqrt{t}$$

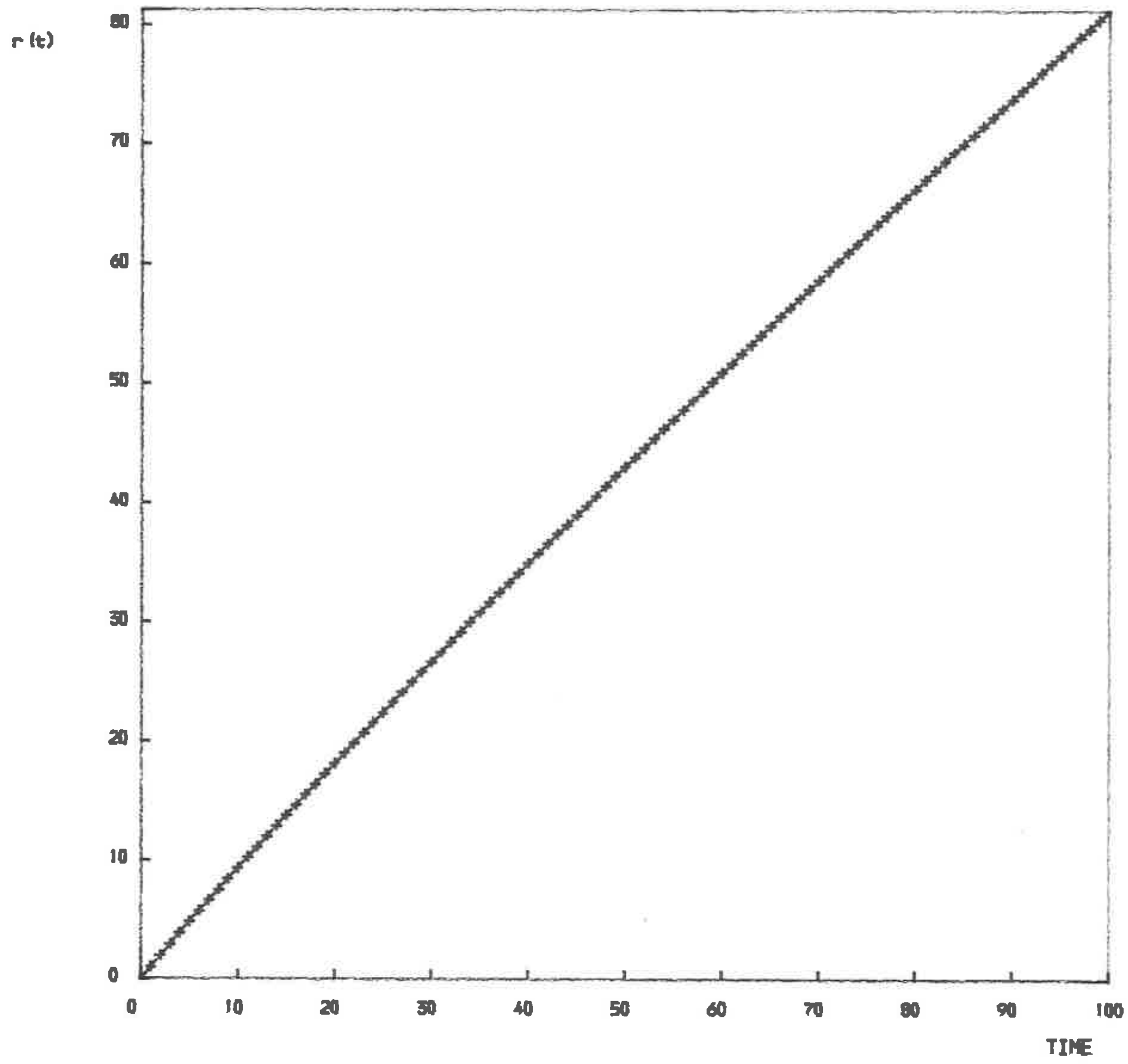
(see graph 1.1),

$$\frac{dr}{dt} = e^{x^2} \operatorname{erfc} x \quad \text{where} \quad x = 0{\cdot}03\sqrt{t} \tag{1.22}$$

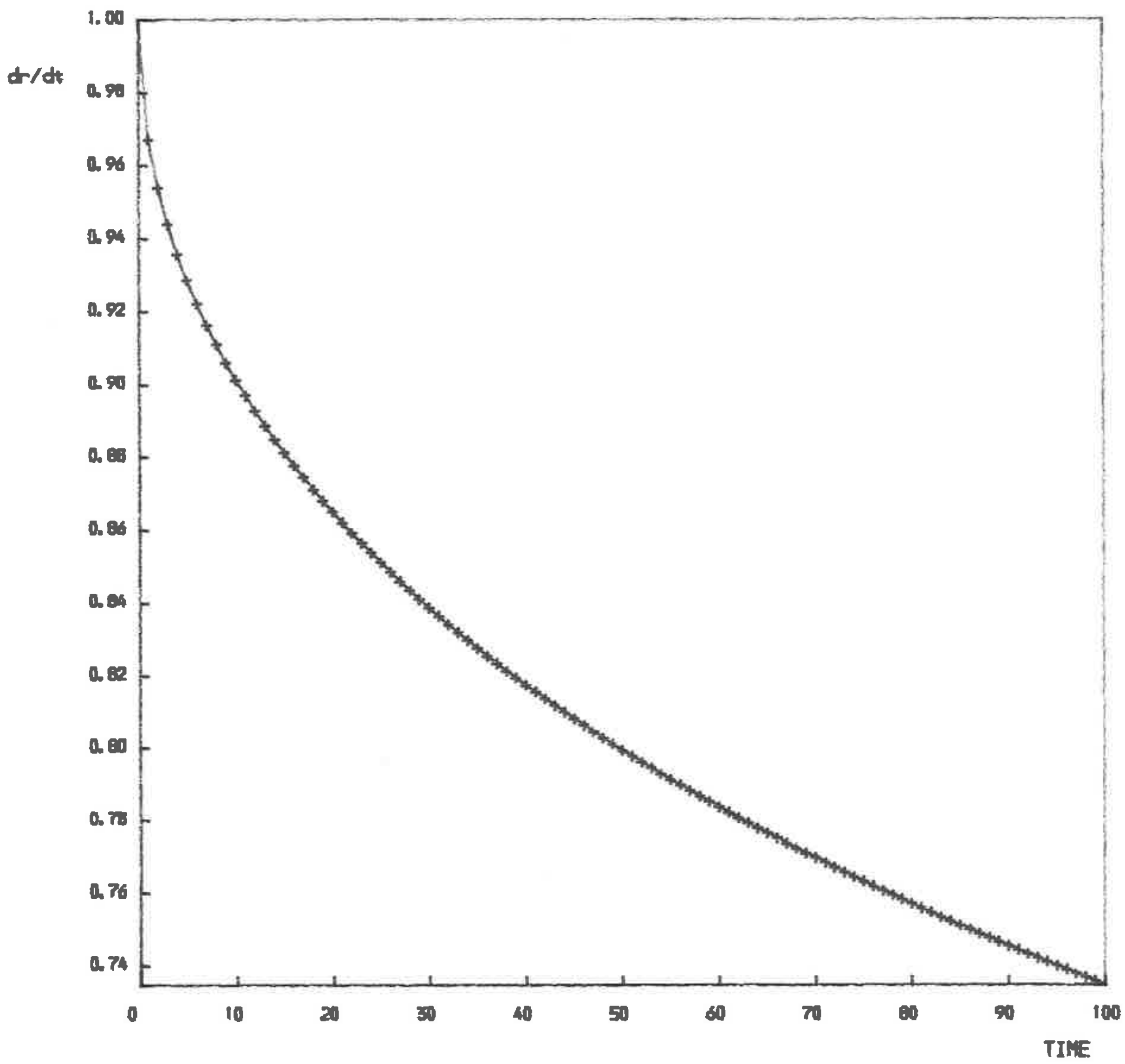
(see graph 1.2)

$$\frac{dr}{dt} \leq 33333\, e^{y^2} \operatorname{erfc} y \quad \text{where} \quad y = 1414\sqrt{t} \quad . \tag{1.23}$$

Combining (1.22) and (1.23) results in the model being valid for $t \leq 784$ where t is dimensionless (see [1] for table of values of $e^{x^2} \operatorname{erfc} x$) .



GRAPH 1.1


dr/dt
1.00
0.98
0.96
0.94
0.92
0.90
0.88
0.86
0.84
0.82
0.80
0.78
0.76
0.74
0
10
20
30
40
50
60
70
80
90
100
TIME

GRAPH 1.2

CHAPTER TWO

THE CONTROL PROBLEM

In this chapter we shall consider control aspects of a single steam injection period. We shall investigate two separate problems:

Problem A in which the length of the injection period is fixed.

Problem B in which the length of the injection period is not fixed, leaving the final time to be determined as part of the optimisation.

There are two approaches to the control that can be taken within each problem.

Approach I - Fix the total amount of steam to be injected and maximize the length of the steam zone at the final time t_f .

Approach II - Fix the length of the steam zone at $t = T$ and minimize the total amount of steam injected.

It is to be expected that the problems arising from these two individual approaches are equivalent in some sense.

SECTION A - NUMERICAL EXPERIMENTS

The aim of these experiements was to obtain some idea of how a fixed amount of steam should be injected in order to obtain the maximum steam zone length after some fixed period of injection, for example, is it better to inject the majority of the steam towards the beginning or end of the injection period?

From chapter one we know that the rate of growth of the steam zone is governed by the non-dimensionalised integral equation

$$F(t) = \frac{dr}{dt} + \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} \frac{dr}{ds}\, ds \quad . \qquad (2.1)$$

Our aim is to maximize $r(t_f)$, given that the total amount of steam to be injected is fixed, where t_f is the final time of injection, i.e. $\max r(t_f)$ subject to $\int_0^{t_f} F(t)dt = I$ where I is fixed.

Equation (2.1) was solved numerically using a second-order Runge-Kutta method (see [6],[7],[8]) with $F(t)$ taken to be a step function with upper and lower bounds given by $0{\cdot}5 \leq F(t) \leq 3{\cdot}0$. The time of injection was chosen such that $t_f = 1$ and σ was determined from the sample data as shown in table 1, page 42. Typical results obtained are shown in graphs 2.1-2.12.

These results suggest that for a fixed amount of steam to be imputted in the form of a step function it is better to step up the rate of injection as time goes on rather than to step down.

The results also suggest that we should perhaps use the idea of injecting as much steam as late as possible in order to maximize the distance, and hence area, penetrated by the steam at the end of the fixed period of injection.

SECTION B - FORMULATING THE CONTROL PROBLEM

We have two problems, A and B as previously described, to consider and within each problem there are two approaches that can be taken. Whichever problem or approach we take the following conditions have to be satisfied:-

$$\frac{dr}{dt} = F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} \frac{dr}{ds}\, ds \qquad (2.2)$$

$$\frac{dr}{dt} \leq \frac{1}{\alpha} w(t) - \sqrt{\frac{\beta}{\alpha^2 \pi}} \int_0^t \frac{1}{\sqrt{t-s}} \frac{dr}{ds}\, ds \qquad (2.3)$$

$r(0) = 0$, initially the steam zone is of zero length. (2.4)

$F(t)$, which represents the rate of steam injection, has upper and lower limits. (2.5)

The form of the non-dimensional ratio $F(t)$ (see equation (1.11)) forces the condition $F(0) = 1$ and so $1 \leq F(t) \leq F_\mu$ is the fixed upper limit.

Here we have assumed, based on the previously described numerical experiments, that we shall increase the rate of injection from its initial value, not decrease it.

We now consider both problems and approaches in turn.

Problem A - Approach I

Fix $\int_0^{t_f} F(t)dt = I$ and $\max\left\{r(t_f) = \int_0^{t_f} \frac{dr}{dt}\,dt\right\}$ subject to the constraints $\frac{dr}{dt} = F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^{t} \frac{1}{\sqrt{t-s}} \frac{dr}{ds}\,ds \quad 1 \leq F(t) \leq F_\mu$.

We assume that the final time t_f , which is fixed, is such that condition (2.3) is satisfied. From Section B of Chapter one we know that, for a piecewise constant increasing injection rate, $\frac{dr}{dt}$ will be monotonically increasing and hence (2.3) will be satisfied for $0 \leq t \leq t_f$.

If we put $y(t) = \frac{dr}{dt}$ we can write the problem as

$$\text{fix} \int_0^{t_f} F(t)dt = I \text{ and } \max \int_0^{t_f} y(t)dt$$

subject to the constraints

$$y(t) = F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^{t} \frac{1}{\sqrt{t-s}}\, y(s)ds$$

$$1 \leq F(t) \leq F_\mu \quad .$$

The Lagrangian is now formed as

$$L(F) = \int_0^{t_f} \left\{ y(t) + \lambda_1(t)\left[y(t) + \sqrt{\frac{\sigma}{\pi}} \int_0^{t} \frac{1}{\sqrt{t-s}}\, y(s)ds - F(t) \right] \right\} dt \qquad (2.6)$$

$$+ \mu_1 \left[\int_0^{t_f} F(t)dt - I \right]$$

where μ_1 is a constant.

