REGULARITY AND ROBUST POLE ASSIGNMENT IN SINGULAR CONTROL SYSTEMS

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ABSTRACT

Necessary conditions are given for the problem of pole assignment by state feedback in singular linear systems (descriptor systems) to have a solution which is regular and non-defective. For a robust solution, such that the assigned closed-loop poles are insensitive to perturbations in the system data, the same conditions must hold. It can be shown that these conditions are also sufficient for the existence of a feedback which assigns the maximum possible number of finite poles with regularity. These results provide the basis of a procedure for constructing closed-loop semi-state systems with given poles, guaranteed regularity and maximum robustness.

Keywords: automatic control, generalized state-space, semi-state, singular linear multi-variable systems, descriptor systems, pole assignment, state feedback, inverse generalised eigenvalue problem.

1. INTRODUCTION

In singular, or degenerate, time-invariant multi-input linear control systems (desciptor systems), pole assignment by feedback requires not only that the closed loop system have prescribed poles, but also that it is regular, and that it is robust, in the sense that its assigned poles are as insensitive as possible to perturbations in the system data. In this paper we give a detailed derivation of results which we have previously reported [6] on conditions for the pole assignment problem to have a regular, non-defective solution. For the existence of a robust solution the same conditions must hold. These results form the basis of numerical procedures for generating robust feedback systems with prescribed poles (also reported in [6]). The procedures are extensions of earlier techniques which we have developed for robust pole assignment in non-degenerate systems [5],[8] and are described in detail in [7].

Here we begin by examining open-loop singular systems, and in §2 we derive conditions for the regularity of a non-degenerate matrix pencil. In §3 we apply these results to closed loop systems and develop the conditions required for pole assignment with regularity. The theoretical development is completed in §4, where conditions are derived for non-defective eigenstructure assignment with regularity. The consequence of these results for the computation of a robust solution to the state feedback pole assignment problem is also discussed.

2. OPEN-LOOP REGULARITY

We first consider systems described by the dynamic equations

$$E\mathcal{D}\underline{x} = A\underline{x} \tag{2.1}$$

where $E, A \in \mathbb{R}^{n\times n}$ and rank $(E) = q \leq n$. Here $\mathcal{D}$ denotes the differential operator d/dt for continuous systems, or the delay operator for discrete systems. We are specifically interested in the singular, or degenerate, case where $q < n$. The behaviour of system (2.1) is governed by the poles, or generalised eigenvalues, of the matrix pencil $A - \lambda E$, denoted by (A,E). Solutions to the equations (2.1) which satisfy given initial conditions are unique provided the pencil (A,E) is regular, that is

$$\det (A - \lambda E) \not\equiv 0, \tag{2.2}$$

(regarded as a polynomial in λ). It is well-known [13] that a regular pencil has at most q finite eigenvalues and that the number of finite eigenvalues is given precisely by $r = \deg \det (A - \lambda E)$. Furthermore, the pencil (E,A) then has precisely $n-r$ zero eigenvalues. This is shown in the following Lemma.

Lemma 1 Assume (A,E) regular. Then (E,A) has precisely $n-r$ zero eigenvalues, where $r = \deg \det (A - \lambda E)$.

Proof: We let $p(\lambda) = \det (A - \lambda E)$ and $\hat{p}(\lambda) = \det (E - \lambda A)$. Then, since

$$\det (A - \lambda E) = \det (-\lambda(E - \frac{1}{\lambda} A)) = (-\lambda)^n \det (E - \frac{1}{\lambda} A) \quad ,$$

we have $p(\lambda) = (-\lambda)^n \hat{p}\left(\frac{1}{\lambda}\right)$. Moreover, (E,A) has precisely $n-r$ zero eigenvalues if and only if $\hat{p}(\lambda) = \lambda^{n-r} t(\lambda)$ where $t(0) \neq 0$. It follows that $p(\lambda) = (-\lambda)^r t\left(\frac{1}{\lambda}\right)$ and $p(\lambda)$ is of exact degree r. □

The eigenvectors of the pencil (E,A) associated with the zero eigenvalues must belong to the null space $\mathcal{N}\{E\}$ which has dimension q. Thus it follows from Lemma 1 that the regular pencil (A,E) has q finite eigenvalues if and only if the zero eigenvalues of (E,A) are non-defective. We have thus shown

