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Interim report on the application of non-linear optimal control techniques in open channel water management

by

I. Villanueva

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APPLICATION OF NON LINEAR OPTIMAL CONTROL TECHNIQUES IN OPEN CHANNEL WATER MANAGEMENT

by

I. VILLANUEVA [1]

Supervisors

PROF. N. K. NICHOLS and PROF. M. J. BAINES [2]

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[1] Dpt. Fluid Mechanics, C.P.S., U. of Zaragoza, Spain
[2] Dpt. Mathematics, U. of Reading, U.K.

Abstract

Numerical techniques are applied to solve an optimal control problem occurring in water supply in a channel. The problem considered here is that of minimizing the difference between the water delivery requirements and the real quantity of water supplied. The numerical model includes the action of gates and offtakes in several pools. The dynamic of the channel is governed by the Saint Venant's equations, and the flow through the gates and offtakes by non-linear functions. A numerical algorithm which couples a gradient optimisation technique with a finite difference solution of the flow and associated adjoint equations is described.

1   Introduction

We present here a mathematical model to assess the best operating strategies in an open channel with several pools, gates and offtakes, used for delivering water.

In order to use an Optimal Nonlinear Control Theory applied to the Channel Operations which includes the complete solution of the Saint Venant's equations of the flow, and does not discretize the objective function from the beginning, the methods and conclusions of previous works [1],[2],[3],[4], which were applied for maximizing the output of a tidal power station in a estuary, have been adapted to the problem of minimizing the difference between the required and delivered quantities of water through some offtakes.

We have studied two models, the first one with the purpose of understanding the Optimisation Problem, and the second one for applying it to a more realistic case.

2   The First Model

2.1   The Conservation Law

We consider that we have a channel with N flat pools, each one with a height of water h_i, one input flow gate, one output flow gate and one offtake, as we see in the figure 1.

The state variables are the heights h_i, and the control variables are α_{1i}, α_{2i}, being respectively the ratio of maximum opening of the offtake and input gate of the pool i , in such a way that

$$0 \leq \alpha_{1i} \leq 1 \tag{1}$$

$$0 \leq \alpha_{2i} \leq 1 \tag{2}$$

The mass conservation equation for each pool is the ODE

$$S_i \dot{h}_i = \alpha_{2i} Q i_i + \alpha_{2i+1} Q o_i - \alpha_{1i} Q t_i \tag{3}$$

where S is the surface, Qi is the input flow, Qo the output flow and Qt the flow trough the take according to :

$$Q i_i = C g_i A g_i B g_i \sqrt{2g|h_{i-1} - h_i|} sgn(h_{i-1} - h_i) \tag{4}$$

$$Q o_i = C g_{i+1} A g_{i+1} B g_{i+1} \sqrt{2g|h_{i+1} - h_i|} sgn(h_{i+1} - h_i) \tag{5}$$

which verifies $Q o_i = -Q i_{i+1}$

$$Q t_i = C t_i A t_i B t_i \sqrt{2g(h_i - H t_i)} \tag{6}$$

where Cg, Ct are coefficients of discharge, Ag, At the maximum openings, Bg, Bt the width of the gates, offtakes, and Ht is a minimum height required for having flow through the offtake.

In order to solve the evolution of this system we need to know the initial value of all the state variables h_i and the boundary conditions imposed for them, i.e. $h_1(t)$, $h_N(t)$. We will assume that all of these functions are periodic in time, with a period T.

2.2   Optimal Control Problem

The optimisation problem is then to determine the control functions $\alpha_{1i}(t)$, $\alpha_{2i}(t)$, which minimize the quantity

$$\overline{M} = \frac{1}{T} \sum_i \int_0^T w_i (\alpha_{1i} Q t_i - D_i)^2 dt \tag{7}$$

subject to constraints (1), (2), and equations (3), (4), (5), (6), and where w_i is a weight value, and D_i is the out fixed demand of water, typically a step function of time.

For studying the necessary conditions for having a unique solution for $\underline{\alpha} = (\alpha_{11}, \alpha_{21}, \cdots, \alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{1N}, \alpha_{2N})^T$ in this model see [1] and [2].

2.3   Necessary Conditions for the Optimal

We define the Lagrangian functional associated with the optimal control problem as

$$L(\underline{\alpha}) = \sum_i \int_0^T \left[w_i(\alpha_{1i}Qt_i - D_i)^2 + \lambda_i(-S_i\dot{h}_i + \alpha_{2i}Qi_i + \alpha_{2i+1}Qo_i - \alpha_{1i}Qt_i) \right] dt \tag{8}$$

Where λ_i is a Lagrange multiplier known as the adjoint state variable.

For $\underline{\alpha}$ to be minimal it is necessary that the first variation $\delta L(\underline{\alpha}, \delta\underline{\alpha})$ is positive, where δL is linear in $\delta\underline{\alpha}$.

We can write this variation in the form :

$$\delta L(\underline{\alpha}, \delta\underline{\alpha}) = \sum_i \int_0^T \left[\delta\alpha_{1i}Qt_i \left(2w_i(\alpha_{1i}Qt_i - D_i) - \lambda_i\right) + \delta\alpha_{2i}Qi_i(\lambda_i - \lambda_{i-1}) \right] dt \tag{9}$$

after using integration by parts and taking λ_i to satisfy :

$$\lambda_i(t) = \lambda_i(t + T) \tag{10}$$

$$
\begin{aligned}
\dot{\lambda}_i &= \frac{\lambda_i}{S_i}\left(\alpha_{1i}\frac{\partial Qt_i}{\partial h_i} - \alpha_{2i}\frac{\partial Qi_i}{\partial h_i} - \alpha_{2i+1}\frac{\partial Qo_i}{\partial h_i}\right) \\
&+ \frac{\lambda_{i+1}}{S_{i+1}}\left(\alpha_{2i+1}\frac{\partial Qo_i}{\partial h_i}\right) \\
&+ \frac{\lambda_{i-1}}{S_{i-1}}\left(\alpha_{2i}\frac{\partial Qi_i}{\partial h_i}\right) \\
&- \left(2w_i(\alpha_{1i}Qt_i - D_i)\alpha_{1i}\frac{\partial Qt_i}{\partial h_i}\right)
\end{aligned}
\tag{11}
$$