We may reverse the order of integration in (2.6) to give

$$\int_0^{t_f}\int_0^{t} \frac{\lambda_1(t)y(s)}{\sqrt{t-s}}\,ds\,dt = \int_0^{t_f}\int_s^{t_f} \frac{\lambda_1(t)y(s)}{\sqrt{t-s}}\,dt\,ds \quad .$$

We can now write (2.6) in the form

$$L(F) = \int_0^{t_f}\left\{1 + \lambda_1(s) + \sqrt{\frac{\sigma}{\pi}}\int_s^{t_f} \frac{\lambda_1(t)}{\sqrt{t-s}}\,dt\right\}y(s)ds \qquad (2.7)$$

$$+ \mu_1\int_0^{t_f} F(t)dt - \mu_1 I - \int_0^{t_f} \lambda_1(t)F(t)dt \quad .$$

Taking variations in F and y, i.e. $F = F^* + \epsilon\delta F$, $y = y^* + \epsilon\delta y$, we obtain on substitution into (2.7)

$$L(F) = \int_0^{t_f}\left\{1 + \lambda_1(s) + \sqrt{\frac{\sigma}{\pi}}\int_0^{t_f} \frac{\lambda_1(t)}{\sqrt{t-s}}\,dt\right\}\left[y^*(s) + \epsilon\delta y(s)\right]ds \qquad (2.8)$$

$$+ \mu_1\int_0^{t_f}\left[F^*(s) + \epsilon\delta F(s)\right]ds - \mu_1 I$$

$$- \int_0^{t_f}\lambda_1(s)\left[F^*(s) + \epsilon\delta F(s)\right]ds \quad .$$

We require $L(F)$ to be a maximum with respect to the variations δy and δF and so if we write

$$L(F^* + \epsilon\delta F) - L(F^*) = \epsilon\int_0^{t_f} \alpha\ \delta y ds + \epsilon\int_0^{t_f} \beta\ \delta F ds$$

where

$$\alpha = 1 + \lambda_1(s) + \sqrt{\frac{\sigma}{\pi}} \int_s^{t_f} \frac{\lambda_1(t)}{\sqrt{t-s}} dt \quad \text{and}$$

$$\beta = \mu_1 - \lambda_1(s)$$

then since δy is unconstrained we require that

$$\alpha \equiv 0 \quad . \tag{2.10}$$

Now δF is not unconstrained as $F^* + \epsilon\delta F \in X$ where X is the convex set $[1, F_\mu]$ but we require that $L(F^* + \epsilon\delta F) - L(F^*) \leq 0 \quad \forall\, \epsilon > 0$ and so

$$\int_0^{t_f} \beta\, \delta F\, ds \leq 0 \tag{2.11}$$

given that condition (2.10) is satisfied. (2.11) implies that

$$\mu_1 - \lambda_1(s) \geq 0 \Rightarrow \delta F \leq 0 \quad \text{and} \tag{2.12}$$

$$\mu_1 - \lambda_1(s) \leq 0 \Rightarrow \delta F \geq 0 \quad \text{where} \tag{2.13}$$

$$\delta F = F - F^* \ .$$

(2.12) and (2.13) give us the conditions

$$F^* = F_\mu \quad \text{when} \quad \mu_1 - \lambda_1(s) \geq 0 \quad \text{and} \tag{2.14}$$

$$F^* = 1 \quad \text{when} \quad \mu_1 - \lambda_1(s) \leq 0 \ .$$

Note that there is no need to find the constant μ_1, the switch point can be determined from the condition that $\int_0^{t_f} F(t)dt = I$ as will be shown later.

We can find $\lambda_1^*(s)$ from equation (2.10) using the same idea as used in Chapter One for finding the solution of (1.14) but this time defining the operator K by

$$(K_f)(s) = \int_s^{t_f} \frac{f(t)}{\sqrt{t-s}}\,dt \quad .$$

In this way we eventually reach the solution

$$\lambda_1(s) = -e^{-x^2}\text{erfcx} \quad \text{where} \quad x = \sqrt{\sigma(t_f - s)} \quad .$$

$\lambda_1(s)$ is a monotonic decreasing function and so from conditions (2.12) and (2.13) we can see that there can be at most one switch point. If we let the switchpoint be at $t = t_s$ then from (2.14) and the fact that $\lambda_1(s)$ is decreasing we can see that

$$F^* = 1 \qquad t \in [0, t_s)$$

$$F^* = F_\mu \qquad t \in (t_s, t_f] \quad . \tag{2.15}$$

Note that (2.15) satisfies the initial condition that $F^*(0) = 1$. However, we still have to satisfy the condition that

$$\int_0^{t_f} F(t)dt = I \quad \text{and so}$$

$$t_s + (t_f - t_s)F_\mu = I$$

$$t_s = \frac{t_f F_\mu - I}{F_\mu - 1} \quad . \tag{2.16}$$

From (2.16) we are able to determine the position of the switch point provided that I, t_f and F_μ are such that $t_s \in (0, t_f]$. Note that t_s cannot be zero as this would contradict the initial condition $F^*(0) = 1$.

Problem A - Approach II

This problem can be stated as

$$\text{fix} \quad r(tf) = r_T = \int_0^{t_f} \frac{dr}{dt}\, dt \quad \text{and} \quad \min \int_0^{t_f} F(t)dt$$

subject to the constraints

$$\frac{dr}{dt} = F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} \frac{dr}{ds}\, ds$$

$$1 \leq F(t) \leq F_\mu \quad .$$

Again we make the assumption that t_f is such that condition (2.3) is satisfied and put $y(t) = \frac{dr}{dt}$. We can now write the problem in the form

$$\text{fix} \quad r_T = \int_0^{t_f} y(t)dt \quad \text{and} \quad \min \int_0^{t_f} F(t)dt$$

subject to the constraints

$$y(t) = F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}}\, y(s)ds$$

$$1 \leq F(t) \leq F_\mu \quad .$$

This time on forming the Lagrangian we obtain

$$L(F) = \int_0^{t_f} \left\{- F(t) + \lambda_2(t)\left[F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}}\, y(s)ds - y(t)\right]\right\}dt$$

$$+ \mu_2\left[\int_0^{t_f} y(s)ds - r_T\right]$$

where μ_2 is a constant.

Again reversing the order of integration and taking variations in F and y we obtain

$$L(F^* + \epsilon\delta F) = L(F^*) =$$

$$\epsilon\int_0^{t_f}\Big[\lambda_2(s) - 1\Big]\delta F\, ds - \epsilon\int_0^{t_f}\Big[\sqrt{\frac{\sigma}{\pi}}\int_0^{t_f}\frac{\lambda_2(t)}{\sqrt{t-s}}\,dt + \lambda_2(s) - \mu_2\Big]\delta y ds \ .$$

Using a similar argument to that used in approach one we have the conditions

$$\sqrt{\frac{\sigma}{\pi}}\int_s^{t_f}\frac{\lambda_2(t)}{(t-s)^{1/2}}\,dt + \lambda_2(s) - \mu_2 = 0 \qquad (2.17)$$

and

$$\begin{aligned} \lambda_2(s) - 1 \geq 0 &\Rightarrow F^* = F_\mu \\ \lambda_2(s) - 1 \leq 0 &\Rightarrow F^* = 1 \ . \end{aligned} \qquad (2.18)$$

Again we can write down the exact solution for $\lambda(s)$,

i.e. $\lambda_2(s) = \mu_2 e^{x^2}\operatorname{erfc}x$ where $x = \sqrt{\sigma(t_f-s)}$.

The constant μ_2 must be positive in order for (2.18) to be feasible and so $\lambda_2(s)$ is a monotonic increasing function and once again we have just one switch point t_s where

$$\begin{aligned} F^* &= 1 \qquad t \in [0,\ t_s] \\ F^* &= F_\mu \qquad t \in (t_s,\ t_f] \ . \end{aligned} \qquad (2.19)$$

This time the position of the switch point has to be determined from the condition that $\int_0^{t_f} y(t)dt = r_T$.

To do this we may use the fact that we know the exact solution $r(t)$. We know from (2.19) that

$$\frac{dr}{dt} = 1 - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \frac{dr}{ds}\, ds \qquad t \in [0,\ t_s)$$

$$\frac{dr}{dt} = F_\mu - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \frac{dr}{ds}\, ds \qquad t \in (t_s,\ t_f]$$

i.e.

$$r(t) = \frac{1}{\sigma}\left[e^{x^2} \operatorname{erfc} x + \frac{2x}{\sqrt{\pi}} - 1\right] \qquad t \in [0,\ t_s)$$

$$r(t) = \frac{F_\mu}{\sigma}\left[e^{x^2} \operatorname{erfc} x + \frac{2x}{\sqrt{\pi}}\right] + \frac{C}{\sigma} \qquad t \in (t_s,\ t_f] \tag{2.20}$$

where $x = \sqrt{\sigma t}$ and C is a constant (see equation 1.19).

Now $r(t)$ must be continuous at $t = t_s$ and so from (2.20)

$$F_\mu\left[e^{X_s^2} \operatorname{erfc} X_s + \frac{2X_s}{\sqrt{\pi}}\right] + C = e^{X_s^2} \operatorname{erfc} X_s + \frac{2X_s}{\sqrt{\pi}} - 1$$

where $X_s = \sqrt{\sigma t_s}$ and thus

$$C = (1 - F_\mu)\left[e^{X_s^2} \operatorname{erfc} X_s + \frac{2X_s}{\sqrt{\pi}}\right] - 1 \; . \tag{2.21}$$

Substituting (2.21) into (2.20) and enforcing the condition $r(t_f) = r_T$ we obtain

$$r_T = \frac{F_\mu}{\sigma}\left[e^{X^2} \operatorname{erfc} X + \frac{2X}{\sqrt{\pi}}\right] + \frac{(1 - F_\mu)}{\sigma}\left[e^{X_s^2} \operatorname{erfc} X_s + \frac{2X_s}{\sqrt{\pi}}\right] - \frac{1}{\sigma} \tag{2.22}$$

where $X = \sqrt{\sigma t_f}$.

It is from equation (2.22) we expect to find the value of X_s and hence the position of the switch point t_s .

<u>**Comments On The Two Approaches Applied To Problem A**</u>

1. Both approaches have led to conditions of the form

$$a + \lambda(s) + \sqrt{\frac{\sigma}{\pi}} \int_s^{t_f} \frac{\lambda(t)}{\sqrt{t-s}}\, dt = 0$$

$$b - \lambda(s) \geq \Rightarrow F^* = \min F, \max F$$

$$b - \lambda(s) \leq \Rightarrow F^* = \max F, \min F$$

where a and b are constants.

2. It is much easier to find the position of the switch point using approach I than it is using approach II as in approach II it proves to be awkward to satisfy the boundary condition $r(t_f) = r_T$.

<u>Problem B - Approach I</u>

Fix $\int_0^{t_f} F(t)dt = I$ and $\max\left\{r(t_f) = \int_0^{t_f} \frac{dr}{dt}\, dt\right\}$ subject to the constraints

$$\frac{dr}{dt} = F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}}\, ds \tag{2.23}$$

$$\frac{dr}{dt} \leq \frac{1}{\alpha} w(t) - \sqrt{\frac{\beta}{\alpha^2 \pi}} \int_0^t \frac{1}{\sqrt{t-s}} \frac{dr}{ds}\, ds \tag{2.24}$$

$$1 \leq F(t) \leq F_\mu$$

where the final time t_f is now free to be chosen.

Equations (2.23) and (2.24) can be combined to give the condition

$$\left[1 - \frac{1}{\alpha}\sqrt{\frac{\beta}{\sigma}}\right] \frac{dr}{dt} \leq \frac{1}{\alpha} w(t) - \frac{1}{\alpha}\sqrt{\frac{\beta}{\sigma}}\, F(t) \quad . \tag{2.25}$$

From (1.12) and (1.13)

$$\frac{1}{\alpha}\sqrt{\frac{\beta}{\sigma}} = 1 + \overline{p_1 C_1 T_1} \;/\; \phi p_{st} L_v \overline{S}_{st} > 1$$

and so (2.25) can be rewritten

$$\frac{dr}{dt} \geq \frac{1}{\alpha\sqrt{\sigma} - \sqrt{\beta}}\left[\sqrt{\sigma}\, w(t) - \sqrt{\beta}\, F(t)\right] .$$

If we consider the case where no water is being injected, just steam then from (1.12) we can see that $F(t) = w(t)$ and so the above inequality can be written

$$\frac{dr}{dt} \geq \gamma F(t) \quad \text{where} \quad \gamma = \frac{\sqrt{\sigma} - \sqrt{\beta}}{\alpha\sqrt{\sigma} - \sqrt{\beta}} .$$

Writing $y(t) = \frac{dr}{dt}$ as before and forming the Lagrangian for the problem we obtain

$$L(F) = \int_0^{t_f} y(t) + \lambda_3(t)\left[F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}}\, y(s)ds - y(t)\right]dt$$

$$+ \int_0^{t_f} \lambda_4(t)\left[y(t) - \gamma F(t)\right]dt + \mu_3\left[\int_0^{t_f} F(t)dt - I\right] \qquad (2.26)$$

where μ_3 is a constant and where

$$\lambda_4 = 0 \quad \text{when} \quad y(t) - \gamma F(t) > 0$$

$$\lambda_4 \neq 0 \quad \text{when} \quad y(t) - \gamma F(t) = 0 \qquad (2.27)$$

(see [5]).