Lemma 2 The pencil (A,E) is regular and has $q \equiv \text{rank}(E)$ finite eigenvalues if and only if

$$\underline{v}^T E = 0 \text{ and } \underline{v}^T A = \underline{z}^T E \text{ for any } \underline{z} \in \mathbb{C}^n \Rightarrow \underline{v} = 0, \tag{2.3}$$

or, equivalently,

$$E\underline{v} = 0 \text{ and } A\underline{v} = E\underline{z} \text{ for any } \underline{z} \in \mathbb{C}^n \Rightarrow \underline{v} = 0. \tag{2.4}$$

□

We now derive a stronger condition which guarantees that if the pencil (A,E) has q non-defective finite eigenvalues, then it is regular. We write

$$E = [R_E, 0]\ [S_E, S_\infty]^T \tag{2.5}$$

where $R_E \in \mathbb{R}^{n\times q}$, R_E is of full rank, and the matrix $[S_E, S_\infty]$ is orthogonal. Then the columns of S_∞ and S_E give orthonormal bases for $\mathcal{N}\{E\}$ and $\mathcal{R}\{E^T\}$, respectively, where $\mathcal{N}\{\cdot\}$ denotes null space and $\mathcal{R}\{\cdot\}$ denotes range. We use the following simple Lemma.

Lemma 3 If $X_q \in \mathbb{C}^{n\times q}$, where $q \equiv \text{rank}(E)$, then the conditions

(i) $\text{rank}(X_q) = q$, $\text{rank}(EX_q) = q$

(ii) $\text{rank}([X_q, S_\infty]) = n$

(iii) $\text{rank}(S_E^T X_q) = q$

are all equivalent.

Proof: Since the matrix $[S_E, S_\infty]$ is orthogonal we may write

$$[X_q, S_\infty] = [S_E, S_\infty] \begin{bmatrix} S_E^T X_q & 0 \\ S_\infty^T X_q & I \end{bmatrix}, \tag{2.6}$$

and it follows that $X_E \equiv S_E^T X_q$ is non-singular $\Leftrightarrow [X_q, S_\infty]$ is non-singular $\Leftrightarrow R_E X_E \equiv EX_q$ and X_q have full rank. □

We next give a <u>necessary</u> condition for a regular pencil to have precisely $q \equiv \text{rank}(E)$ non-defective finite eigenvalues (multiple or simple).

<u>Lemma 4</u> If the pencil (A,E) is regular and there exists $X_q \in \mathbb{C}^{n\times q}$ with $\text{rank}(X_q) = q \equiv \text{rank}(E)$ such that

$$AX_q = EX_q\Lambda_q, \qquad \Lambda_q = \{\lambda_1,\lambda_2,..\lambda_q\} \tag{2.7}$$

where $\lambda_j \in \mathbb{C}$ $\forall_j$, then $\text{rank}(EX_q) = q$, or, equivalently, $\text{rank}([X_q,S_\infty]) = n$.

<u>Proof:</u> The first part follows by contradiction. If $\text{rank}(X_q) = q$ and $\text{rank}(EX_q) < q$, then there exists $\underline{w} \neq 0$ such that $\underline{v} = X_q\underline{w} \neq 0$ and $E\underline{v} = 0$. Then for $\underline{z} = X_q\Lambda_q\underline{w}$ we have

$$A\underline{v} = AX_q\underline{w} = EX_q\Lambda_q\underline{w} = E\underline{z} \tag{2.8}$$

and the condition of Lemma 2 is violated. The last part follows from Lemma 3. □

This lemma implies that if the regular pencil (A,E) has q independent eigenvectors, then these eigenvectors must remain independent under the application of E, or equivalently, no linear combination of them lies in the null space of E. This lemma also gives, therefore, a necessary condition for a regular pencil to have $q \equiv \text{rank}(E)$ <u>distinct</u> (simple) finite eigenvalues.

We now give the main result of this section.

<u>Theorem 1</u> If there exists $X_q \in \mathbb{C}^{n\times q}$ such that $[X_q,S_\infty]$ is non-singular and X_q satisfies (2.7), then the pencil (A,E) is regular <u>if and only if</u>

$$\text{rank}(E + AS_\infty S_\infty^T) = n. \tag{2.9}$$

<u>Proof:</u> From Lemma 3, we find that $\text{rank}([X_q,S_\infty]) = n$ implies $X_E \equiv S_E^T X_q$ is invertible, and then from (2.7) we obtain

$$A - \lambda E = (E + AS_\infty S_\infty^T)[S_E,S_\infty]\begin{bmatrix} X_E\Lambda_q X_E^{-1} - \lambda I & 0 \\ -S_\infty^T X_q X_E^{-1} & I \end{bmatrix}[S_E,S_\infty]^T. \tag{2.10}$$