For a given control $\underline{\alpha}$ the minimum condition is

$$< \nabla \overline{M}(\underline{\alpha}), \underline{\beta} - \underline{\alpha} > \geq 0 \qquad (12)$$

for all admissible controls $\underline{\beta}$

Where the inner product $< \cdot, \cdot >$ is defined as $< a, b >= \int_0^T a^T b\, dt$

In our case we may write from (7),(8),(9),

$$< \nabla \overline{M}(\underline{\alpha}), \underline{\beta} - \underline{\alpha} >= \delta L(\underline{\alpha}, \delta \underline{\alpha})$$

with

$$\nabla \overline{M}(\underline{\alpha}) = \begin{pmatrix} \cdots \\ Qt_i \left(2w_i(\alpha_{1i} Qt_i - D_i) - \lambda_i\right) \\ Qi_i(\lambda_i - \lambda_{i-1}) \\ \cdots \end{pmatrix} \qquad (13)$$

For a given control $\underline{\alpha}$, state and adjoint variables, the gradient vector can be calculated from (13), and since the values of the controls belong to a closed interval $[0,1]$, the inequality (12) is easily tested. Gradient methods can therefore be applied to determine numerical approximations to the optimal control problem.

2.4   The Numerical Method

We propose a constrained optimisation technique for determining the control, together with a finite difference method for solving the state and adjoint problems.

As it was concluded in [4], and during the course of this research, the Conditional Gradient Method was selected for the sluice controls. It generates a sequence of piecewise continuous controls $\underline{\alpha}^k(t)$ approximating to $\underline{\alpha}(t)$ such that

$$\overline{M}(\underline{\alpha}^{k+1}) < \overline{M}(\underline{\alpha}^k)$$

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and

$$\underline{\alpha}^{k+1} = \underline{\alpha}^k + \theta \delta \underline{\alpha}^k$$

with $0 < \theta \leq 1$.

The iterative algorithm stops when the absolute value of the measure

$$minimum\left(< \nabla \overline{M}(\underline{\alpha}^k), \underline{\beta} - \underline{\alpha}^k >\right), \quad \underline{\beta} \in U$$

is less than a given tolerance .

2.4.1 Basic Algorithm

Supposing that we discretize the time T and all the functions depending on it in Z intervals, we use the following steps :

1. Obtain the Kth approximation $\underline{\alpha}^k(t)$ to $\underline{\alpha}(t)$, i.e. to know the Z vectors

$$\underline{\alpha}^k(t) = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \cdots \\ \alpha_{1i} \\ \alpha_{2i} \\ \cdots \\ \alpha_{1N} \\ \alpha_{2N} \end{pmatrix} \tag{14}$$

   If $k = 0$ this matrix is an initial input value and we also must choose $\theta \in (0, 1]$

2. Solve the conservation of mass for the pools, i.e. for $j = 1, \cdots, Z - 1$

$$\begin{pmatrix} h_1^j \\ \cdots \\ h_i^j \\ \cdots \\ h_N^j \end{pmatrix} \Longrightarrow \begin{pmatrix} h_1^{j+1} \\ \cdots \\ h_i^{j+1} \\ \cdots \\ h_N^{j+1} \end{pmatrix} \tag{15}$$

3. Solve the adjoint equation for the pools (backward in time), i.e. for $j = Z, \cdots, 2$

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$$\begin{pmatrix} \lambda_1^j \\ \cdots \\ \lambda_i^j \\ \cdots \\ \lambda_N^j \end{pmatrix} \Longrightarrow \begin{pmatrix} \lambda_1^{j-1} \\ \cdots \\ \lambda_i^{j-1} \\ \cdots \\ \lambda_N^{j-1} \end{pmatrix} \tag{16}$$

4. Evaluation of the quality $Q(\underline{\alpha}^k)$ of the Kth approximation, i.e.

$$Q(\underline{\alpha}^k) = minimum \left(< \nabla \overline{M}(\underline{\alpha}^k), \underline{\beta} - \underline{\alpha}^k > \right), \quad \underline{\beta} \in U \tag{17}$$

This minimum and $\underline{\beta}$ are obtained following the next criteria

$$\beta_{1i}^j = \begin{cases} 0 & \text{if } \left(\nabla \overline{M}(\underline{\alpha}^k) \right)_{1i}^j \geq 0 \\ 1 & \text{otherwise} \end{cases} \tag{18}$$

$$\beta_{2i}^j = \begin{cases} 0 & \text{if } \left(\nabla \overline{M}(\underline{\alpha}^k) \right)_{2i}^j \geq 0 \\ 1 & \text{otherwise} \end{cases} \tag{19}$$

Stop if $|Q(\underline{\alpha}^k)| < tolerance$, if not continue with the next steps

5. If $\overline{M}(\underline{\alpha}^k) > \overline{M}(\underline{\alpha}^{k-1})$ then $\theta = \frac{\theta}{2}$

6. $\underline{\alpha}^{k+1} = \underline{\alpha}^k + \theta(\underline{\beta}^k - \underline{\alpha}^k)$

   $k = k + 1$

   go to step 1

As it was well explained in [4], the Conditional Gradient Algorithm can be improved with slight changes, and we have used that modified version, which only allows a new iteration for $\underline{\alpha}$ if its function $\overline{M}$ is less than the previous one .

2.4.2   Finite Difference Scheme for the State and Adjoint variables

Both of the equations (3),(11), are solved using a fixed iteration method. That implies that for a system of the kind $\underline{\dot{H}} = \underline{F}(\underline{H})$ where $\underline{H}$ is the vector of variables and $\underline{F}$ a functional vector of it, for each interval of time n of the discretization $n = 1, \cdots, Z - 1$ we have

$$\frac{\underline{H}^{n+1} - \underline{H}^n}{\Delta t} = (1 - \psi)\underline{F}^n(\underline{H}^n) + \psi\underline{F}^{n+1}(\underline{H}^{n+1}), \quad 0 \leq \psi \leq 1$$

At the beginning of each step n we impose $\underline{F}^{n+1} = \underline{F}^n$, $\underline{H}^{n+1} = \underline{H}^n$ and with the new obtained value of $\underline{H}^{n+1}$ we continue the iteration process until the desired convergence for $\underline{H}^{n+1}$.