Reversing the order of integration (2.26) can be written

$$L(F) = \int_0^{t_f}\left[1 - \lambda_3(s) + \lambda_4(s) - \sqrt{\frac{\sigma}{\pi}}\int_s^{t_f}\frac{\lambda_3(t)}{\sqrt{t-s}}\,dt\right]y(s)ds$$

$$+ \int_0^{t_f}\left[\mu_3 + \lambda_3(s) - \gamma\lambda_4(s)\right]F(s)ds - \mu_3 I \quad . \tag{2.28}$$

Remembering that t_f is now free to be chosen we must consider variations in t_f as well as in F and y. Substituting $t = t_f^* + \epsilon\delta t$, $y = y^* + \epsilon\delta y$ and $F = F^* + \epsilon\delta F$ into (2.28) and subtracting (2.28) we obtain

$$L(F^* + \epsilon\delta F) - L(F^*) =$$

$$\int_{t_f^*}^{t_f+\epsilon\delta t}\left[1 - \lambda_3(s) + \lambda_4(s) - \sqrt{\frac{\sigma}{\pi}}\int_s^{t^*}\frac{\lambda_3(t)}{\sqrt{t-s}}\,dt\right]y^*(s)ds$$

$$+ \epsilon\int_0^{t_f^*+\epsilon\delta t}\left[1 - \lambda_3(s) + \lambda_4(s) - \sqrt{\frac{\sigma}{\pi}}\int_s^{t_f^*}\frac{\lambda_3(t)}{\sqrt{t-s}}\,dt\right]\delta y(s)ds$$

$$- \sqrt{\frac{\sigma}{\pi}}\int_0^{t_f^*+\epsilon\delta t}\left[\int_{T^*}^{t_f^*+\epsilon\delta t}\frac{\lambda_3(t)}{\sqrt{t-s}}\,dt\right]\left[y^*(s) + \epsilon\delta y(s)\right]ds$$

$$+ \epsilon\int_0^{t_f^*+\epsilon\delta t}\left[\mu_3 + \lambda_3(s) - \gamma\lambda_4(s)\right]\delta F(s)ds$$

$$+ \int_{t_f^*}^{t_f^*+\epsilon\delta t}\left[\mu_3 + \lambda_3(s) - \gamma\lambda_4(s)\right]F^*(s)ds \quad .$$

We require $L(F^* + \epsilon\delta F) - L(F^*) \leq 0 \quad \forall \epsilon > 0$. δy is unconstrained and so we must have that

$$1 - \lambda_3(s) + \lambda_4(s) - \sqrt{\frac{\sigma}{\pi}} \int_s^{t_f^*} \frac{\lambda_3(t)}{\sqrt{t-s}} dt = 0 . \qquad (2.29)$$

Using the same argument as before for δF not being unconstrained we have

$$\mu_3 + \lambda_3(s) - \gamma\lambda_4(s) \geq 0 \implies F^* = F_\mu$$
$$(2.30)$$
$$\mu_3 + \lambda_3(s) - \gamma\lambda_4(s) \leq 0 \implies F^* = 1 .$$

Now for any function $f(s)$

$$\int_{t_f^*}^{t_f^*+\epsilon\delta t} f(s)ds \sim \epsilon\delta t_f(t_f^*) \quad \text{as} \quad \epsilon \longrightarrow 0$$

and so

$$\int_0^{t_f^*} \frac{\lambda_3(t_f^*)}{\sqrt{t_f^* - s}} y^*(s)ds = 0 \qquad (2.31)$$

and

$$\left[\mu_3 + \lambda_3(t_f^*) - \gamma\lambda_4(t_f^*)\right]F^*(t_f^*) = 0 \qquad (2.32)$$

(2.31) $\implies \lambda_3(t_f^*) = 0$

(2.32) $\implies \mu_3 - \gamma\lambda_4(t_f^*) = 0$

$\lambda_3(t_f^*) = 0 \implies \lambda_4(t_f^*) = -1$ (see (2.29))

$\lambda_4(t_f^*) = -1 \implies y(t_f^*) - \gamma F(t_f^*) = 0$ (see (2.27)). (2.33)

We know from the exact solution that $y(t)$ is a monotonic function and thus that there will be only one value of t_f^* such that (2.33) is satisfied, and hence that $\lambda_4(t) = 0 \quad t \in [0, t_f^*)$. Using this in equation (2.29) we have the condition

$$1 - \lambda_3(s) - \sqrt{\frac{\sigma}{\pi}} \int_s^{t_f^*} \frac{\lambda_3(t)}{\sqrt{t-s}} dt = 0 \qquad s \in [0, t_f^*) \qquad (2.34)$$

and from (2.30)

$$\lambda_3(s) + \mu_3 \geq 0 \Rightarrow F^* = F_\mu$$

$$\lambda_3(s) + \mu_3 \leq 0 \Rightarrow F^* = 1 \quad .$$

From (2.34) $\lambda_3(s) = e^{x^2} \mathrm{erfc}x$ where $x = \sqrt{\sigma(t_f^*-s)}$ and so $\lambda_3(s)$ is a monotonic increasing function. It now follows that

$$F^* = 1 \qquad t \in [0, t_s)$$

$$F^* = F_\mu \qquad t \in (t_s, t_f^*]$$

where t_s is the switchpoint determined as before in problem A.

<u>Problem B - Approach II</u>

Fix $\int_0^{t_f} y(t)dt = r_T$ and $\min \int_0^{t_f} F(t)dt$ subject to

$$y(t) = F(t) - \sqrt{\frac{\sigma}{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} y(s)ds$$

$$y(t) \geq \gamma F(t)$$

$$1 \leq F(t) \leq F_\mu$$

where $y(t) = \dfrac{dr}{dt}$.

We can form the Lagrangian for this problem

$$L(F) = \int_0^{t_f}\left\{- F(t) + \lambda_6(t)\left[F(t) - \sqrt{\frac{\sigma}{\pi}}\int_0^{t} \frac{1}{\sqrt{t-s}}\, y(s)ds - y(t)\right]\right\}dt$$

$$+ \int_0^{t_f} \lambda_7(t)\left[y(t) - \gamma F(t)\right]dt + \mu_4\left[\int_0^{t_f} y(t)dt - r_T\right] \qquad (2.35)$$

where

$$\lambda_7 = 0 \text{ when } y(t) - \gamma F(t) > 0$$

$$\lambda_7 \neq 0 \text{ when } y(t) - \gamma F(t) = 0 \qquad (2.36)$$

(see [5]).

Reversing the order of integration (2.35) can be written

$$L(F) = \int_0^{t_f}\left[\mu_4 - \lambda_6(s) + \lambda_7(s) - \sqrt{\frac{\sigma}{\pi}}\int_s^{t_f} \frac{1}{\sqrt{t-s}}\, \lambda_6(t)dt\right]y(s)ds$$

$$+ \int_0^{t_f}\left[\lambda_6(s) - 1 - \gamma\lambda_7(s)\right]F(s)ds - \mu_4 r_T \quad .$$

Taking variations in t_f , y and F as before we obtain

$$L(F^* + \epsilon\delta F) - L(F^*) =$$

$$\epsilon\int_0^{t_f^*}\left[\mu_4 - \lambda_6(s) + \lambda_7(s) - \sqrt{\frac{\sigma}{\pi}}\int_s^{t_f^*} \frac{\lambda_6(t)}{\sqrt{t-s}}\, dt\right]\delta y(s)ds$$

$$+ \int_{t_f^*}^{t_f^*+\epsilon\delta t}\left[\mu_4 - \lambda_6(s) + \lambda_7(s) - \sqrt{\frac{\sigma}{\pi}}\int_s^{t_f^*} \frac{\lambda_6(t)}{\sqrt{t-s}}\, dt\right]\left[y^*(s) + \epsilon\delta y(s)\right]ds$$

$$- \int_0^{t_f^*+\epsilon\delta t}\left[\sqrt{\frac{\sigma}{\pi}}\int_{t_f^*}^{t_f^*+\epsilon\delta t} \frac{\lambda_6(t)dt}{\sqrt{t-s}}\right]\left[y^*(s) + \epsilon\delta y(s)\right]ds$$

$$+ \int_{t_f^*}^{t_f^*+\epsilon\delta t} \left[\lambda_6(s) - 1 - \gamma\lambda_7(s)\right]\left[F^*(s) + \epsilon\delta F(s)\right]ds$$

$$+ \epsilon\int_0^{t_f^*} \left[\lambda_6(s) - 1 - \gamma\lambda_7(s)\right]\delta F(s)ds \quad .$$

Finally, using similar arguments to those in the previous approach, we obtain

$$\mu_4 - \lambda_6(s) + \lambda_7(s) - \sqrt{\frac{\sigma}{\pi}} \int_s^{t_f^*} \frac{\lambda_6(t)}{\sqrt{t-s}} dt \;=\; 0 \qquad (2.37)$$

$$\lambda_6(t_f^*) \;=\; 0 \qquad (2.38)$$

$$\lambda_6(t_f^*) - 1 - \gamma\lambda_7(t_f^*) \;=\; 0 \qquad (2.39)$$

$$\left.\begin{array}{l} \lambda_6(s) - 1 - \gamma\lambda_7(s) \geq 0 \implies F^* = F_\mu \\ \lambda_6(s) - 1 - \gamma\lambda_7(s) \leq 0 \implies F^* = 1 \quad . \end{array}\right. \qquad (2.40)$$

Substituting (2.38) into (2.39) results in the condition that $\lambda_7(t_f^*) = -\frac{1}{\gamma} \neq 0$ and so from (2.36) $y(t_f^*) - \gamma F(t_f^*) = 0$. We know from the exact solution that $y(t)$ is a monotonic function and hence that $y(t) - \gamma F(t) > 0$ for $t \in [0, t_f^*)$. We can now write conditions (2.37) and (2.40) as

$$\mu_4 - \lambda_6(s) - \sqrt{\frac{\sigma}{\pi}} \int_s^{t_f^*} \frac{\lambda_6(t)}{\sqrt{t-s}}\, dt = 0 \qquad t \in [0,\ t_f^*) \tag{2.41}$$

$$\begin{aligned} \lambda_6(s) - 1 \geq 0 &\Rightarrow F^* = F_\mu \\ \lambda_6(s) - 1 \leq 0 &\Rightarrow F^* = 1 \ . \end{aligned} \tag{2.42}$$

Combining (2.37), (2.38) and (2.39) results in $\mu_4 = \frac{1}{\gamma} > 0$ (from (1.14)). $\lambda_6(s)$ is therefore a monotonic increasing function and hence

$$F^* = 1 \qquad t \in [0,\ t_s)$$

$$F^* = F_\mu \qquad t \in (t_s,\ t_f^*]$$

where t_s is the switch point which has to be determined by a similar method to that in approach two of problem A.

<u>**Comments On The Two Approaches Applied To Problem B**</u>

1. Both approaches have led to conditions of the form

$$a + \lambda(s) + \sqrt{\frac{\sigma}{\pi}} \int_s^{T} \frac{\lambda(t)}{\sqrt{t-s}}\, dt = 0$$

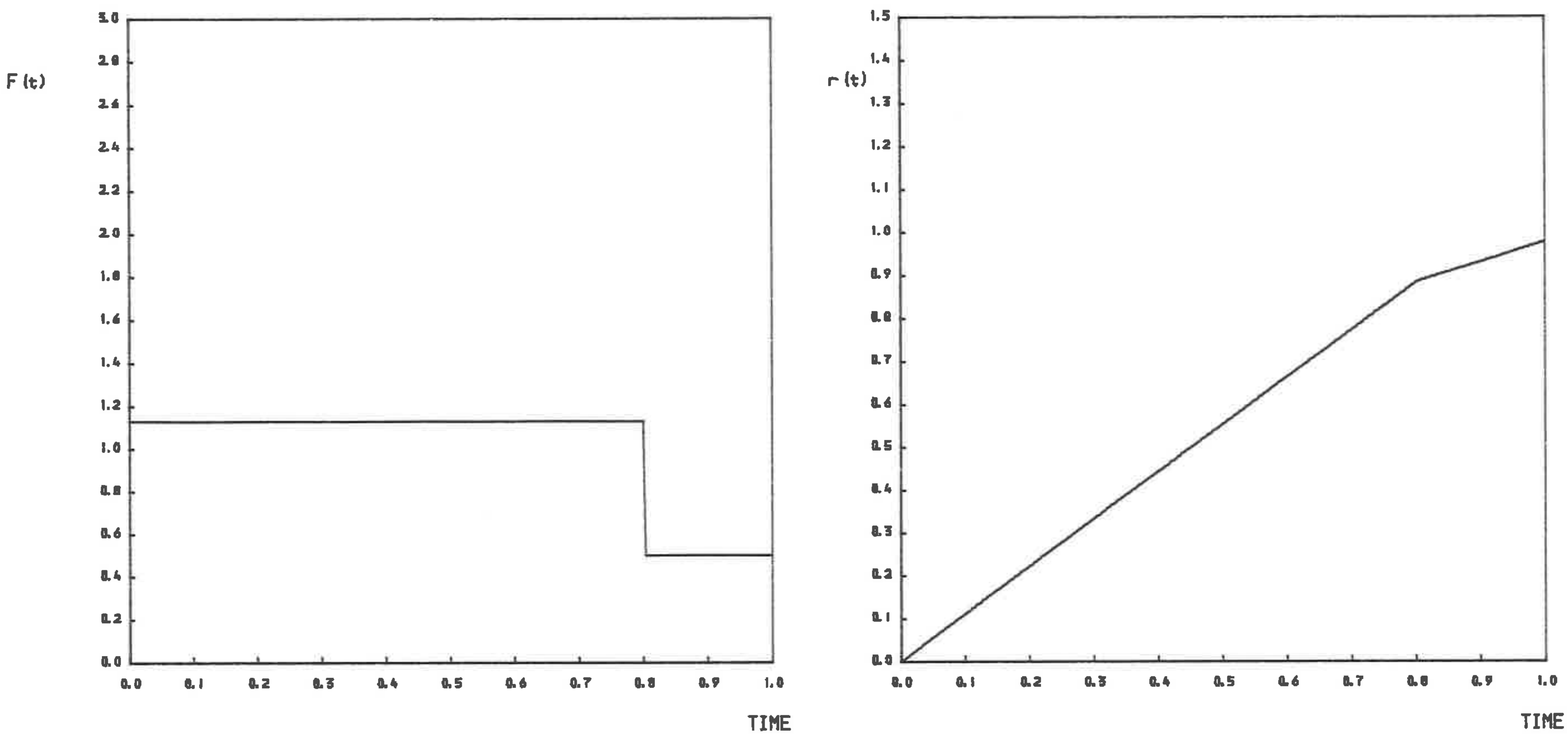
$$b - \lambda(s) \geq 0 \Rightarrow F^* = \min F$$

$$b - \lambda(s) \leq 0 \Rightarrow F^* = \max F \ .$$

2. Both approaches have given the condition on t_f^* to be $y(t_f^*) - \gamma F(t_f^*) = 0$. This condition means that the optimal final time is the greatest one for which the model is valid. This