To demonstrate this result we observe that

$$(A - \lambda E)[S_E, S_\infty] = [(A - \lambda E)S_E,\ AS_\infty], \tag{2.11}$$

$$(E + AS_\infty S_\infty^T)[S_E, S_\infty] = [ES_E, AS_\infty]\ , \tag{2.12}$$

and, since $S_E S_E^T = I - S_\infty S_\infty^T$ and $ES_E S_E^T = E$, we have

$$ES_E(X_E \Lambda_q X_E^{-1} - \lambda I) - AS_\infty S_\infty^T X_q X_E^{-1} =$$

$$(EX_q \Lambda_q - AX_q)X_E^{-1} + (A - \lambda E)S_E\ . \tag{2.13}$$

The equation (2.10) then follows directly from (2.7), (2.11) and (2.12). It is clear from (2.10) that rank $(A - \lambda E) = n$ for $\lambda \neq \lambda_j$ if and only if (2.9) holds and the theorem is proved. □

In the next section we apply Theorem 1 to obtain conditions for the existence of regular solutions to the problem of pole assignment in singular systems.

3. POLE ASSIGNMENT IN SINGULAR SYSTEMS

We now consider singular control systems governed by the open loop equations

$$E\mathcal{D}\underline{x} = A\underline{x} + B\underline{u} \tag{3.1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, rank $(E) = q < n$ and rank $(B) = m$. (Here $\mathcal{D}$ again denotes either the continuous differential or the discrete delay operator). The poles, or generalised eigenvalues of the pencil (A,E) govern the behaviour of the system and may be modified by state feedback. The pole assignment problem is specified as follows.

Problem 1 Given real matrices E, A, B, where $E, A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, rank (E) = q < n, and rank (B) = m, and an arbitrary set of q self-conjugate complex numbers $L = \{\lambda_1, \lambda_2, .. \lambda_q\}$, find $F \in \mathbb{R}^{mxn}$ such that

$$\det (A + BF - \lambda E) = 0, \qquad \forall \lambda \in L \quad , \tag{3.2}$$

and such that

$$\det (A + BF - \lambda E) \neq 0 \qquad \forall \lambda \notin L \quad . \tag{3.3}$$

□

The equation (3.2) implies that $\lambda_j \in L$ is a generalised eigenvalue of the pencil (M,E), where M = A + BF, and equation (3.3) guarantees that the pencil is regular.

The following two conditions are easily shown to be necessary for the pole assignment problem, Problem 1, to have a solution for any arbitrary self-conjugate set L of q eigenvalues.

Condition C1 : If $\underline{v}^T A = \mu \underline{v}^T E$ and $\underline{v}^T B = \underline{0}$, then $\underline{v} = 0$.

Condition C2 : If $\underline{v}^T E = 0$, $\underline{v}^T B = 0$ and $\underline{v}^T A = \underline{z}^T E$, then $\underline{v} = 0$.

If Condition C1 does not hold then there exists a vector $\underline{v}$ such that $\underline{v}^T (A + BF) = \mu \underline{v}^T$ for any choice of matrix F, and hence both (3.2) and (3.3) cannot be satisfied unless $\mu \in L$ and the problem cannot be solved for arbitrary L. Similarly, if C2 is not satisfied then there exists $\underline{v} \neq 0$ and vector $\underline{z}$ such that $\underline{v}^T E = 0$ and $\underline{v}^T (A + BF) = \underline{z}^T E$ for any choice of F, and, by Lemma 2, a regular solution to the feedback problem cannot exist.

The conditions C1 and C2 are thus necessary for the existence of a solution to the pole assignment problem, Problem 1 (see also [1], [2], [3], [9], [12],[14]). According to Fletcher [3], these two conditions are also sufficient for the existence of a feedback which assigns precisely q ≡ rank (E) given finite eigenvalues with regularity. Fletcher [3] also points out that when condition C1 holds but C2 does not hold it is still possible to assign fewer than q ≡ rank (E) finite eigenvalues with regularity.

We now derive new conditions (as originally reported in [6]), which are necessary and sufficient for arbitrary pole assignment of $q \equiv \text{rank } E$ finite eigenvalues with regularity. We define

Condition C3 : If $\underline{v}^T(E + AS_\infty S_\infty^T) = 0$ and $\underline{v}^T B = 0$, then $\underline{v} = 0$.

The following theorem then gives the result.

Theorem 2 The pole assignment problem, Problem 1, has a solution for an arbitrary self-conjugate set of poles L if and only if Conditions C1 and C3 hold.

