The Rate of Convergence of the Viscosity Method

for a Nonlinear Hyperbolic System

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The rate of convergence of the viscosity method for a nonlinear hyperbolic system

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Abstract

This paper consists of two parts. Part I considers the Cauchy problem for the nonstrictly hyperbolic system $\rho_t + (\rho u)_x = 0$, $(\rho u)_t + (\rho u^2)_x + \rho(ke^{\frac{s}{\beta}}\rho^{\gamma-1})_x = 0$, $s_t + us_x = 0$, which is motivated by the non-isentropic equations of polytropic gas in Euler coordinates, and gives the global Holder continuous solution by applying the method of vanishing viscosity. Part II further studies the relation between the exact solution and the viscosity solutions and obtains error estimates.

Introduction

In this paper we consider the nonlinear hyperbolic system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2)_x + \rho(ke^{\frac{s}{\beta}}\rho^{\gamma-1})_x = 0 \\ s_t + us_x = 0, \end{cases} \tag{1}$$

with C^1 initial data

$$(\rho(x,0), u(x,0), s(x,0)) = (\rho_0(x), u_0(x), s_0(x)), \tag{2}$$

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where β, k, γ are positive constants with $1 < \gamma \leq 3$. System (1) is motivated by the non-isentropic equations of polytropic gas in Euler coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + \frac{k}{\gamma} e^{\frac{s}{\beta}} \rho^\gamma)_x = 0 \\ s_t + u s_x = 0, \end{cases} \tag{3}$$

where ρ, u and s are density, velocity and entropy respectively.

When $s =$ is constant, (1) and (3) are equivalent. For this isentropic case, paper [6] considered the global Holder continuous solution of the Cauchy problem. In this paper we report on preliminary research on the system (1) and believe it will later be useful for research into system (3). For similar work of the global smooth solutions to hyperbolic systems, see papers $[1, 3, 4, 5]$.

This paper consists of two parts: Part I considers the existence of the global Holder continuous solution of the Cauchy problem (1), (2). The method used is a variant of the "viscosity" argument $[2, 5, 6]$.

Substituting the first equation of (1) into the second, system (1) is equivalent, for smooth solutions, to the following,

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + (\frac{u^2}{2} + k e^{\frac{s}{\beta}} \rho^{\gamma-1})_x = 0 \\ s_t + u s_x = 0. \end{cases} \tag{4}$$

By simple calculation, system (4) can be rewritten as follows:

$$\begin{cases} (u + \alpha c)_t + (u + c)(u + \alpha c)_x = 0 \\ (u - \alpha c)_t + (u - c)(u - \alpha c)_x = 0 \\ s_t + u s_x = 0, \end{cases} \tag{5}$$

where $\alpha = \frac{2}{\gamma-1}$ and $c = \sqrt{(\gamma - 1) k e^{\frac{s}{\beta}} \rho^{\gamma-1}}$.

Let $w = u + \alpha c, z = u - \alpha c$. We try to construct solutions of (5) as limits $(\epsilon \to 0^+)$ of smooth solutions of the systems

$$\begin{cases} w_t + (u + c)w_x = \epsilon w_{xx} \\ z_t + (u - c)z_x = \epsilon z_{xx} \\ s_t + u s_x = \epsilon s_{xx}, \end{cases} \tag{6}$$