Two things must be noted :

- The adjoint equations are solved backward in time, because their homogeneous solution decays in that way, see [1].

- In the progress of the problem it was said that the state and adjoint variables are periodic in time, that implies that we must repeat the numerical integration over the interval $[0, T]$ until the values of h_i^1 and λ_i^1 differ by less than one small tolerance with h_i^Z and λ_i^Z .

2.5 Additional Term Added to the minimisation function

In order to obtain satisfactory results for the control, in fact we have for N pools, $2N$ control variables, it is better to have more terms depending on the control in equation (7). We added the next

$$v_i \alpha_{2i}$$

which implies that depending of the weight coefficient v_i we minimize the opening time of each gate.

We must also add to equation (13) the term v_i for obtaining

$$\nabla \overline{M}(\underline{\alpha}) = \begin{pmatrix} \cdots \\ Qt_i\left(2w_i(\alpha_{1i}Qt_i - D_i) - \lambda_i\right) \\ Qi_i(\lambda_i - \lambda_{i-1}) + \mathbf{v_i} \\ \cdots \end{pmatrix} \tag{20}$$

2.6 A practical example

Just for checking these techniques we propose an example which has six pools, with some demands of water. In the first pool there is an increase of the height for satisfying that demands.

The input data are :

Parameter	Pool 1	Pool 2	Pool 3	Pool 4	Pool 5	Pool 6
Surface (m^2)	60	60	60	60	60	60
h^1 (m)	1	1	1	1	1	1
λ^Z	0	0	0	0	0	0
α_1	0.5	0.5	0.5	0.5	0.5	0.5
α_2	0.5	0.5	0.5	0.5	0.5	0.5
Ag (m)	0.5	0.5	0.5	0.5	0.5	0.5
Bg (m)	1	1	1	1	1	1
Cg	0.7	0.7	0.7	0.7	0.7	0.7
At (m)	0.5	0.5	0.5	0.5	0.5	0.5
Bt (m)	1	1	1	1	1	1
Ct	1	1	1	1	1	1
Ht (m)	1	1	1	1	1	1
w_i	1	1	1	1	1	1
v_i	0	0	0	0	0	0
t_0 demand	0	0.2	0.7	0.8	0.9	0
t_f demand	0	0.3	0.8	0.9	1.0	0
demand m^3/s	0	1.5	0.5	0.5	0.5	0

The period of time is $T = 600$ s, the boundary conditions for the state and adjoint variables are $h_1(t) = 1 + \sin(\frac{2\pi}{T})$, $h_6(t) = 1$ and $\lambda_1(t) = \lambda_6(t) = 0$.

We made the calculations using a number of intervals of $Z = 600$, a coefficient $\psi = 0$ for the finite difference scheme, $\theta = 1$ as initial value for the gradient search, and 100 iterations with a final quality of 0.0016.

The most representative graphics with the heights, movements of the gates and offtakes are in figures (3),(4),(5),(6), where we remark that the demands are satisfied for each pool at the right times.

3 The Second Model

3.1 The Conservation Laws

In our second model we want to discretize in space the pools and use the conservation of mass and momentum for a more detailed description of the channel dynamics, including the effects of wave propagations, friction and mass loss. We use the 1D shallow water or Saint Venant's equations for each pool

$$A_t + Q_x = -q_l \tag{21}$$

$$Q_t + \left(\frac{Q^2}{A} + gI_1 \right)_x = gA(S_o - S_f) \tag{22}$$

This is a conservative formulation where the state variables are $A = A(x,t)$, the cross sectional area and $Q = Q(x,t)$, the discharge. The term q_l is the lateral outflow loss per unit of length, in such a way that if Xt_i is the location of the offtake i and its flow is Qt_i from equation (6), $q_l(Xt_i) = \alpha_{1i}Qt_i/\Delta x$. I_1 is the hydrostatic pressure force term (function of A), S_o is the bed slope and S_f the friction term (function of Q), as described in [5].

A finite difference method is applied for solving (21),(22), in each pool, and we use the next notation for the discretized state variables

$$A_{ij}^k, Q_{ij}^k, \quad i = 1, \cdots, N \quad j = 1, \cdots, S \quad k = 1, \cdots, Z$$

with N pools, S space intervals and Z steps of time.

For each bound of each pool we need two more equations for solving the state variables. If the bound is a gate we use the equation (4), for its discharge Qg_{i+1}

$$Qg_{i+1}^k = Cg_{i+1}Ag_{i+1}Bg_{i+1}\sqrt{2g|h_{iL}^k - h_{i+1R}^k|}sgn(h_{iL}^k - h_{i+1R}^k) \tag{23}$$

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and

$$Q_{iL}^k = Q_{i+1R}^k = Qg_{i+1}^k \qquad (24)$$

where L, and R, denote the left neighbour point of the i+1 th gate, and the right one, respectively. The other equation comes from the theory of characteristics, see [5].

For the first and last bound of the channel we must impose the area or the discharge, or a relation between them, and use the theory of characteristics.

Basically the theory of characteristics for our hyperbolic system (21),(22), implies that along the curves in the $x - t$ plane

$$\frac{dx}{dt} = u \pm c$$

it verifies that

$$\frac{d(u \pm 2c)}{dt} = g(S_o - S_f)$$

where $u = \frac{Q}{A}$, $c = \sqrt{\frac{gA}{\sigma}}$ and σ is the width of the section. We can use these curves for obtaining a relation between the state variables A_{ij}^{n+1} and Q_{ij}^{n+1} knowing their values at time n.

3.2   Necessary conditions for the optimal

The lagrangian functional associated with the optimal control problem is now defined by

$$
\begin{aligned}
L(\underline{\alpha}) &= \sum_i \int_0^T \left[w_i(\alpha_{1i}Qt_i - D_i)^2 \right] dt \\
&+ \sum_i \int_0^T \left[\gamma_{1i}(Q(Xf_i) - \alpha_{2i+1}Qg_{i+1}) + \gamma_{2i}(Q(Xf_i) - Q(Xi_{i+1})) \right] dt
\end{aligned}
$$

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$$+ \sum_i \int_0^T \int_{Xi_i}^{Xf_i} \lambda \left(-A_t - Q_x - q_l \right) dx_i dt$$

$$+ \sum_i \int_0^T \int_{Xi_i}^{Xf_i} \mu \left(-Q_t - \left(\frac{Q^2}{A} + gI_1 \right)_x + gA(S_o - S_f) \right) dx_i dt \quad (25)$$

Being Xf the downstream end of the pool and Xi the upstream one. We remark that the Lagrange multipliers γ_{1i}, γ_{2i}, depend only on time whereas λ and μ depend on x and t.