Proof: The necessity of condition C1 has already been established. If C3 does not hold, then there exists $\underline{v} \neq 0$ such that $\underline{v}^T(E + (A+BF)S_\infty S_\infty^T) = 0$ for all choices of F and hence $E + MS_\infty S_\infty^T$, where $M = A + BF$, is singular for all matrices F. By Theorem 1 we cannot, therefore, assign q finite non-defective eigenvalues with regularity, and, in particular, we cannot assign $q = \text{rank } (E)$ distinct eigenvalues with regularity. Furthermore, since $\underline{v} \neq 0$, $\underline{v}^T E = 0$, $\underline{v}^T B = 0$ and $\underline{v}^T A = \underline{z}^T E$ implies that $\underline{v}^T(E + AS_\infty S_\infty^T) = \underline{z}^T ES_\infty S_\infty^T = 0$, then condition C3 implies C2 and by [3] condition C3 together with C1 is also sufficient. □

We remark that conditions C1 and C3 are equivalent, respectively, to the conditions

Condition C1' : $\text{rank } ([B, A - \lambda E]) = n, \qquad \forall \lambda \in \mathbb{C}$.

Condition C3' : $\text{rank } ([B, E + AS_\infty S_\infty^T]) = n$.

Condition C1 (or C1') corresponds to the "controllability" condition of [11]. Condition C3 (or C3') implies the "infinite controllability" condition C2, also given by [1],[2],[9],[12] but it is not equivalent to C2. The condition C3 is a stronger condition than C2 and essentially guarantees regularity. At the same time, C3 is necessary for the assignment of $q \equiv \text{rank } (E)$ finite non-defective eigenvectors (with regularity). These results are demonstrated in the next section. We remark that systems which are defective are well-known [11], [13] to be less robust than those which are not, in the sense that the poles of defective systems are more sensitive to perturbations in the system data than those of non-defective systems. In practice, therefore, we are interested in constructing feedback matrices which give non-defective, as well as regular, closed-loop matrix pencils.

4. EIGENSTRUCTURE ASSIGNMENT IN SINGULAR SYSTEMS

In non-singular systems, pole assignment by state feedback can be achieved by assigning the eigenvectors associated with the assigned eigenvalues of the closed loop system. The selected eigenvectors then uniquely determine the required feedback matrix [8], [10]. In singular systems eigenvalue-eigenvector assignment alone is not sufficient to determine the feedback. Furthermore to obtain regularity of the closed loop pencil, certain restrictions on the eigenstructure must be satisfied, as shown in §3. For robustness we also require the eigenstructure to be non-defective. In this section we derive conditions for determining a feedback such that the closed loop system has a specified non-defective eigenstructure and is regular.

We first give two necessary conditions for non-defective eigenstructure assignment with regularity. From the proof of Theorem 2 we have immediately

Lemma 5 If there exists $F \in \mathbb{R}^{m\times n}$, such that the pencil $(A + BF, E)$ is regular, and $X_q \in \mathbb{C}^{n\times q}$, such that $\operatorname{rank}(X_q) = n$ and

$$(A + BF)X_q = EX_q\Lambda_q, \quad \Lambda_q = \{\lambda_1, \lambda_2, \ldots \lambda_q\}, \tag{4.1}$$

where $\lambda_j \in \mathbb{C}$, $\forall_j$, then condition C3 (equivalently, C3') holds.

From Lemma 4 we obtain directly

Lemma 6 If the conditions of Lemma 5 are met, then the matrix $[X_q, S_\infty]$ (equivalently, EX_q) is of full rank. □

The main result is then as follows.

Theorem 3 Given $\Lambda_q = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$, $\lambda_j \in L$, and matrix X_q such that $[X_q, S_\infty]$ is non-singular, then there exists F satisfying (4.1) and such that the pencil $(A + BF, E)$ is regular if and only if

$$U_1^T(AX_q - EX_q\Lambda_q) = 0, \tag{4.2}$$

and

$$U_1^T(E + AS_\infty S_\infty^T) \text{ has full rank,} \tag{4.3}$$

where

$$B = [U_0, U_1] \begin{bmatrix} Z \\ 0 \end{bmatrix}, \tag{4.4}$$

with $U = [U_0, U_1]$ orthogonal and Z non-singular. Then F is given explicitly by

$$F = Z^{-1}[U_0^T(EX_q\Lambda_q - AX_q), W]\,[X_q, S_\infty]^{-1} \tag{4.5}$$

where W is any matrix such that

$$\text{rank}\,(E + AS_\infty S_\infty^T + U_0 W S_\infty^T) = n \quad . \tag{4.6}$$

Proof: The assumption that B is of full rank implies the existence of decomposition (4.4). From (4.1) F must satisfy

$$BFX_q = EX_q\Lambda_q - AX_q, \tag{4.7}$$

and pre-multiplication by U^T gives

$$ZFX_q = U_0^T\,(EX_q\Lambda_q - AX_q) \tag{4.8}$$

and

$$0 = U_1^T\,(EX_q\Lambda_q - AX_q) \tag{4.9}$$

from which (4.2) follows.