with initial data

$$\begin{aligned} (w(x, 0), z(x, 0), s(x, 0)) &= (w_0^\epsilon(x), z_0^\epsilon(x), s_0(x)) \\ &= (w_0(x) + \delta(\epsilon), z_0(x) + \delta(\epsilon), s_0(x)), \end{aligned} \tag{7}$$
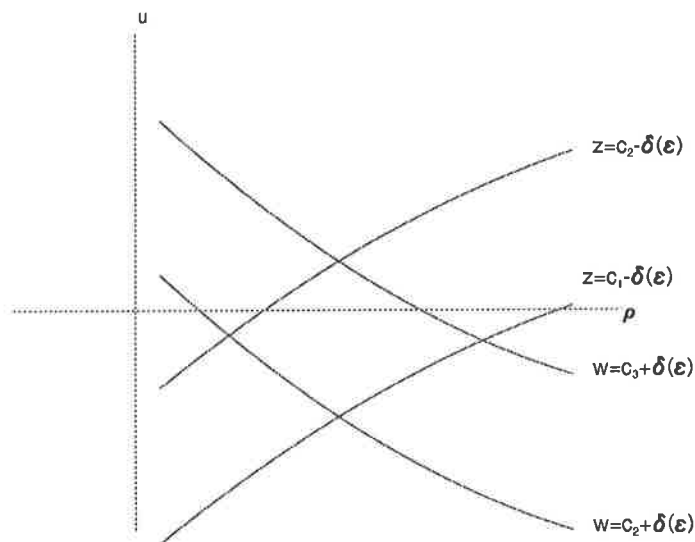
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Figure 1: An illustration in the (ρ, u) plane.

where $\delta(\epsilon)$ is a positive, bounded function of ϵ and defined later.

We first show that for every fixed $\epsilon > 0$, the Cauchy problem $(6), (7)$ has a bounded solution (w, z, s) whose differentials with respect to x, t are also bounded independent of ϵ. We show that there is a subsequence

$$\{\rho^{\epsilon_n}, u^{\epsilon_n}, s^{\epsilon_n}\}, \quad \epsilon_n \to 0^+ \quad \text{as } n \to \infty,$$

on any bounded region converging uniformly to a Holder continuous solution of the Cauchy problem $(1), (2)$.

Part II further studies the relation between the exact solution (ρ, u, s) and the viscosity solutions $(\rho^\epsilon, u^\epsilon, s^\epsilon)$ and gives the following error estimates

$$|u^\epsilon - u| \le M(T)\epsilon, \ |\rho^\epsilon - \rho| \le M(T)\epsilon, \ |s^\epsilon - s| \le M(T)\epsilon,$$

where $M(T)$ is a positive constant depending only on the time T, being independent of ϵ.

It is worth pointing out that the idea in Part II is motivated in part by Tadmor's paper [8] although in our paper only the maximum principle is used, the E-condition of the forward problem used by Tadmor being unnecessary in our case.

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Part *I*. Holder continuous solutions

In this section we consider the existence of the global Holder continuous solution of the Cauchy problem $(1), (2)$. As discussed in the introduction, we first study, for any fixed $\epsilon > 0$, the existence of solutions to the Cauchy problem $(6), (7)$.

By the representations of w and z, for any fixed s, we can easily depict the lines $w = c_3 + \delta(\epsilon), w = c_2 + \delta(\epsilon), z = c_2 - \delta(\epsilon), z = c_1 - \delta(\epsilon)$ in the plane (ρ, u) as in Figure 1, where c_i $(i = 1, 2, 3)$ are constants satisfying $c_1 < c_2 < c_3$. Moreover we have

$$u = \frac{w + z}{2}, \qquad u + c = \frac{(\gamma + 1)w + (3 - \gamma)z}{4}, \qquad u - c = \frac{(\gamma + 1)z + (3 - \gamma)w}{4}.$$

Define

$$U = \{(w, z, s, w_x, z_x, s_x) \; : \; c_2 + \tfrac{1}{2}\delta(\epsilon) \leq w \leq c_3 + \tfrac{3}{2}\delta(\epsilon),$$

$$c_1 - \tfrac{3}{2}\delta(\epsilon) \leq z \leq c_2 - \tfrac{1}{2}\delta(\epsilon),$$

$$|s| \leq 2M, |w_x| \leq 2M, |z_x| \leq 2M, |s_x| \leq 2M\}$$

and consider the mapping T:

$$T \left\{ \begin{array}{c} w(x, t) \\ z(x, t) \\ s(x, t) \end{array} \right\} = \left\{ \begin{array}{c} w^0(x, t) \\ z^0(x, t) \\ s^0(x, t) \end{array} \right\}$$