Integrating by parts and taking variations for $L(\underline{\alpha})$ we join the terms as follows for making them to vanish :

$$\sum_i \int_{Xi_i}^{Xf_i} [-\lambda \delta A - \mu \delta Q]_0^T \, dx_i \quad (26)$$

$$\sum_i \int_0^T \left[-\lambda \delta Q - \mu \left(2u\delta Q + (c^2 - u^2)\delta A \right) \right]_{Xi_i}^{Xf_i} dt \quad (27)$$

$$\sum_i \int_{Xi_i}^{Xf_i} \int_0^T \delta A \left[\lambda_t + \mu_x(c^2 - u^2) - S_\lambda \right] dx_i dt \quad (28)$$

$$\sum_i \int_{Xi_i}^{Xf_i} \int_0^T \delta Q \left[\mu_t + \mu_x 2u + \lambda_x - S_\mu \right] dx_i dt \quad (29)$$

with

$$S_\lambda = -(-\frac{\lambda \alpha_{1i}}{\Delta x_i} + 2w_i \alpha_{1i}(\alpha_{1i}Qt_i - D_i))\frac{\partial Qt_i}{\partial A}$$

$$- \mu g(S_o - (S_f + A\frac{\partial S_f}{\partial A})) \quad (30)$$

$$S_\mu = \mu g A\frac{\partial S_f}{\partial Q} \quad (31)$$

It has been used that $\frac{\partial I_1}{\partial A} = \frac{A}{\sigma}$, see [5].

Equation (26) vanishes if the state and adjoint variables are periodic in time.

For the downstream bound of the last pool we impose that $Q(Xf_N, t) = 0$ and this leads to $\mu(Xf_N, t) = 0$.

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For the upstream bound of the first pool the area is a known function, so it gives $\lambda(Xi_1, t) + \mu(Xi_1, t)2u(Xi_1, t) = 0$

The terms inside the brackets in equations (28),(29), are a coupled system of hyperbolic differential equations that we must solve, with source terms S_λ and S_μ

For every gate we have the next vanishing sum from (25),(27), following the mentioned notation of L and R, $(Xf_i \equiv L, Xi_{i+1} \equiv R)$

$$
\begin{aligned}
& \delta A_L \left(-\gamma_{1i}\alpha_{2i+1}\frac{\partial Qg_{i+1}}{\partial A_L} - \mu_L(c_L^2 - u_L^2) \right) \\
+ \; & \delta A_R \left(-\gamma_{1i}\alpha_{2i+1}\frac{\partial Qg_{i+1}}{\partial A_R} + \mu_R(c_R^2 - u_R^2) \right) \\
+ \; & \delta Q_L \left(\gamma_{1i} + \gamma_{2i} - \lambda_L - \mu_L 2u_L \right) \\
+ \; & \delta Q_R \left(-\gamma_{2i} + \lambda_R + \mu_R 2u_R \right) = 0
\end{aligned}
\tag{32}
$$

As $\delta Q_L = \delta Q_R$, equation (24), we obtain

$$
\gamma_{1i} = \lambda_L - \lambda_R + 2u_L\mu_L - 2u_R\mu_R
\tag{33}
$$

$$
\gamma_{2i} = \lambda_R + 2u_R\mu_R
\tag{34}
$$

For obtaining the values of the adjoint variables at each bound, we also need to use the theory of characteristics for the hyperbolic adjoint system. These are the same that we use for the conservation laws but with a different propagated quantity

$$
\frac{d(\lambda + (u \pm c)\mu)}{dt} = S_\lambda + S_\mu(u \pm c)
$$

In [5] it is described how to obtain this equation.

After all these considerations we finally write

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$$\delta L(\underline{\alpha}, \delta \underline{\alpha}) = \sum_i \int_0^T \left[\delta \alpha_{1i} Q t_i \left(2 w_i (\alpha_{1i} Q t_i - D_i) - \frac{\lambda(X t_i)}{\Delta x_i} \right) - \delta \alpha_{2i+1} \gamma_{1i} Q g_{i+1} \right] dt$$

$$(35)$$

3.3 The Numerical Method

It has the same steps that we described in the first model, but now in steps 2 and 3 we have two adjoint and two state variables coupled in an hyperbolic differential system that must be solved by a finite difference scheme and the theory of characteristics for the bounds. A schematic picture shows it in the figure 2.

3.3.1 Finite Difference Scheme for the State and Adjoint variables

Both of the systems, shallow water equations (21),(22), and adjoint problem (28),(29), are at first solved using a modified version of the Leap-Frog scheme [1]. It proposes for a system with the aspect :

$$\underline{U}_t + \underline{F}(\underline{U})_x = \underline{H}(\underline{U})$$

the discretization :

$$\frac{\underline{U}_j^{n+1} - \underline{U}_j^{n-1}}{2\Delta t} + \frac{\underline{F}_{j+1}^n - \underline{F}_{j-1}^n}{2\Delta x} = \frac{\underline{H}_j^{n+1} + \underline{H}_j^{n-1}}{2}$$

which is constrained in time by the CFL condition, see [5].

3.4 Practical Example

In order to make a first test of the second model we propose a simple two pools channel with only one offtake in the second pool and a front wave traveling from upstream. The data are :

Geometrical Parameters	Pool 1	Pool 2
Length (m)	1000	1000
Width (m)	10	10
Manning	0.016	0.016
So	0.0008	0.0008
Ag (m)	-	1
Bg (m)	-	2.5
Cg	-	0.75
Xt (m)	-	1500
At (m)	-	1
Bt (m)	-	1.5
Ct	-	1
Ht (m)	-	0
t_0 demand	-	0.625
t_f demand	-	0.75
demand m^3/s	-	15

The period of time is $T = 4800$ s , $\Delta x = 10$ m and $CFL = 0.5$.