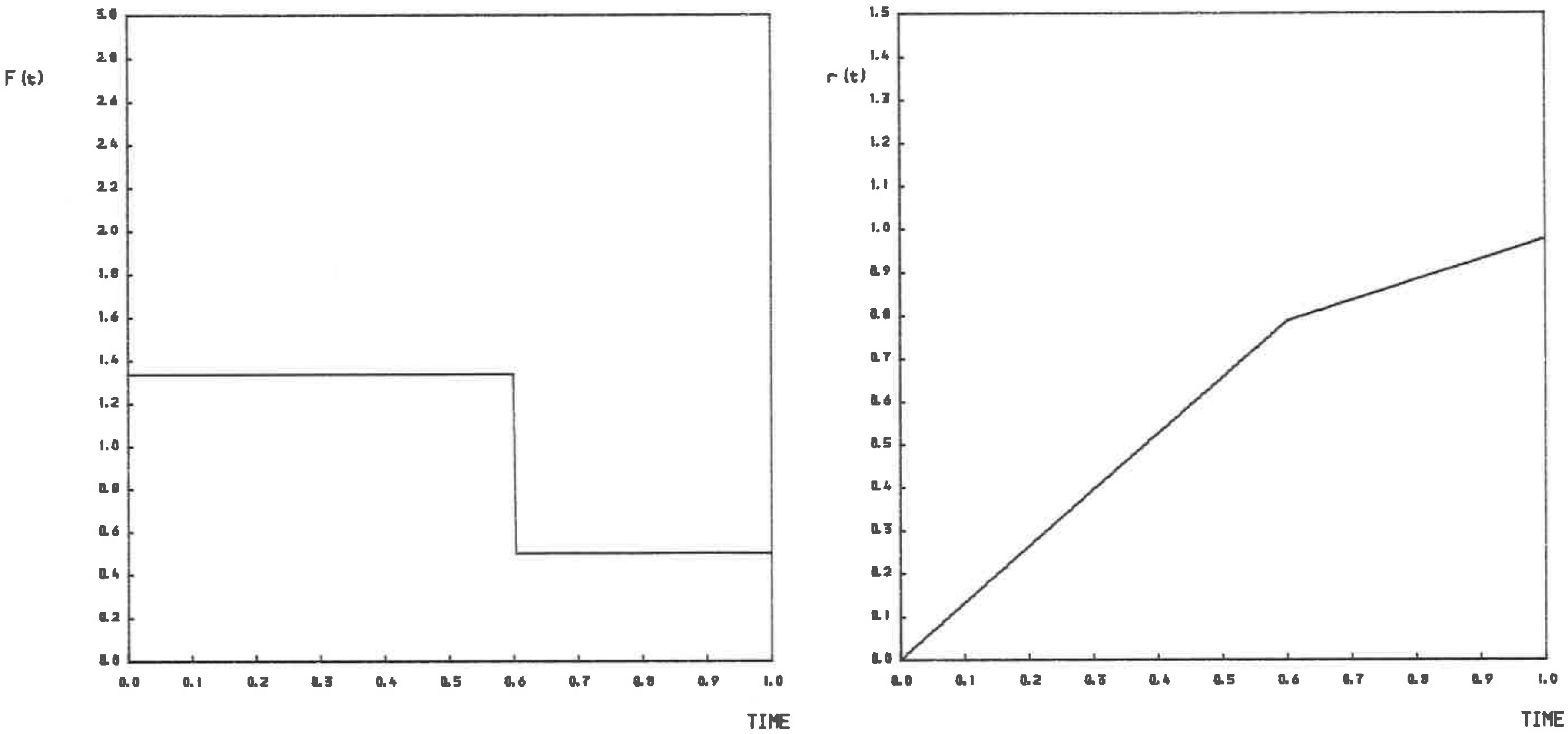
suggests that if the model were valid for all time then there would be no optimal final time and hence no optimal solution for the unfixed end time problem.

Having applied both approaches to the fixed and free end time problem approach one seems the most favourable in that it is far easier to determine the position of the switchpoint.