$$+ \left\{ \begin{array}{c} \displaystyle\int_0^t ds \int_{-\infty}^{\infty} -\frac{(\gamma + 1)w(y, s) + (3 - \gamma)z(y, s)}{4} w_y(y, s)G(x - y, t - s)dy \\ \displaystyle\int_0^t ds \int_{-\infty}^{\infty} -\frac{(\gamma + 1)z(y, s) + (3 - \gamma)w(y, s)}{4} z_y(y, s)G(x - y, t - s)dy \\ \displaystyle\int_0^t ds \int_{-\infty}^{\infty} -\frac{w(y, s) + z(y, s)}{2} s_y(y, s)G(x - y, t - s)dy, \end{array} \right\}$$

where

$$w^0(x, t) = \int_{-\infty}^{\infty} w_0^\epsilon(y)dy, \qquad z^0(x, t) = \int_{-\infty}^{\infty} z_0^\epsilon(y)dy, \qquad s^0(x, t) = \int_{-\infty}^{\infty} s_0^\epsilon(y)dy$$

and G is the appropriate Green's function.

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By applying the contraction mapping principle to the map T, we can easily obtain the following local existence result for the solution to the Cauchy problem (6)(7); (see [5],[7]).

<u>Lemma 1</u> *Let $w_0(x), z_0(x), s_0(x)$ be bounded in C^1 space and satisfy*

$$c_2 \le w_0(x) \le c_3, \qquad c_1 \le z_0(x) \le c_2, \qquad |s_0(x)| \le M,$$

$$|w_{0x}(x)| \le M, \qquad |z_{0x}(x)| \le M, \qquad |s_{0x}(x)| \le M.$$

Then there exists a smooth solution for the Cauchy problem (6),(7) in some $R_S = (-\infty, \infty) \times [0, S)$, which satisfies

$$c_2 + \tfrac{1}{2}\delta(\epsilon) \le w(x,t) \le c_3 + \tfrac{3}{2}\delta(\epsilon),$$

$$c_1 - \tfrac{3}{2}\delta(\epsilon) \le z(x,t) \le c_2 - \tfrac{1}{2}\delta(\epsilon),$$

$$|s(x,t)| \le 2M, \qquad |w_x(x,t)| \le 2M,$$

$$|z_x(x,t)| \le 2M, \qquad |s_x(x,t)| \le 2M.$$

When we have a priori estimate in C^1 space, we can establish the global existence by using the local existence step by step. The framework given in [6] about the maximum principle of nonlinear parabolic system shows the following required estimate.

<u>Lemma 2</u> *Let the conditions in Lemma 1 be satisfied and in addition $w_{0x} \ge 0, z_{0x} \ge 0$. Suppose $(w(x,t), z(x,t), s(x,t))$ is a smooth solution of the Cauchy problem (6),(7) defined in a strip $(-\infty, \infty) \times [0, T]$ with $0 < T < \infty$. Then*

$$\left\{ \begin{array}{l} c_2 + \delta(\epsilon) \le w(x,t) \le c_3 + \delta(\epsilon), \\ c_1 - \delta(\epsilon) \le z(x,t) \le c_2 - \delta(\epsilon), \\ \qquad |s(x,t)| \le M, \end{array} \right. \tag{8}$$

$$0 \le w_x(x,t) \le M, \qquad 0 \le z_x(x,t) \le M, \qquad |s_x(x,t)| \le M. \tag{9}$$

Therefore we obtain the following global existence result.

<u>Theorem 3</u> *Let $w_0(x), z_0(x), s_0(x)$ be bounded in C^1 space and satisfy*

$$c_2 \le w_0(x) \le c_3, \qquad c_1 \le z_0(x) \le c_2, \qquad |s_0(x)| \le M,$$

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$$0 \le w_{0x}(x) \le M, \qquad 0 \le z_{0x}(x) \le M, \qquad |s_{0x}(x)| \le M.$$

Then the Cauchy problem (6), (7) *has a unique global smooth solution satisfying* (8), (9).