The upstream boundary condition is

$$h_{11}(t) = 0.6 + 0.75 \sin \frac{2\pi t}{6000} \quad \text{if } 0 \leq t \leq 3000$$
$$= 0.6 \quad \text{otherwise}$$

For the initial state all the gates and offtakes are closed, and the flow variables are calculated using a Runge-Kutta approximation to the normal depth curve $\frac{dy}{dx} = \frac{S_o - S_f}{1 - F^2}$, $F = \frac{gA}{\sigma}$, with the bound values $h_{2N} = 2$, $Q_{2N} = 0$ and $h_{1N} = 1.4$, $Q_{1N} = 0$

For the initial state for the adjoint variables we use $\lambda_{ij}^Z = \mu_{ij}^Z = 0$

With the initial values of $\alpha_{12} = 0.5$, $\alpha_{22} = 0.5$, $w_2 = 0.01$, $v_2 = 0$, and for an excessive demand of 15 m^3/s , the results are displayed in figures (7),(8),(9). The demand is not satisfied but we observe the movement of the gate for having the maximum quantity at the offtake.

It was checked that with this control obtained for the gate, the amount of water supplied was greater than using $\alpha_{22} = 1$ over all the period, and very similar to the one using $\alpha_{22} = 1$ only if $h_L > h_R$.

We remark that in this case we have used both Conditional Gradient Methods

obtaining similar results. For an initial value for the gradient search $\theta = 1$ and with 18 iterations the final quality was of 0.003597, for the basic version and with 22 iterations and a final quality of 0.003341 for the improved version

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4 Further Research and Comments

Further research :

Before adding new pools to the second model we need to

- Make an analysis of the properties of the adjoint system, specially for the source terms S_λ and S_μ

- Apply another finite difference methods for the flow-adjoint problem, which allow longer steps of time

- Add new terms to the function $\overline{M}$, including penalty terms

- Use a new gradient algorithm search for many controls

Comments :

- Study the behaviour of the gradients nearly 0

- Try the Projected Gradient Method [4] for the smooth variations of the controls

- Try different initial guesses for the controls

- Check if reducing the steps of time the convergence improves

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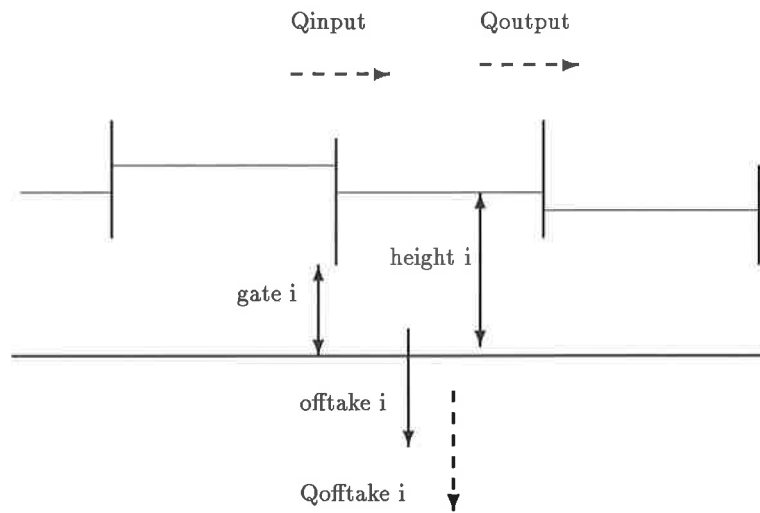
Figure 1: The basic model of N pools
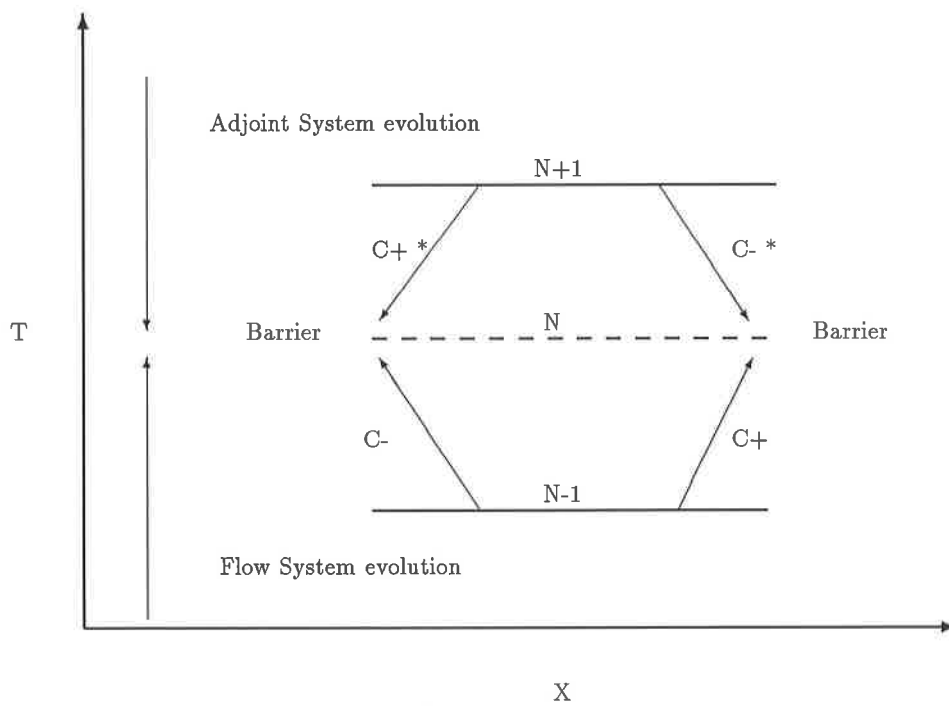
Figure 2: The characteristic curves and time direction for the flow and adjoint systems
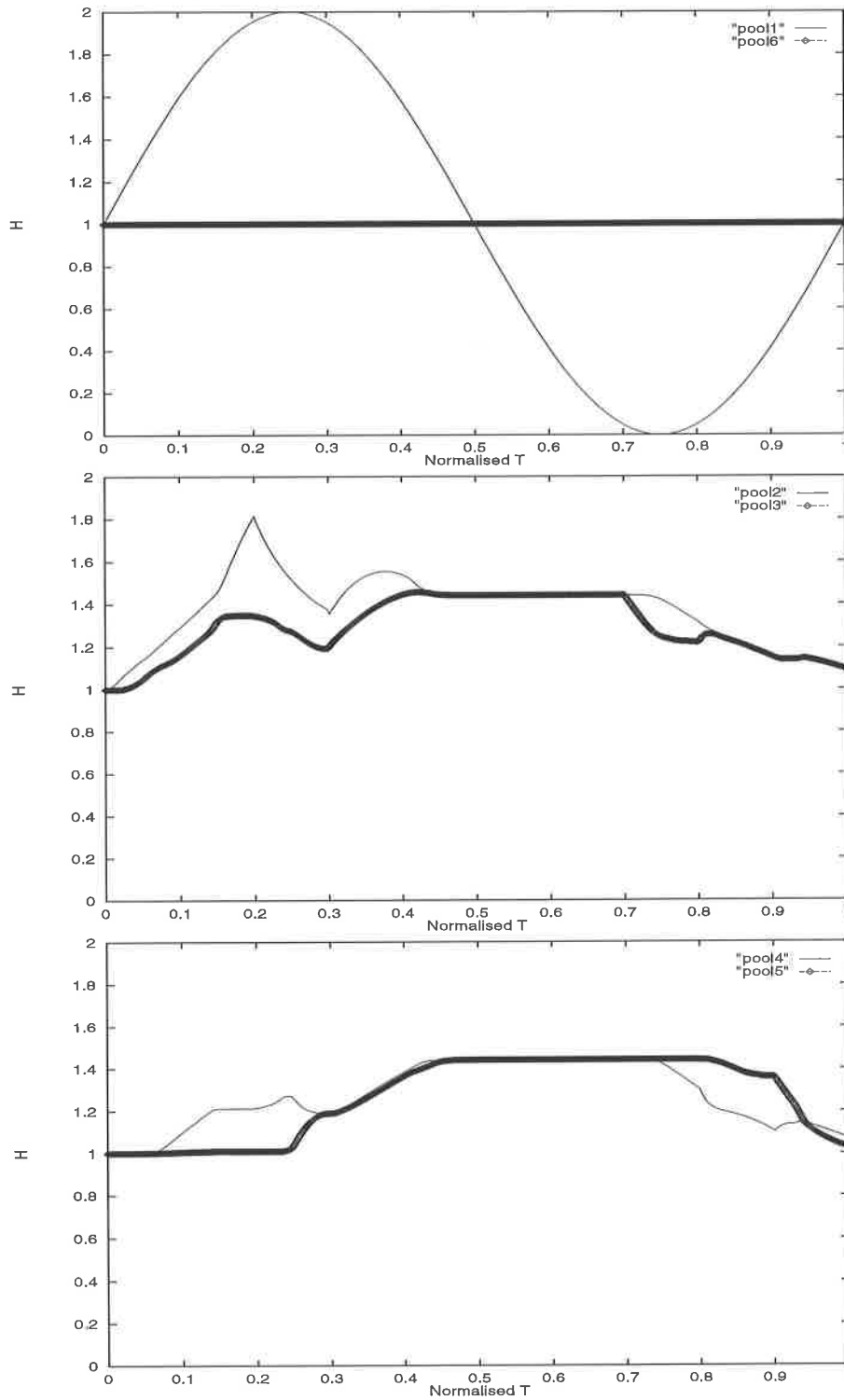
Figure 3: First Model, heights (m) as function of normalised time in the six pools