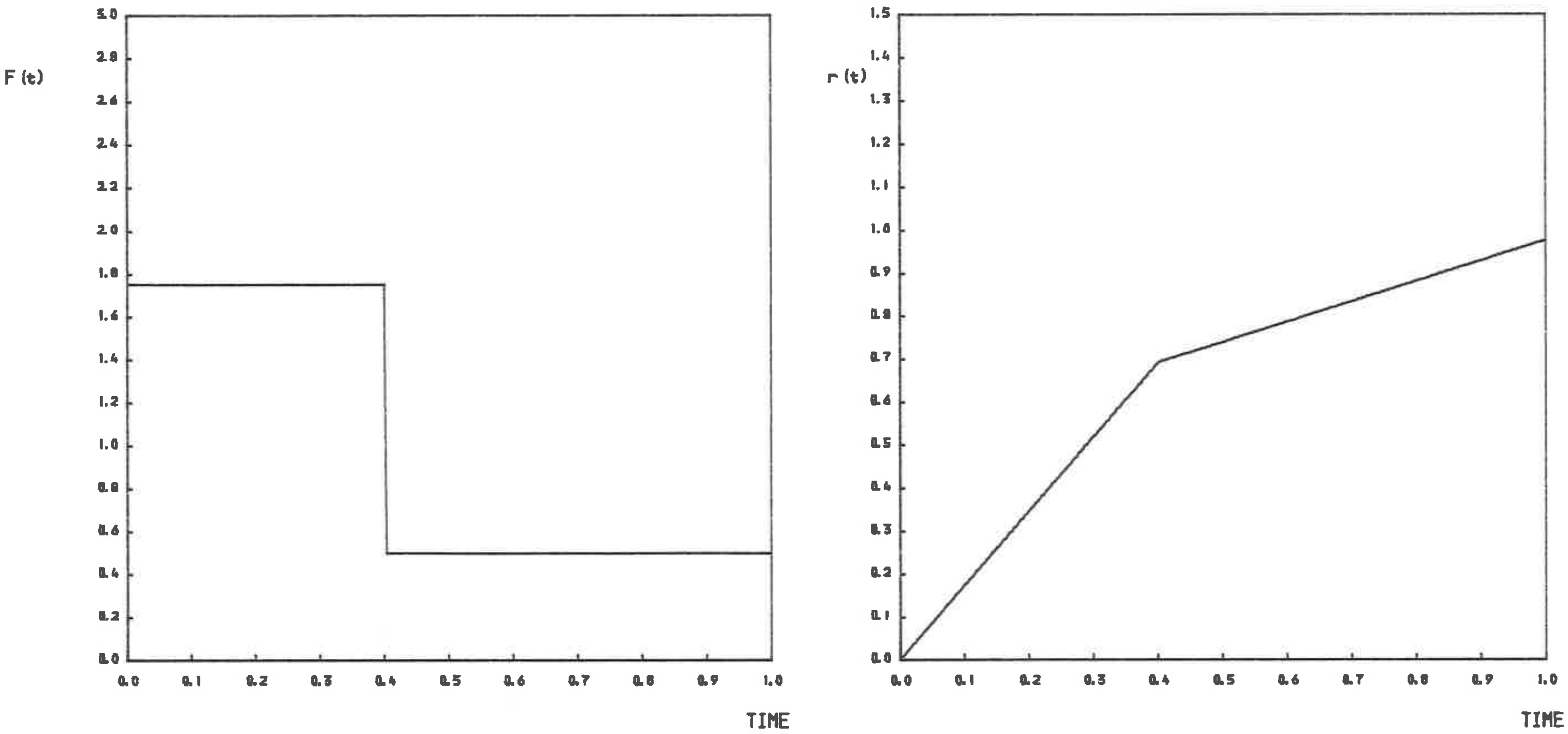
r(1)=0.977387

GRAPH 2.1



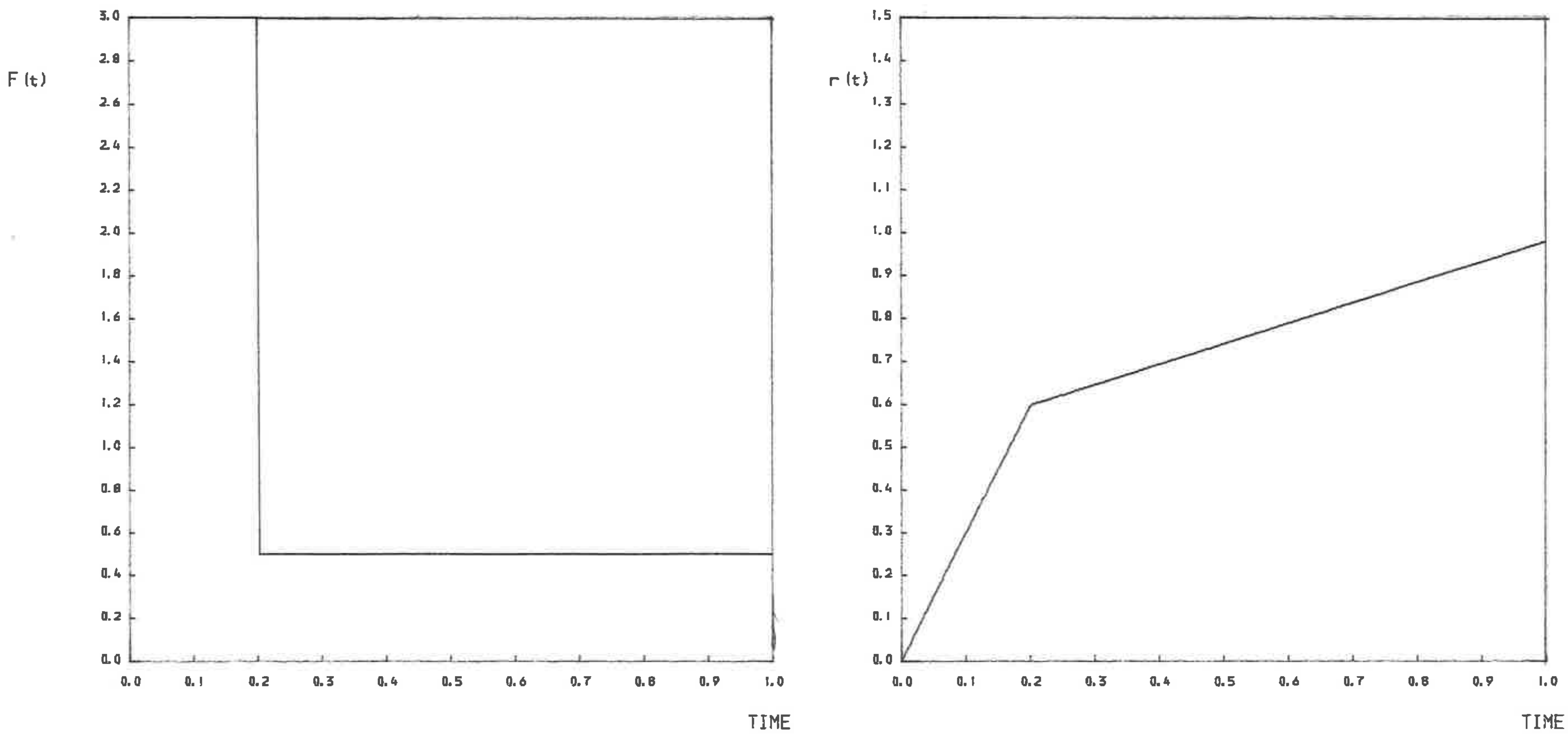
r (1) =0. 976559

GRAPH 2.2



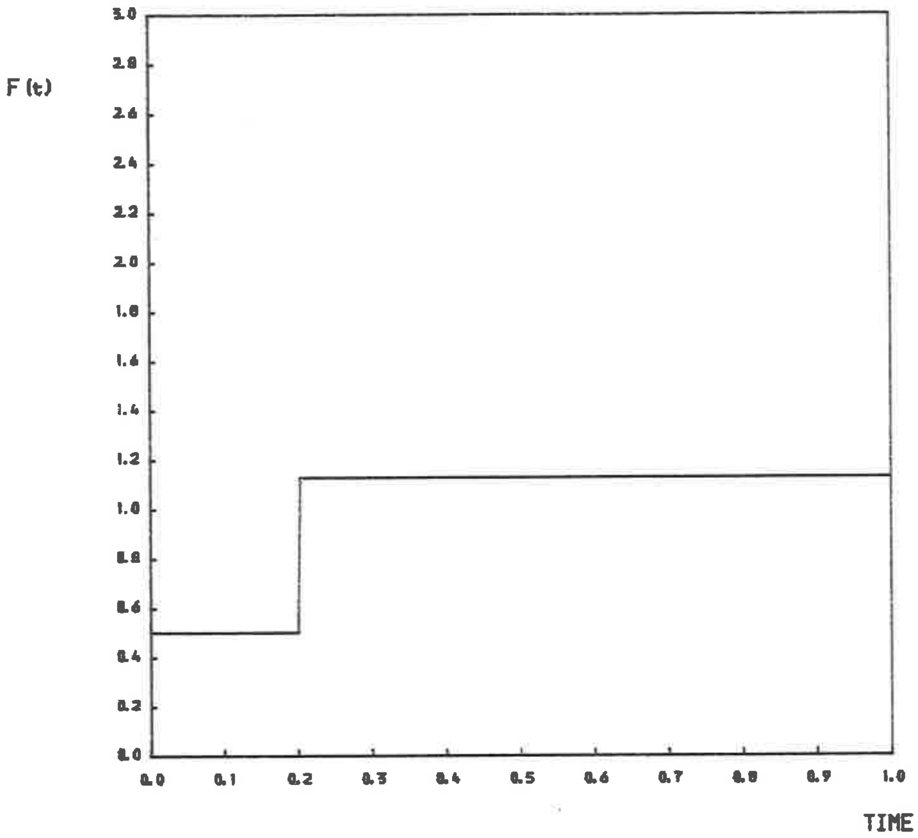
r (1) =0. 976223

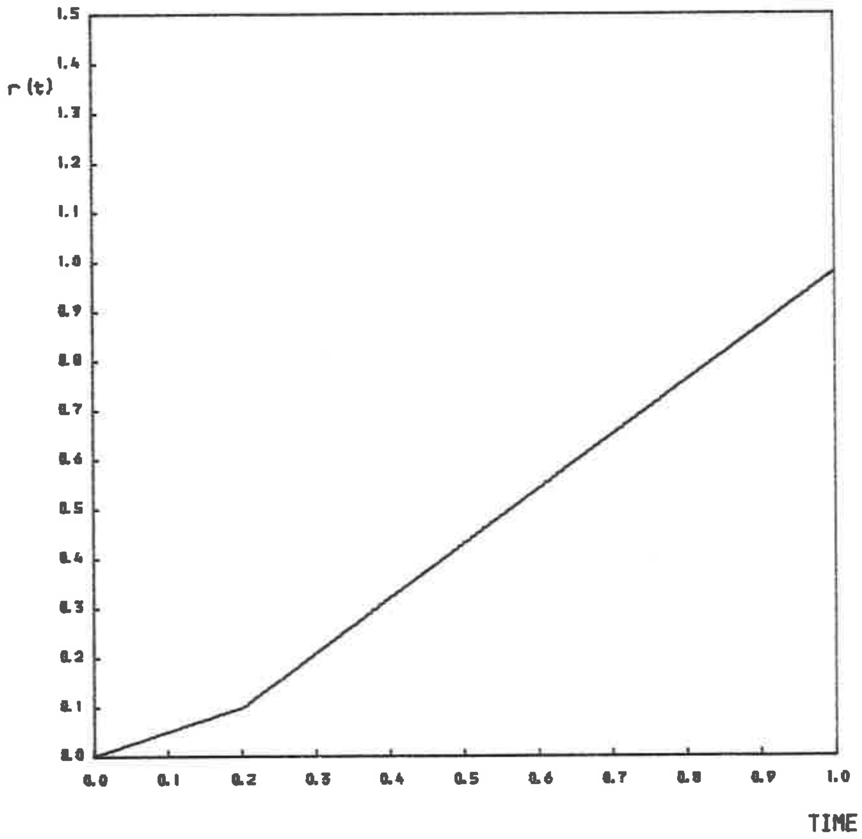
GRAPH 2. 3



r (1) =0.977332

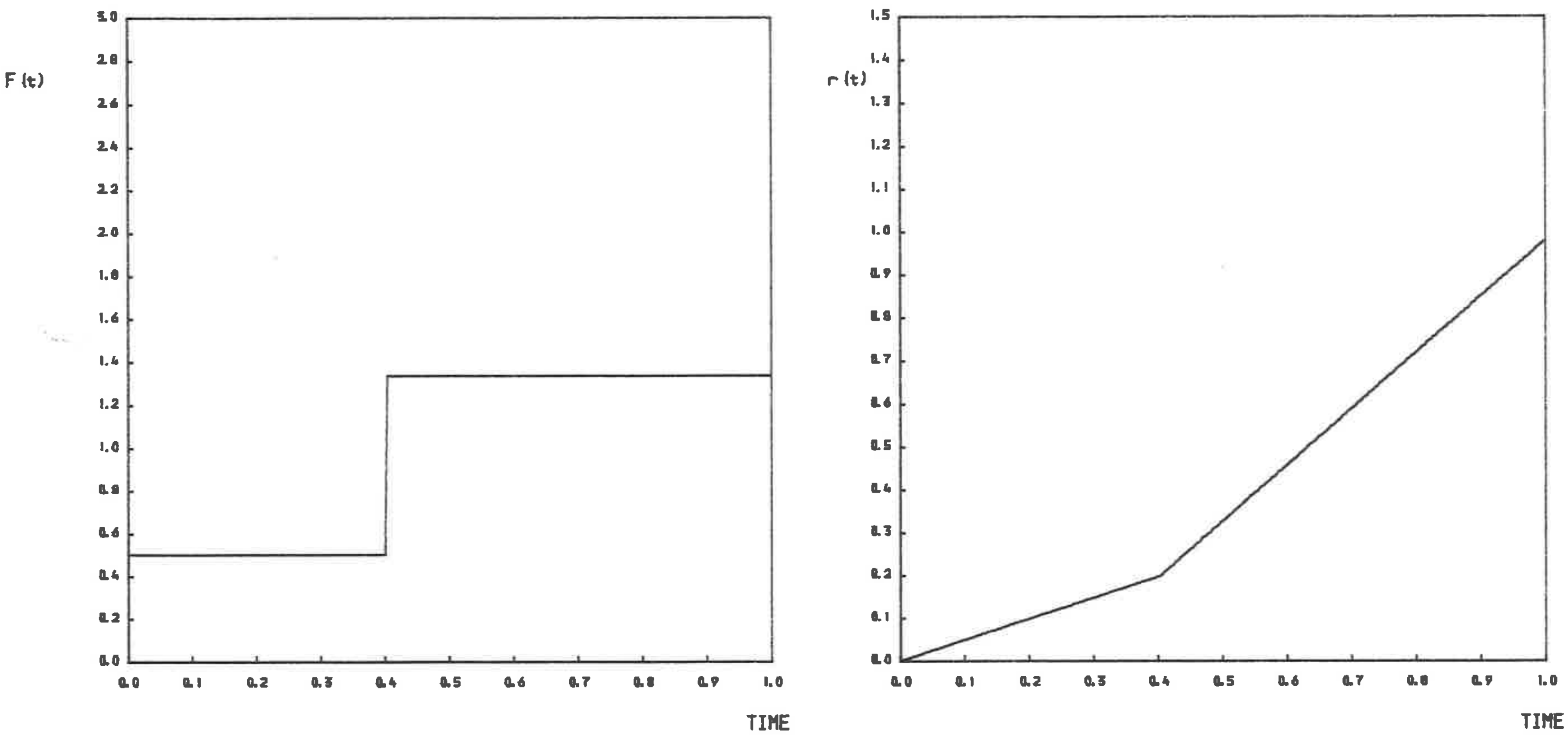
GRAPH 2.4





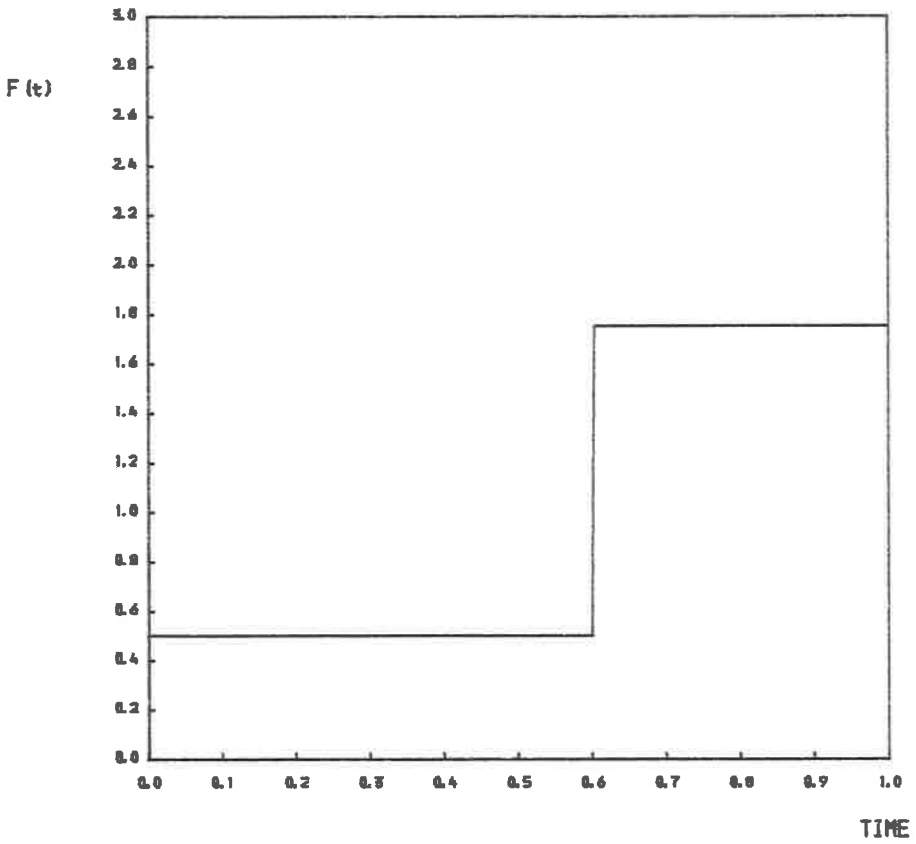
r (1) =0.978012

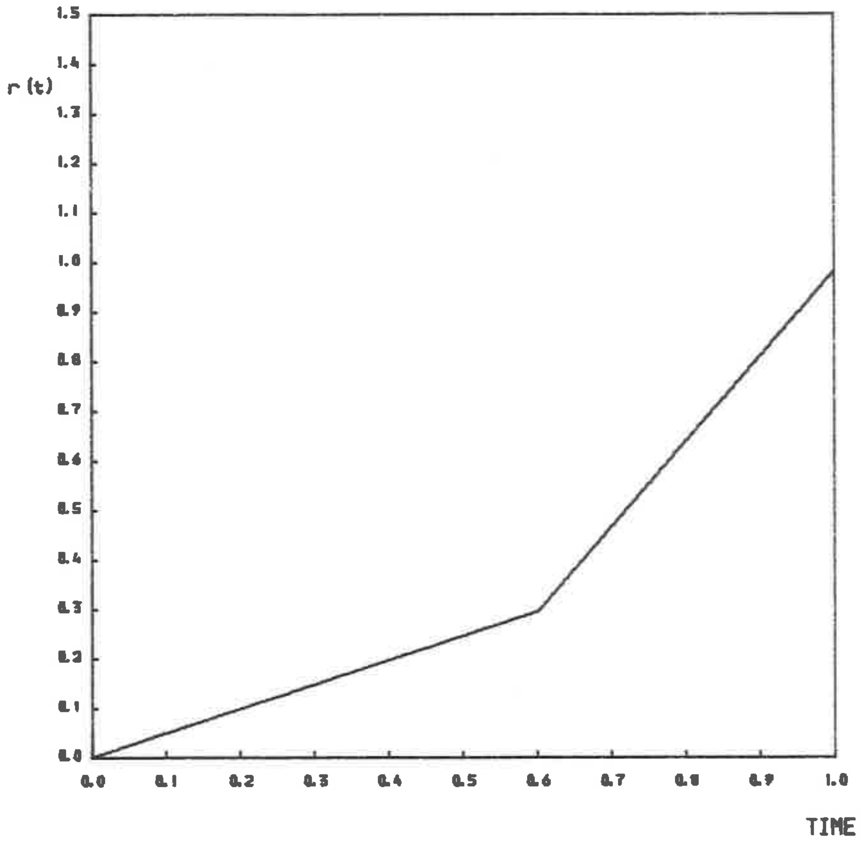
GRAPH 2.5



r (1) =0. 978977

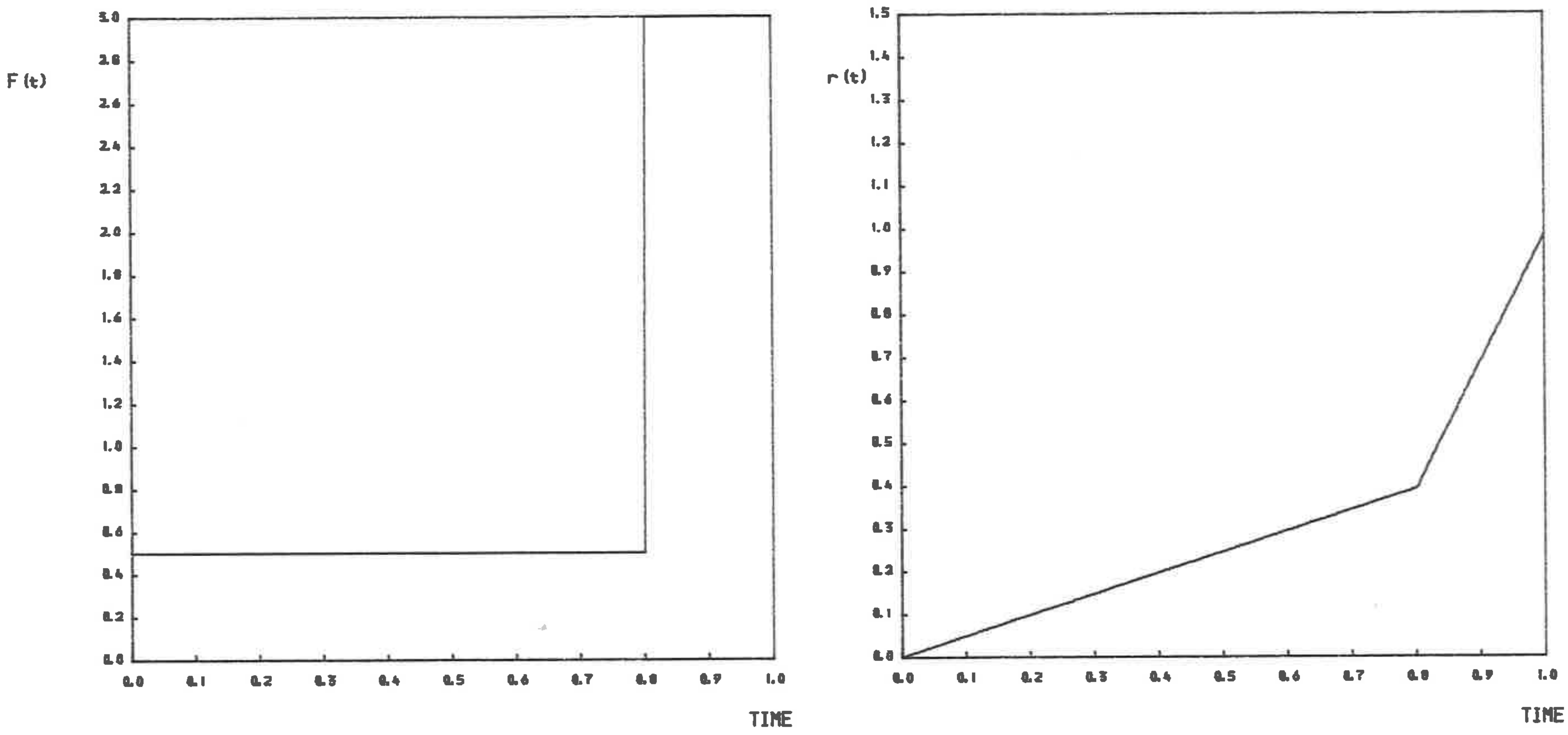
GRAPH 2. 6





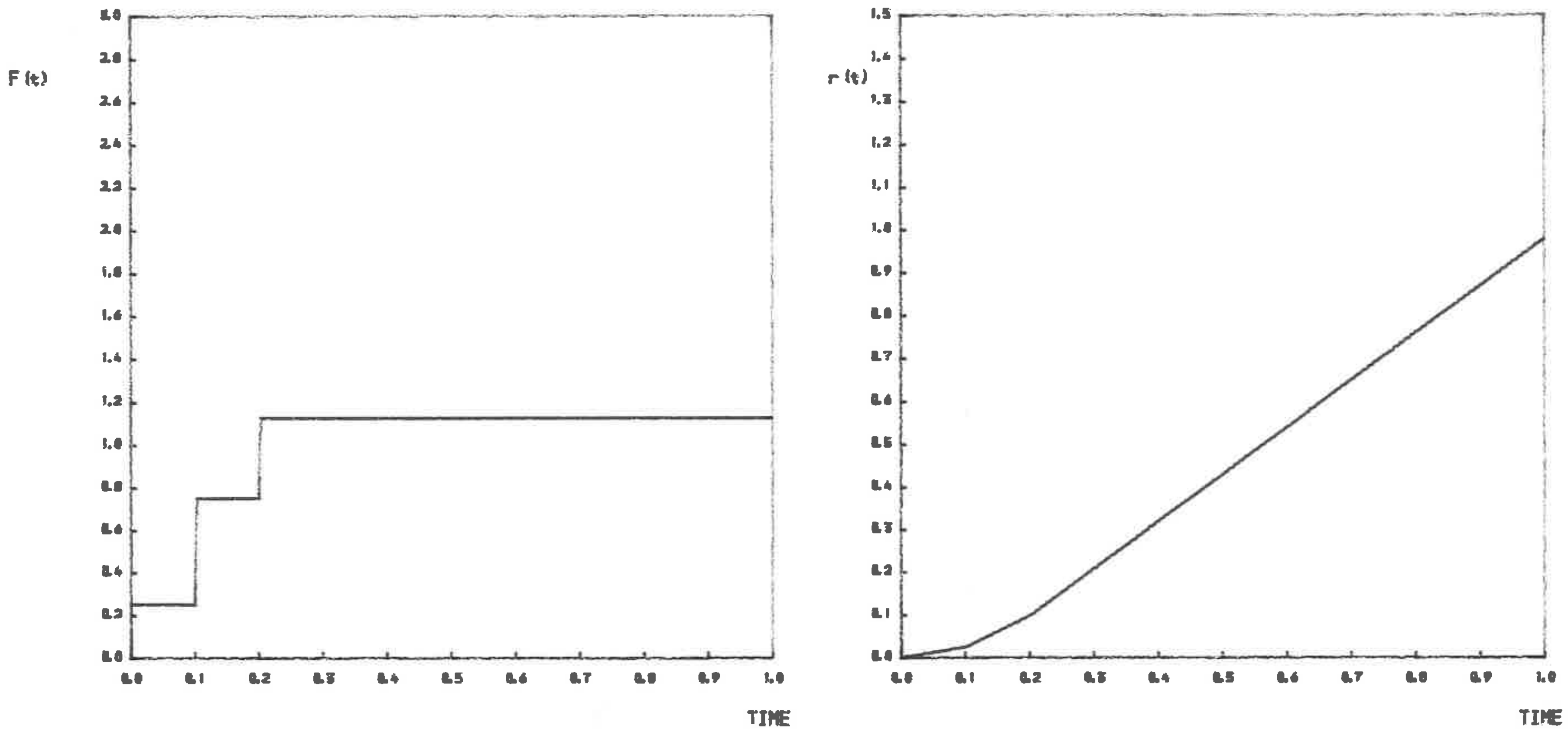
r (1) =0.979851

GRAPH 2.7



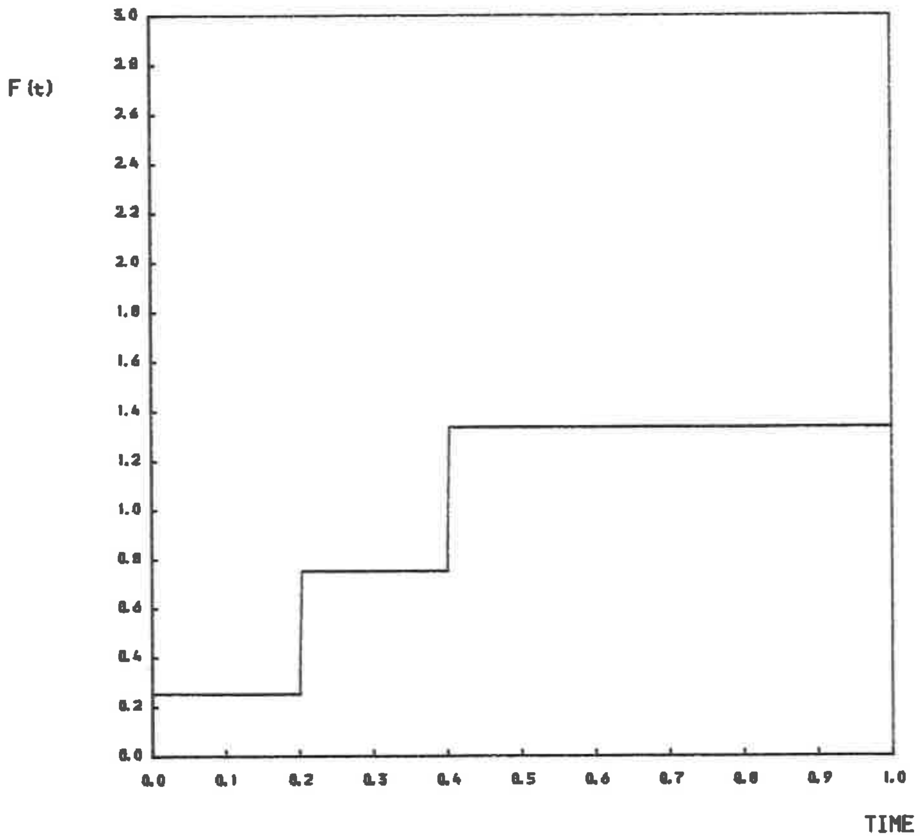
r (1) =0. 979832

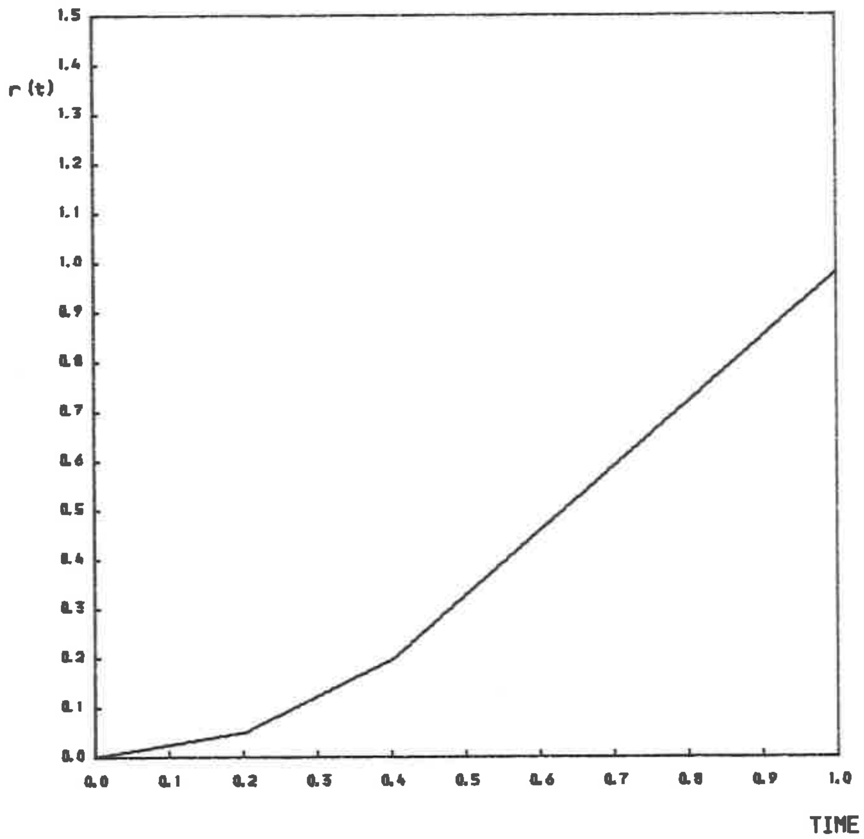
GRAPH 2. 8



r (1) =0. 977651

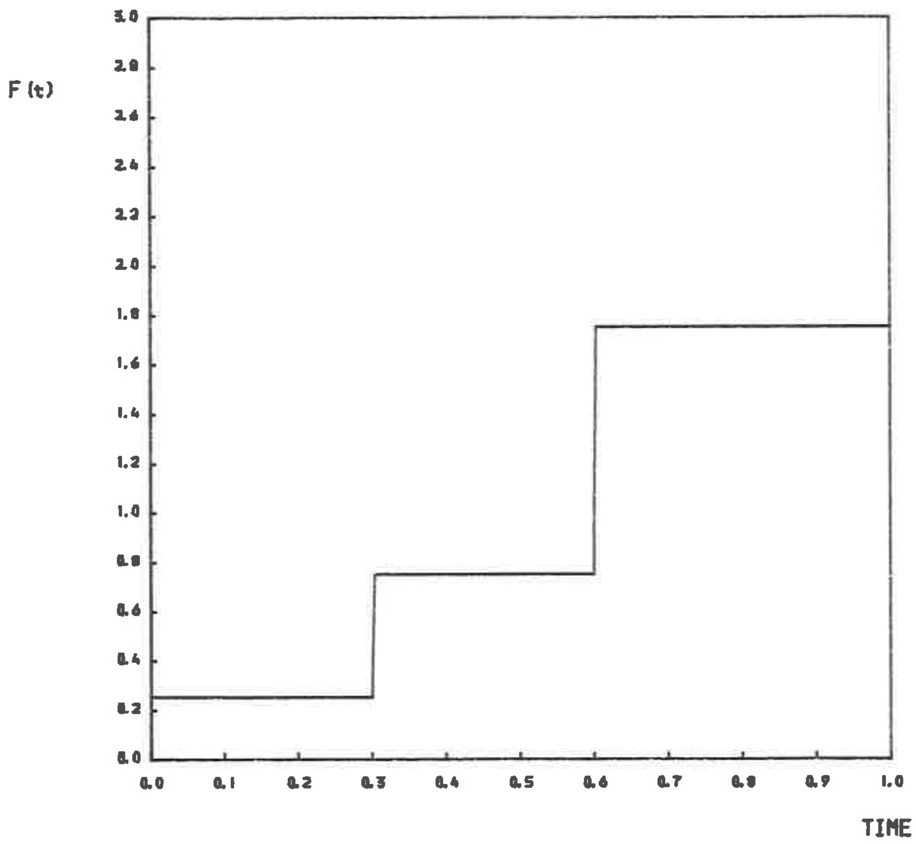
GRAPH 2.9

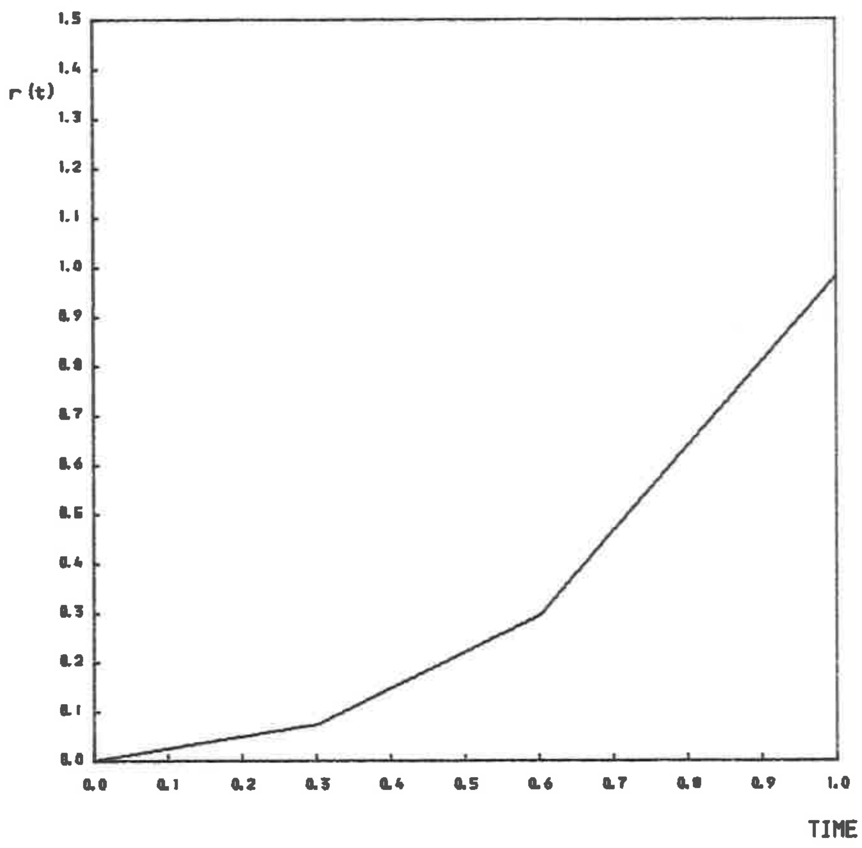




r (1) =0. 978755

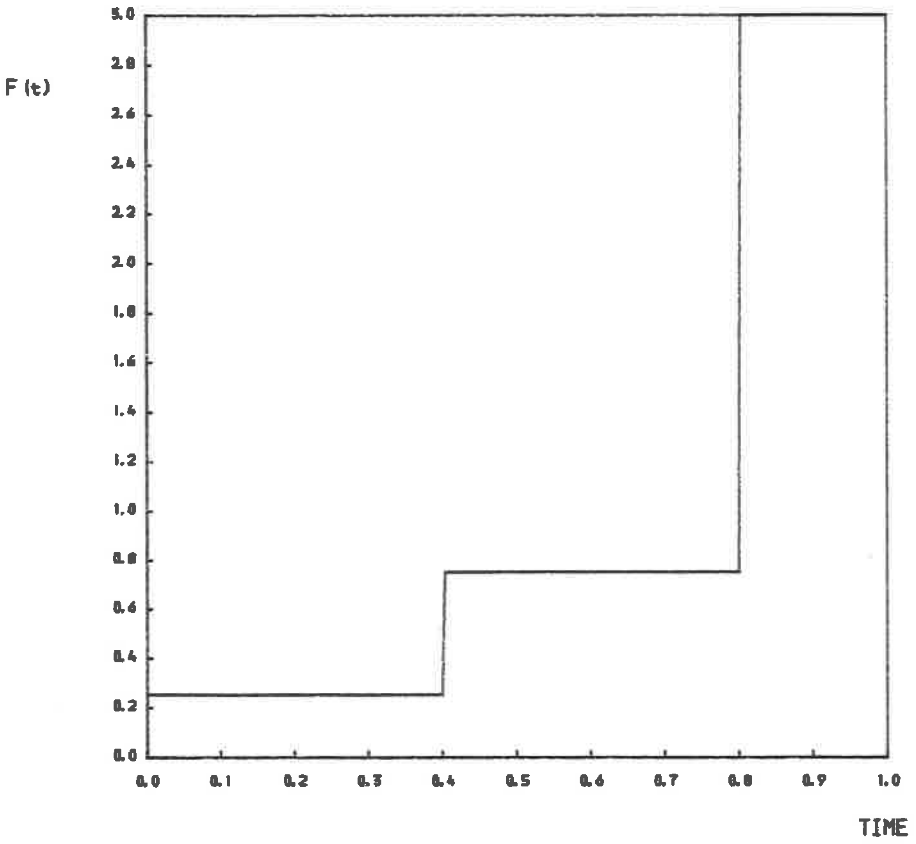
GRAPH 2. 10

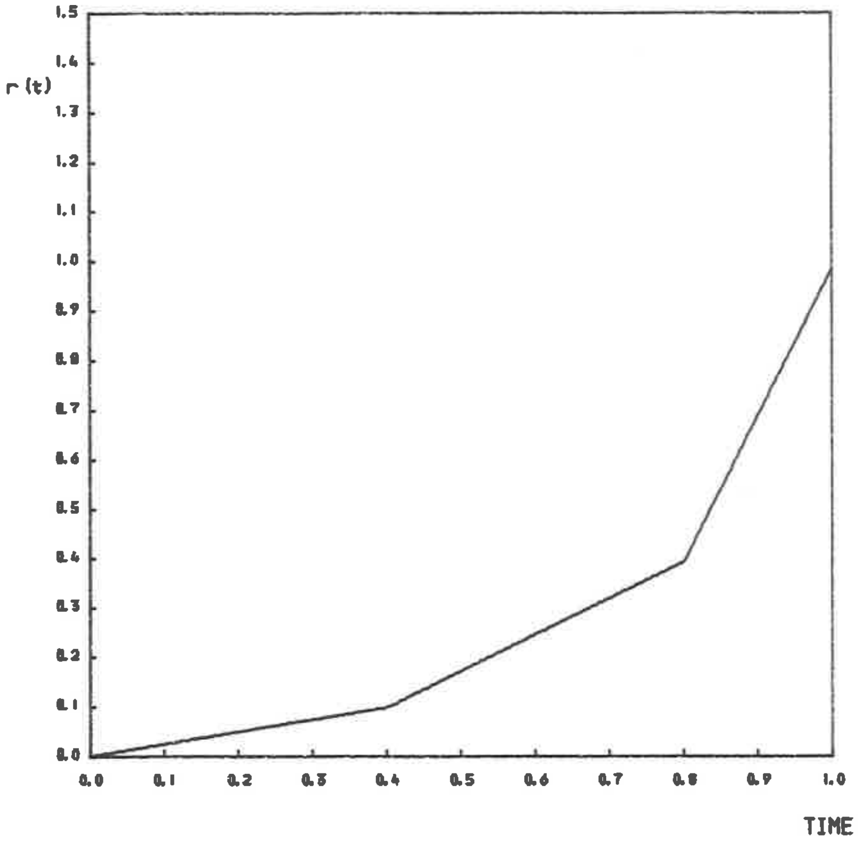




r (1) =0. 979889

GRAPH 2. 11





r (1) =0.980297

GRAPH 2.12

CHAPTER THREE

ECONOMIC LIMITS ON THE MODEL

In the previous chapter we have considered optimizing the area penetrated by the steam and paid no attention to the cost of carrying out such an operation. In practice overburden and underburden heat losses will impose a practical limit on the area which can be "swept out", or heated, from any one injection point for any given combination of heat injection rates and reservoir parameters. Continued heat injection beyond this point imposes an economic liability upon the operation.

A simple profit functional, based largely on the injection period, may take the form

$$P(t) = \begin{bmatrix} \text{value of oil} \\ \text{displaced so far} \end{bmatrix} - \begin{bmatrix} \text{cost of injecting} \\ \text{steam so far} \end{bmatrix} . \qquad (3.1)$$

The theoretical economic limit for sustained heat injection may be defined as the point reached when the net value of the oil displaced per unit time is just equal to the cost of the heat injected per unit time. This can be formulated as the point when $\frac{dP}{dt} = 0$ where P is the profit functional, such as the one expressed by (3.1).

We consider now certain aspects of the production of oil resulting from one steam injection period.

Oil Displacement Rates

Firstly it should be noted that displaced oil, as the term is used here, is not the same as produced oil. Only between 80 and 100 per cent of the displaced oil should ultimately be produced from standard well patterns.

The oil displacement rate, $V_o(t)$, can be expressed in the form