We now give the estimates of w_t, z_t and s_t. Let $X = w_t, Y = z_t, R = s_t$, then

$$
\begin{cases}
X \mid_{t=0} = w_t \mid_{t=0} = \left(\epsilon w_{xx} - \frac{(\gamma+1)w+(3-\gamma)z}{4} w_x \right) \mid_{t=0}, \\[2mm]
Y \mid_{t=0} = z_t \mid_{t=0} = \left(\epsilon z_{xx} - \frac{(3-\gamma)w+(\gamma+1)z}{4} z_x \right) \mid_{t=0}, \\[2mm]
R \mid_{t=0} = s_t \mid_{t=0} = \left(\epsilon s_{xx} - \frac{w+z}{2} s_x \right) \mid_{t=0}.
\end{cases}
\tag{10}
$$

Differentiating (6) with respect to t, we have

$$
\begin{cases}
X_t + \frac{(\gamma+1)w+(3-\gamma)z}{4} X_x + \frac{(\gamma+1)X+(3-\gamma)Y}{4} w_x = \epsilon X_{xx}, \\[2mm]
Y_t + \frac{(3-\gamma)w+(\gamma+1)z}{4} Y_x + \frac{(3-\gamma)X+(\gamma+1)Y}{4} z_x = \epsilon Y_{xx}, \\[2mm]
R_t + \frac{w+z}{2} R_x + \frac{X+Y}{2} s_x = \epsilon R_{xx}.
\end{cases}
\tag{11}
$$

Lemma 4 *Let $w_0(x), z_0(x), s_0(x)$ satisfy the conditions of Theorem 3. In addition, let $\{w_0(x), z_0(x), s_0(x)\} \in C^2$ and*

$$|X_0(x)| \le M, \qquad |Y_0(x)| \le M, \qquad |R_0(x)| \le M,$$

then

$$|X(x,t)| \le M, \qquad |Y(x,t)| \le M, \qquad |R(x,t)| \le M + M^2 T.$$

<u>Proof</u>. Make the transformations

$$-X = \bar{X} + M + \frac{N(x^2 + c_4 L e^t)}{L^2}, \qquad -Y = \bar{Y} + M + \frac{N(x^2 + c_4 L e^t)}{L^2}, \tag{12}$$

where c_4, N are positive constants and N is the upper bound, of w_t, z_t on $(-\infty, \infty) \times [0, T]$, dependent on ϵ. The functions $\bar{X}$ and $\bar{Y}$ are easily seen to

satisfy the equations

$$
\begin{cases}
\bar{X}_t + \dfrac{(\gamma+1)w + (3-\gamma)z}{4}\bar{X}_x + (\dfrac{\gamma+1}{4}\bar{X} - \dfrac{3-\gamma}{4}\bar{Y})w_x \\[2mm]
\quad + (c_4 Le^t + \dfrac{(\gamma+1)w + (3-\gamma)z}{2}x - 2\epsilon)\dfrac{N}{L^2} \\[2mm]
\quad + \dfrac{\gamma-1}{2}(M + \dfrac{N(x^2 + c_4 Le^t)}{L^2})w_x = \epsilon\bar{X}_{xx}, \\[4mm]
\bar{Y}_t + \dfrac{(3-\gamma)w + (\gamma+1)z}{4}\bar{Y}_x + (\dfrac{\gamma+1}{4}\bar{Y} - \dfrac{3-\gamma}{4}\bar{X})z_x \\[2mm]
\quad + (c_4 Le^t + \dfrac{(3-\gamma)w + (\gamma+1)z}{2}x - 2\epsilon)\dfrac{N}{L^2} \\[2mm]
\quad + \dfrac{\gamma-1}{2}(M + \dfrac{N(x^2 + c_4 Le^t)}{L^2})z_x = \epsilon\bar{Y}_{xx},
\end{cases}
\tag{13}
$$

resulting from the first and second equations of (11), and, moreover

$$
\begin{cases}
\bar{X}_0(x) = -X_0 - M - \frac{c_4 LN}{x^2} < 0, \quad \bar{Y}_0(x) = -Y_0 - M - \frac{c_4 LN}{x^2} < 0, \\[2mm]
\bar{X}(+L,t) < 0, \quad \bar{X}(-L,t) < 0, \quad \bar{Y}(+L,t) < 0, \quad \bar{Y}(-L,t) < 0.
\end{cases}
\tag{14}
$$