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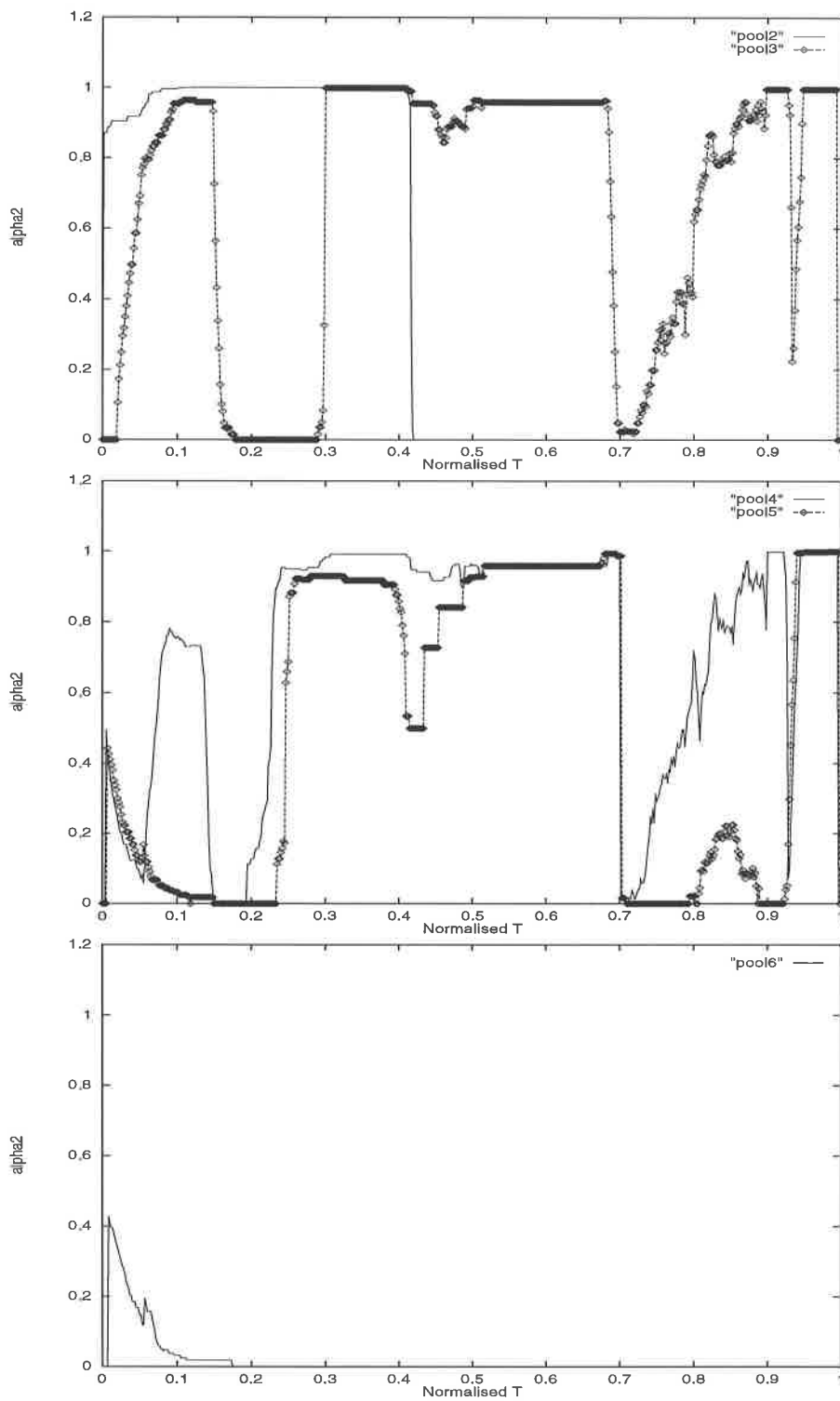
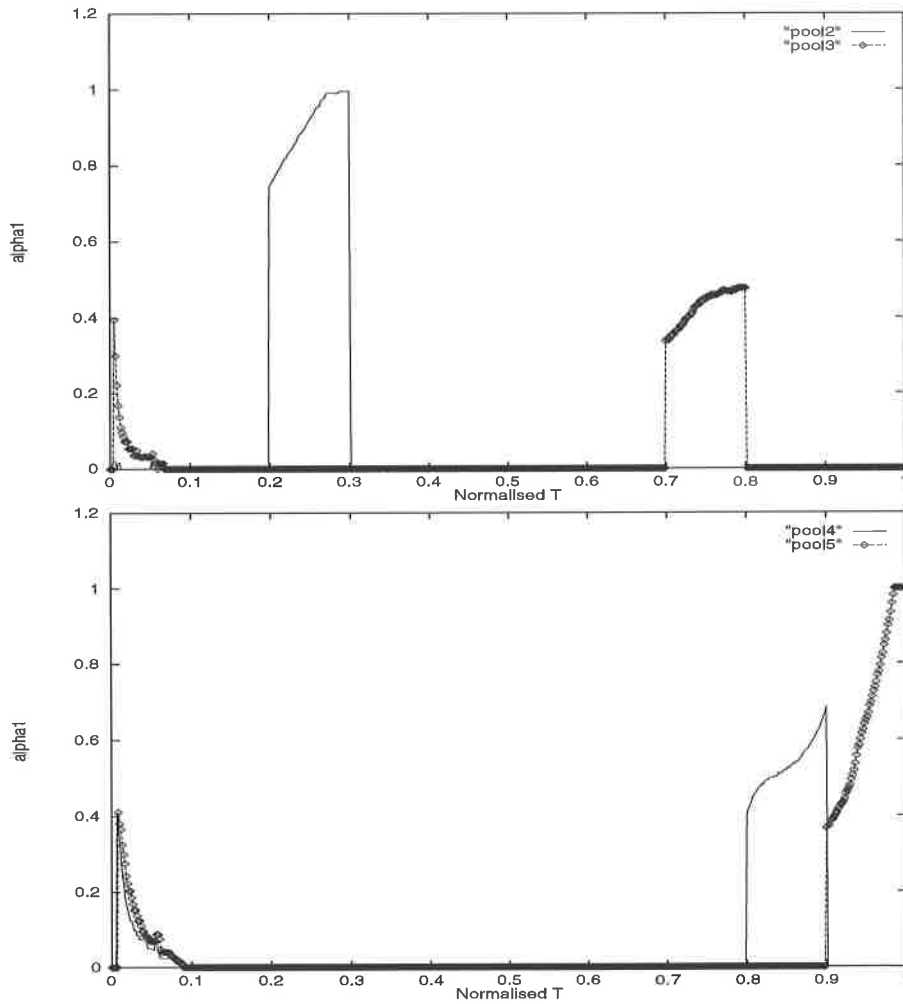Figure 4: First Model, controls for the gates as function of normalised time