$$V_o(t) = \begin{bmatrix}\text{fraction of mobile} \\ \text{oil/unit volume}\end{bmatrix} \begin{bmatrix}\text{rate of change of volume} \\ \text{penetrated by steam}\end{bmatrix}$$

i.e.

$$V_o(t) = h\phi(S_o - S_{or}) \frac{dA}{dt} \tag{3.2}$$

where $A(t)$ is the area penetrated by the steam at time t.

The non-dimensional equation we have been studying,

i.e.

$$\frac{dy_D}{dt_D} = F(t_D) - \sqrt{\frac{\sigma}{\pi}} \int_0^{t_D} \frac{1}{\sqrt{t_D - s_D}} \frac{dy_D}{ds_D} ds_D , \tag{3.3}$$

describes the development of the steam zone. Marx and Langenheim assert that $A(t)$ is given by (3.3) whereas Mandl and Volek imply that it is $r(t)$ that is given by (3.3). This difference leads to two separate expressions for $V_o(t)$ (see (3.2)) and hence for the profit functional itself.

We know, from chapter one, the exact solution to equation (3.3) where $F(t_D) \equiv 1$, i.e.

$$y_D(t_D) = \frac{1}{\sigma}\left[e^{x^2}\mathrm{erfc}x + \frac{2x}{\sqrt{\pi}} - 1\right] \tag{3.4}$$

and

$$\frac{dy_D}{dt_D} = e^{x^2}\mathrm{erfc}x \tag{3.5}$$

where $x = \sqrt{\sigma t_D}$. We may substitute the appropriate solutions into (3.2) to obtain expressions for V_o and hence the profit functional P.

The formula (3.1) may be expressed in the form

$$P(t) = \begin{bmatrix}\text{cost of oil/}\\ \text{unit volume}\end{bmatrix}\begin{bmatrix}\text{volume of oil}\\ \text{displaced}\end{bmatrix} - \begin{bmatrix}\text{steam energy}\\ \text{cost/million Btu}\end{bmatrix}\begin{bmatrix}\text{no. of million Btu}\\ \text{of steam injected}\end{bmatrix}$$

and so for a constant rate of steam injection we have

$$P(t) = Z\int_0^t V_o(s)ds - Y\left[t.Q.h.W_{st}(0) \times 10^{-6}\right] \tag{3.6}$$

where Z = cost of oil/unit volume

and Y = steam energy cost/million Btu.

Differentiating (3.6) results in $\frac{dP}{dt}$ being expressed as

$$\frac{dP}{dt} = Z\,V_o(t) - Y\left[Q.h.W_{st}(0) \times 10^{-6}\right] . \tag{3.7}$$

Substituting values from table 1, page 42, into (3.6) and (3.7) we obtain

$$P(t) = \left[\int_0^t V_o(s)ds - 5.75\,\frac{Y}{Z}\,t\right] \tag{3.8}$$

$$\frac{dP}{dt} = Z\left[V_o(t) - 5.75\,\frac{Y}{Z}\right] . \tag{3.9}$$

We now consider the two cases arising from the two separate interpretations of equation (3.3).

Case I. $y_D = \dfrac{dA_D}{dt_D}$

Remembering that V_o is a dimensional quantity, we have

$$V_o(t) = h\phi(S_o - S_{or})V_i \frac{dA_D}{dt_D} \tag{3.10}$$

where $V_i = \frac{dA}{dt}(0)$.

From equation (1.8) putting $t = 0$ gives

$$V_i = \frac{W_{st}(0)\left[L_v + C_w T_1\right]}{\phi p_{st} L_v \bar{S}_{st} + T_1 \overline{p_1 C_1}} \tag{3.11}$$