From Theorem 1, the pencil $(A + BF, E)$ is regular, under the given conditions, if and only if the matrix $E + (A + BF)S_\infty S_\infty^T$ has full rank, or equivalently $E + AS_\infty S_\infty^T + U_0 W S_\infty^T$ has full rank, where

$$ZFS_\infty = W. \tag{4.10}$$

This condition holds if and only if W can be chosen such that the matrix

$$\begin{bmatrix} U_0^T(E + AS_\infty S_\infty^T) + WS_\infty^T \\ U_1^T(E + AS_\infty S_\infty^T) \end{bmatrix} \tag{4.11}$$

has full rank. Clearly condition (4.3) is necessary and sufficient for this to be possible. The expression (4.5) for the feedback matrix F then follows directly from (4.8)and (4.10), and if W is chosen to satisfy (4.6), the pencil $(A + BF, E)$ has the given finite eigenvalues and is regular. □

The significance of this theorem for the construction of a feedback which achieves pole assignment with regularity is considerable. Condition (4.3) of the theorem holds if and only if Condition C3' holds. This follows since we have C3' if and only if the matrix

$$U^T[B,\ E + AS_\infty S_\infty^T] \begin{bmatrix} Z & U_0^T(E + AS_\infty S_\infty^T) \\ 0 & U_1^T(E + AS_\infty S_\infty^T) \end{bmatrix}$$

has full rank, which holds if and only if (4.3) holds. Condition (4.3) can be tested independently of any choice of F, and if it is not satisfied then a feedback assigning q finite eigenvalues and giving a regular non-defective closed loop pencil cannot be found. Conversely if a set of q independent eigenvectors corresponding to the required closed-loop poles can be selected such that $[X_q, S_\infty]$ is non-singular, then condition (4.3), and hence C3, guarantees that a feedback F can be found such that the pencil $(A + BF, E)$ is regular.

Furthermore, from condition (4.2) of Theorem 3 the eigenvectors corresponding to a distinct closed-loop eigenvalue λ_j must belong to the space

$$S_j = N\{U_1^T(A - \lambda_j E)\} \quad . \tag{4.12}$$

(This, together with the requirement that a closed-loop finite pole must be non-defective, implies a minor restriction on the multiplicity of λ_j). It follows that, given set $L = \{\lambda_j\}$, if we select q independent vectors $\underline{x}_j$, such that $\underline{x}_j \in S_j$, and $E\underline{x}_j$ are independent, $j = 1,2,...,q$, and a matrix W such that (4.6) holds, then the feedback matrix F given by (4.5) with $X_q = [\underline{x}_1,\underline{x}_2,...,\underline{x}_q]$, solves the pole assignment problem, Problem 1, and regularity of the closed-loop pencil is guaranteed.

Moreoever, since the robustness of the closed loop system depends on the selected eigenvectors, we may select the set $\{\underline{x}_j\}$ such as to optimize robustness. In [7] we describe a measure of robustness and give an explicit algorithm for selecting the set $\{\underline{x}_j\}$ and the matrix W such as to obtain a robust feedback solution to the pole assignment problem.

We remark that Theorem 3 gives conditions for assigning a maximum number of finite poles, $q \equiv$ rank (E), with regularity. In the case where fewer finite poles can be assigned with regularity, similar results hold (see [4]).

5. CONCLUSIONS

Novel necessary conditions for the solution of the pole assignment problem by state feedback in singular systems are given in this paper. These conditions must be satisfied in order to assign the maximum possible number of finite poles by feedback and also obtain a closed-loop system pencil which is regular and non-defective. It can be shown that these conditions are also sufficient for the existence of a feedback which assigns q finite poles with regularity. The prime significance of these results is that they provide conditions for the construction of a feedback which assigns given poles with guaranteed regularity, and such that the closed-loop system is robust, in the sense that its poles are insensitive to perturbations in the system data.

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