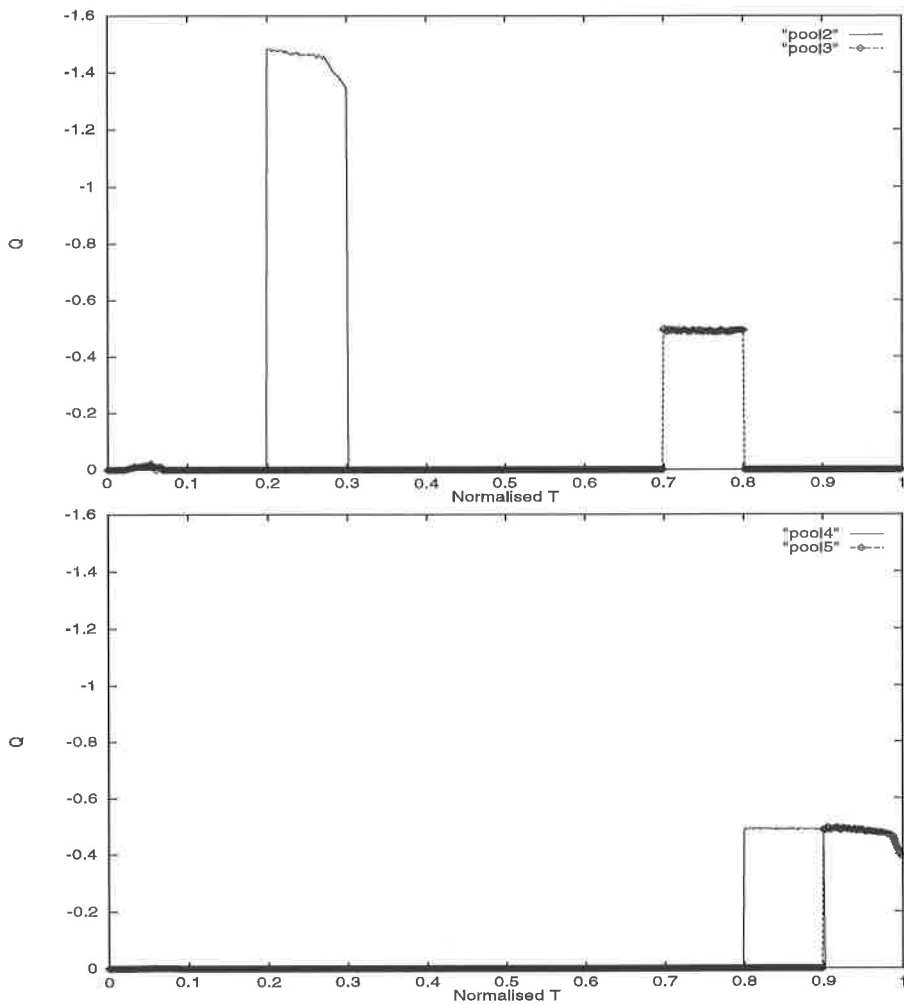Figure 5: First Model, controls for the offtakes as function of normalised time