where $W_w(t) \equiv 0$.

Substituting expressions (3.11) and (3.5) into (3.10) and using values as shown in table 1, page 42, results in

$$V_o(t) = 65\, e^{x^2} \operatorname{erfc} x \tag{3.12}$$

where $x = \sqrt{\sigma t_D} = 0.016\sqrt{t}$.

Substituting (3.12) into (3.8) and (3.9) and taking values from table 1 we obtain the following expressions for P and $\frac{dP}{dt}$,

$$P(t) = \frac{Z}{[0.016]^2}\left[e^{x^2}\operatorname{erfc} x + \frac{2x}{\sqrt{\pi}} - 1 - 0.0015\,\frac{Y}{Z}\,t\right]$$

$$\frac{dP}{dt} = Z\left[e^{x^2}\operatorname{erfc} x - 5.75\,\frac{Y}{Z}\right] .$$

Case II $\quad y_D \equiv \frac{dr_D}{dt_D}$

We know that $A(t) = \pi[r(t)]^2$ and so we obtain from (3.2)

$$V_o(t) = 2\pi h\phi(S_o - S_{or})r(t)\frac{dr}{dt} .$$

Again we have to remember that V_o is a dimensional quantity and thus can be expressed in the form

$$V_o(t) = 2\pi h\phi(S_o - S_{or})LV_i\, r_D(t_D)\frac{dr_D}{dt_D} \qquad (3.13)$$

As in case I we are able to use the exact solution of equation (3.3) for the special case of $F(t_D) \equiv 1$ and therefore obtain, for the data shown in table 1, the following expression for V_o,

$$V_o(t) = 40840705\left[e^{x^2}\text{erfc}x + \frac{2x}{\sqrt{\pi}} - 1\right]e^{x^2}\text{erfc}x \qquad (3.14)$$

where $\quad x = 0.016\sqrt{t}$.

We can integrate the above expression for V_o and this results in

$$\int_0^t V_o(s)ds = 8.0 \times 10^{10}\left[e^{x^2}\text{erfc}x + \frac{2x}{\sqrt{\pi}} - 1\right]^2 . \qquad (3.15)$$

Substituting (3.14) and (3.15) into (3.9) and (3.8) respectively results in the following expressions for P and $\frac{dP}{dt}$,

$$P(t) = 8.0 \times 10^{10}Z\left\{\left[e^{x^2}\text{erfc}x + \frac{2x}{\sqrt{\pi}} - 1\right]^2 - 7.2 \times 10^{-10}\frac{Y}{Z}t\right\}$$

$$\frac{dP}{dt} = 40940705Z\left\{\left[e^{x^2}\text{erfc}x + \frac{2x}{\sqrt{\pi}} - 1\right]e^{x^2}\text{erfc}x - 1.4 \times 10^{-7}\frac{Y}{Z}\right\} .$$

We now have two separate expressions, arising from the two cases, for the profit functional P , i.e.,

Case I

$$P(t) = \frac{Z}{[0.016]^2}\left[e^{x^2}\mathrm{erfc}x + \frac{2x}{\sqrt{\pi}} - 1 - 0.0015\,\frac{Y}{Z}\,t\right] . \qquad (3.16)$$

Case II

$$P(t) = 8.0 \times 10^{10} Z\left\{\left[e^{x^2}\mathrm{erfc}x + \frac{2x}{\sqrt{\pi}} - 1\right]^2 - 7.2 \times 10^{-10}\,\frac{Y}{Z}\,t\right\} . \qquad (3.17)$$

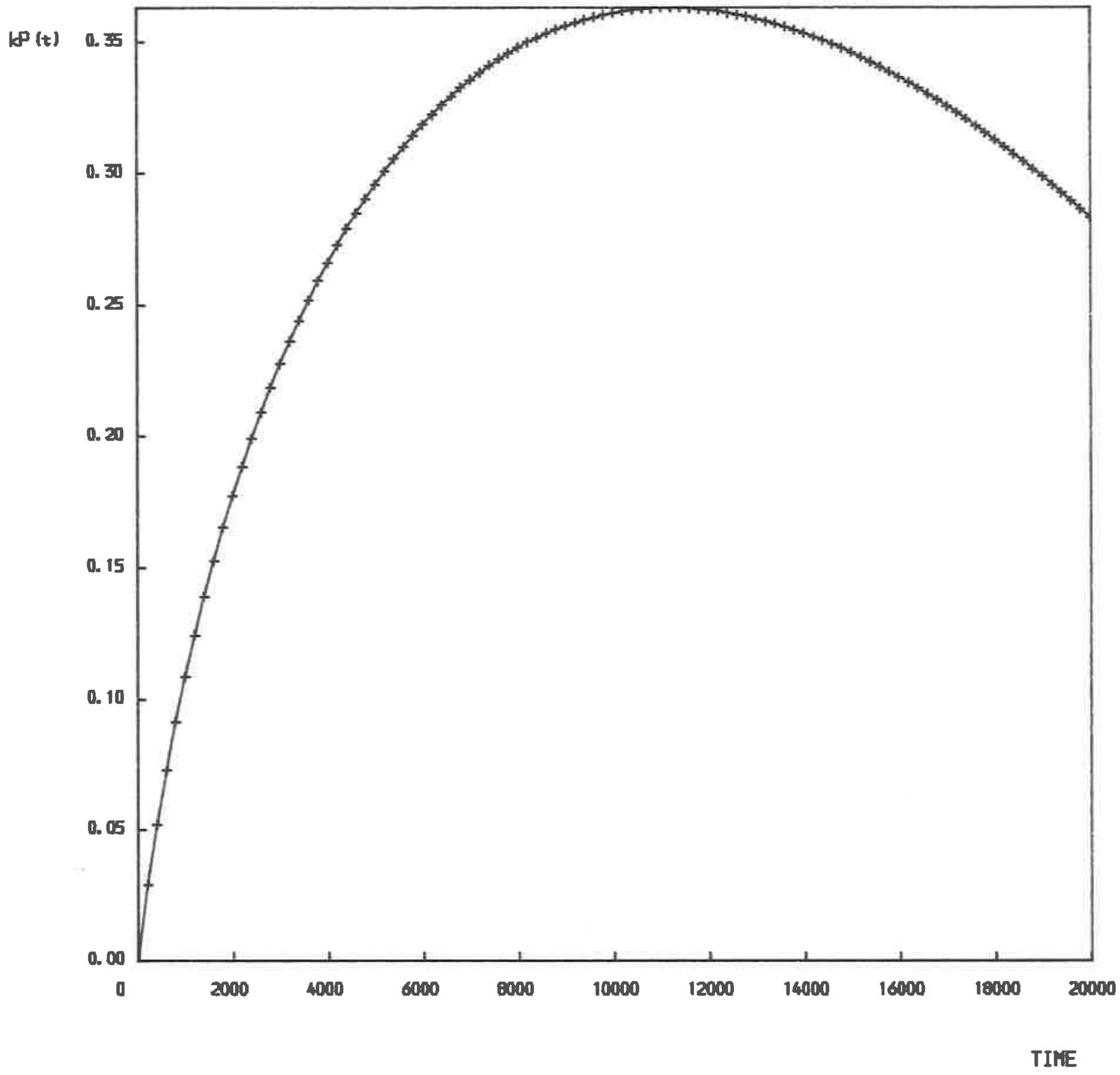
We are interested in finding where (3.16) and (3.17) reach their maximum. However, first we make the following observations.

Comments

1. In order that P given by (3.16) should have a maximum the condition $\frac{Y}{Z} < 0.17$ must be satisfied as this forces $\frac{dP}{dt}(0) > 0$ which ensures that the one extremum of P is a maximum and not a minimum.
2. For $\frac{Y}{Z}$ of O(1) it appears, from studying (3.17), that for case II the functional P does not have a maximum and that $\frac{dP}{dt}(0) \approx 0$ corresponds to P having its minimum at $t = 0$, i.e. $P(0) = 0$.

As a consequence of this second comment, only the profit functional corresponding to case I was plotted to investigate the effects of changing the ratio $\frac{Y}{Z}$. This can be seen in graphs 3.1, 3.2 and 3.3.

By comparing these graphs it appears that as the ratio $\frac{Y}{Z}$ decreases the time at which the profit reaches its maximum increases. As would be expected, the value taken by the profit functional at its maximum also increases as $\frac{Y}{Z}$ decreases.



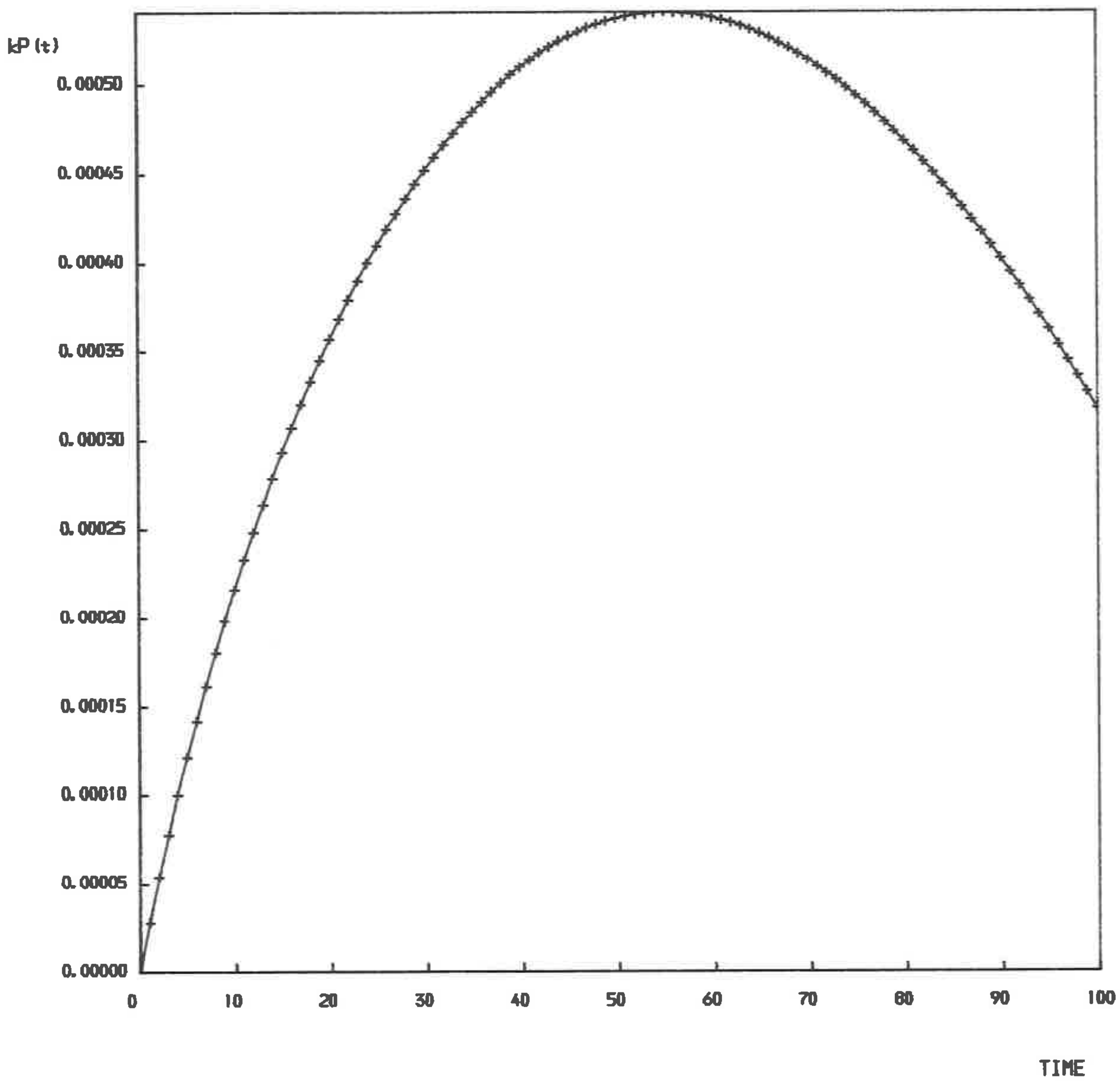
Y/Z=0.05

TIME OF MAXIMUM ~11760

VALUE OF MAXIMUM ~0.36 k

k=3906.25 Z

GRAPH 3.1



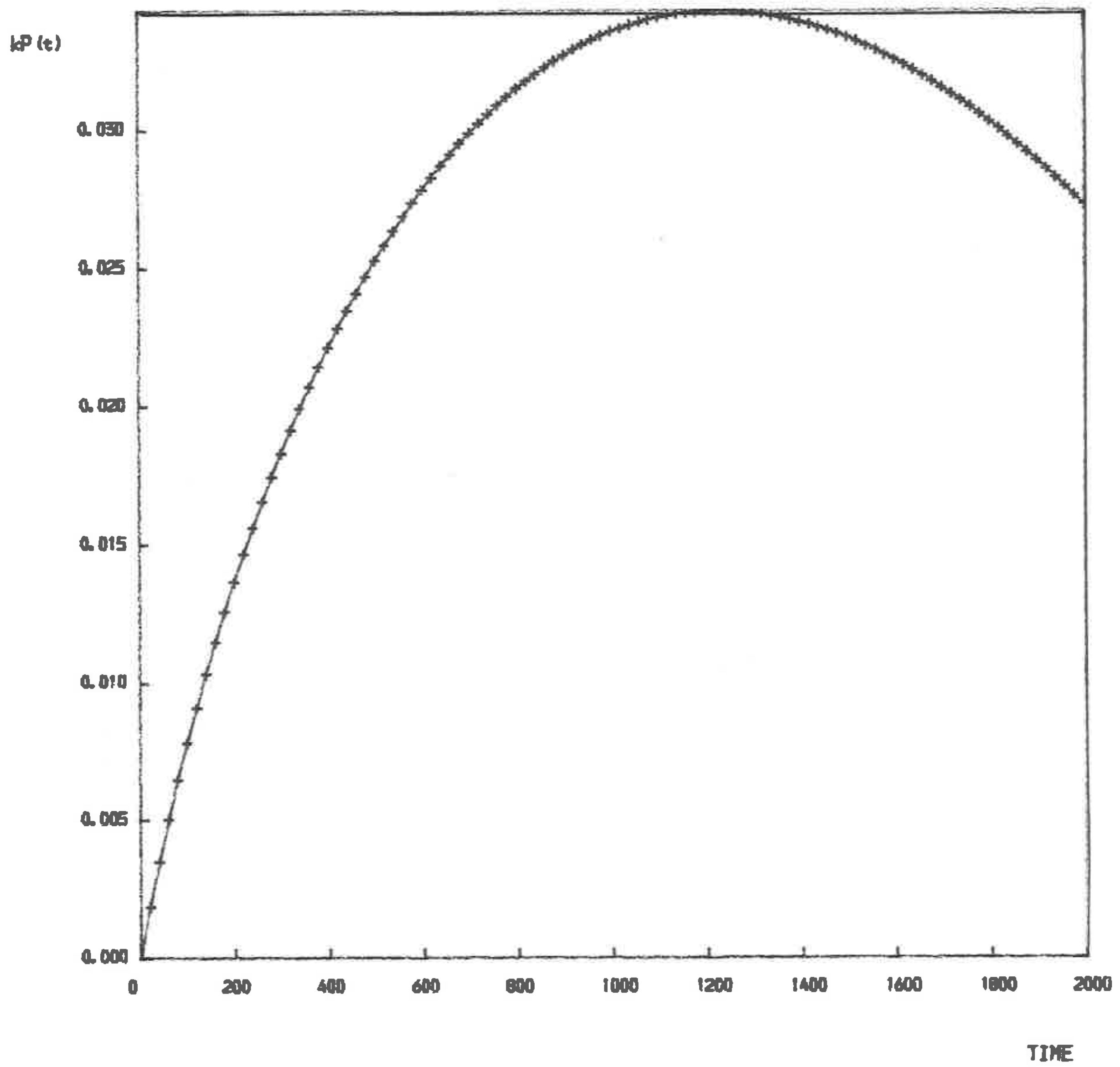
GRAPH 3.3

Y/Z=0.15

TIME OF MAXIMUM ~74

VALUE OF MAXIMUM ~0.005 k

k=3906.25 Z



Y/Z=0.1

TIME OF MAXIMUM ~1344

VALUE OF MAXIMUM ~0.034 k

k=3906.25 Z

GRAPH 3.2

CONCLUSION

In chapter one a very simple model has been developed for the steam injection process around which, in chapter two, two separate, but related, control problems have been applied.

These control problems are

1. Fix the total amount of steam to be injected and maximize the area penetrated by the steam.

2. Fix the area to be penetrated by the steam and minimize the total amount of steam injected.

The results obtained indicate that we should use the hypothesis of "injecting as much steam as late as possible". Numerical experiments also suggest that this is the correct approach to take. It was found to be much easier to satisfy all the conditions in the first control problem rather than the second.

Finally in chapter three we looked at a very simple profit functional based largely on the injection period. The results obtained emphasize how simple the model is. Taking two different interpretations of the quantity $r(t)$, a radius or area, resulted in two totally different shapes of profit functional. The first, as expected, reached a maximum but the second was found to be unrealistic, being monotonically increasing throughout.

A great deal of further work needs to be carried out in this area. A soak and production period has to be attached to the injection period to form one complete cycle. Then the problem of optimisation where two or more of these cycles are linked together, and how previous cycles affect the present one, has to be investigated.

SAMPLE DATA (see [1], [2])

Reservoir porosity	ϕ	=	0.25
Oil saturation	$\bar{S}_{01}$	=	0.60
Residual oil saturation	S_{or}	=	0.10
Steam saturation	$\bar{S}_{st}$	=	0.20
Water saturation	$\bar{S}_{w1}$	=	0.20
Relative steam temperature	T_1	=	390°F
Initial reservoir temperature	T_{RI}	=	80°F
Steam injection rate	$W_{st}(0)$	=	250 lb/hr - ft^2
Reservoir thickness	h	=	20 ft
Length of reservoir	L	=	90 ft

Physical Constants

Specific heat, rock	C_s	=	0.21 Btu/lb - °F
Specific heat, water	C_w	=	1.0 Btu/lb - °F
Specific heat, oil	C_0	=	0.5 Btu/lb - °F
Specific heat, cap rock	C_f	=	0.20 Btu/lb - °F
Density, rock	p_s	=	167 lb/ft^3
Density, water	p_w	=	62.4 lb/ft^3
Density, oil	p_0	=	50 lb/ft^3
Density, steam	p_{st}	=	0.006 lb/ft^3
Density, cap rock	p_f	=	137 lb/ft^3
Thermal conductivity, cap rock	K_{hf}	=	1.00 Btu/lb - hr°F
Latent heat of steam	L_v	=	908.8 Btu/lb
Available heat of steam at 470°F, 500 p.s.i.g.	Q	=	1,150 Btu/lb

TABLE 1

Glossary Of Symbols

C_f	specific heat of cap and base rock
C_0	specific heat of oil
C_s	specific heat of rock
C_w	specific heat of water
H_1	heat content per unit volume of steam zone
H_2	heat content per unit volume of liquid zone
K_{hf}	thermal conductivity in cap and base rock
L	radius of reservoir
L_v	latent heat per unit mass of steam
$\dot{Q}_{st}$	rate of heat loss from steam zone to cap and base rock
r	radius of steam zone
S_{01}	saturation of oil in steam zone
S_{02}	saturation of oil in liquid zone
S_{st}	saturation of steam
S_{w1}	saturation of water in steam zone
S_{w2}	saturation of water in liquid zone
T_{R1}	initial reservoir temperature
T_1	temperature of steam zone relative to initial reservoir temperature
T_2	Temperature of liquid zone relative to initial reservoir temperature
t	time
t_f	final time of injection period
U_{h1}	heat flux in steam zone
U_{h2}	heat flux in liquid zone
V	rate of growth of steam zone radius
V_i	initial rate of growth of steam zone radius

W_{st}	mass rate of steam injection through unit cross-section
W_w	mass rate of hot water injection through unit cross-section
ϕ	porosity
p_f	density of cap and base rock
p_O	density of oil
p_s	density of rock
p_{st}	density of steam
p_w	density of water

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