Figure 6: First Model, offtake discharges (m^3/s) as function of normalised time. The step required demands were $1.5, 0.5, 0.5, 0.5$ respectively
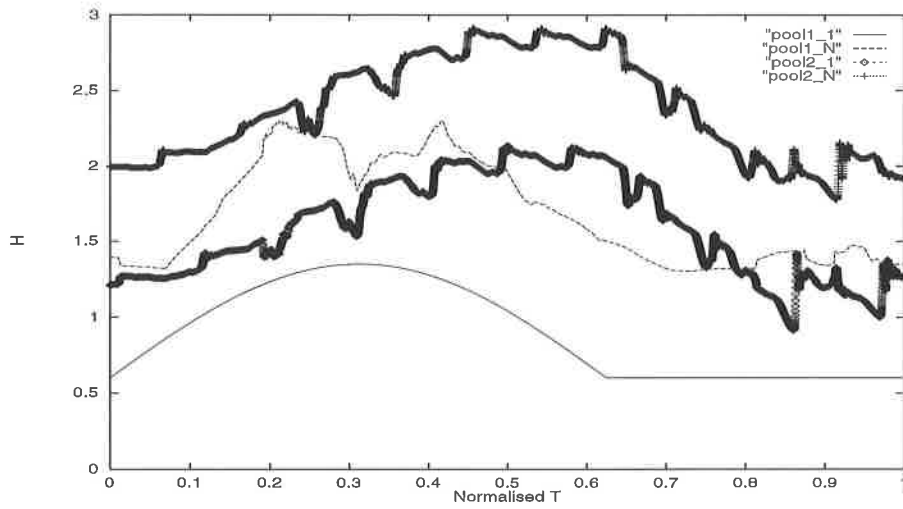
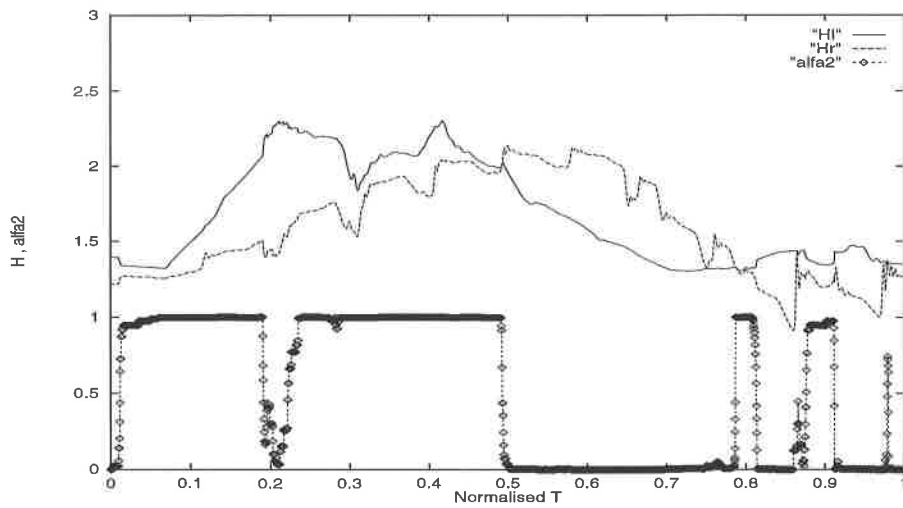Figure 7: Second Model, heights (m) as function of normalised time at the bounds of each pool

Figure 8: Second Model, heights (m) just before and after the gate and its control, as function of normalised time
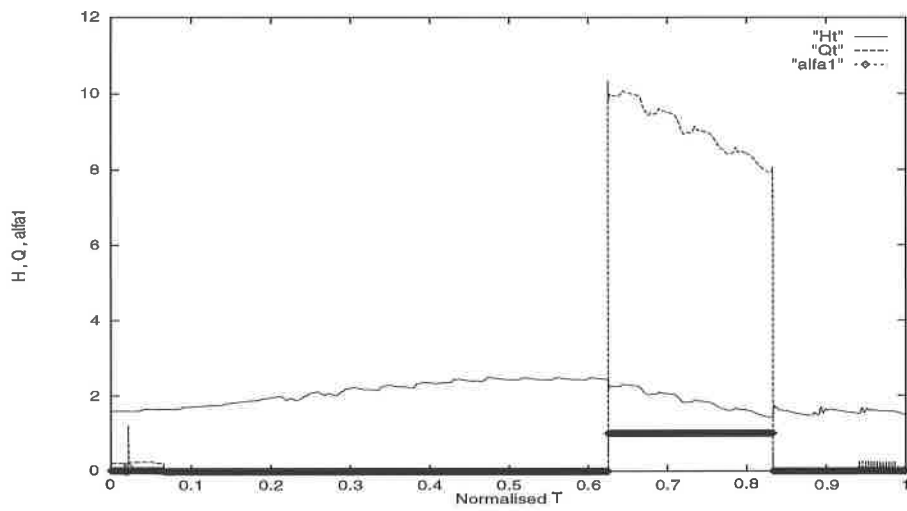
Figure 9: Second Model, height (m), discharge m^3/s and control at the offtake, as function of normalised time