We have then from $(13), (14)$

$$
\bar{X}(x,t) < 0, \quad \bar{Y}(x,t) < 0, \text{ on } (-L,L) \times (0,T).
\tag{15}
$$

Since if (15) is not valid, then at least one of $\bar{X}$ and $\bar{Y}$, say $\bar{X}$, is non-positive at a point (x,t) in $(-L,L) \times (0,T)$. Let $\bar{t}$ be the least upper bound of values of t at which $\bar{X} > 0$, then by the continuity we see that $\bar{X} = 0$ at some point $(\bar{x}, \bar{t}) \in (-L,L) \times (0,T)$. So $\bar{X}_t \geq 0, \bar{X}_x = 0$ and $-\epsilon\bar{X}_{xx} \geq 0$ at $(\bar{x}, \bar{t})$, namely

$$
\bar{X}_t + \frac{(\gamma+1)w + (3-\gamma)z}{4}\bar{X}_x - \epsilon\bar{X}_{xx} \geq 0, \text{ at } (\bar{x}, \bar{t}).
\tag{16}
$$

But if we choose sufficiently large c_4 such that

$$
\begin{cases}
c_4 L + \frac{(\gamma+1)w + (3-\gamma)z}{2}x - 2\epsilon > 0, \\[2mm]
c_4 L + \frac{(\gamma+1)z + (3-\gamma)w}{2}x - 2\epsilon > 0
\end{cases}
$$

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on $[-L, L] \times (0, T)$, noticing that $\bar{Y} \leq 0$ at $(\bar{x}, \bar{t})$ and w_x, z_x are nonnegative, then the first equation of (13) gives a conclusion contradicting (16). So (15) is proved. Thus for any point (x_0, t_0) in $(-L, L) \times (0, T)$,

$$\begin{cases} X(x_0, t_0) \geq -(M + \frac{N(x_0^2 + c_4 L e^{t_0})}{L^2}), \\ \\ Y(x_0, t_0) \leq (M + \frac{N(x_0^2 + c_4 L e^{t_0})}{L^2}). \end{cases} \tag{17}$$

Letting $L \uparrow \infty$ in (17), we have $X(x, t) \geq -M, Y(x, t) \leq M$ on $(-\infty, \infty) \times (0, T)$. Similarly $X(x, t) \leq M, Y(x, t) \geq -M$.

To give the estimate of $R(x, t)$, we make the transformation

$$\bar{R} = R - bt - \frac{N(x^2 + L e^{c_4 t})}{L^2} - M$$

where $b = \max_{(-\infty, \infty) \times [0, T)} (\frac{X+Y}{2} s_x)$.

Noticing that $|s_x| \leq M, |X| \leq M, |Y| \leq M$, and using the same technique to prove X, Y, we obtain $R \leq bT + M \leq M^2 T + M$. Similarly we have $R \geq -M^2 T - M$ and Lemma 4 is proved.

We now give the Holder continuous solution of the Cauchy problem $(1), (2)$. First, the following estimates about (ρ, u, s) can be obtained from the above estimates.

Lemma 5 *If the conditions of Lemma 4 are satisfied, then*

$$\begin{cases} \dfrac{c_1 + c_2}{2} \leq u(x, t) \leq \dfrac{c_2 + c_3}{2}, \qquad |s(x, t)| \leq M, \\ \\ \sqrt{\dfrac{\gamma - 1}{4k}} \delta(\epsilon) e^{-\frac{M}{2\beta}} \leq \rho^{\frac{\gamma-1}{2}} \leq \sqrt{\dfrac{\gamma - 1}{16k}} (c_3 - c_1 - 2\delta(\epsilon)) e^{\frac{M}{2\beta}}, \\ \\ 0 \leq u_x \leq M, \quad |s_x| \leq M, \quad |\rho^{\frac{\gamma-3}{2}} \rho_x| \leq \sqrt{\dfrac{1}{4k(\gamma - 1)}} (1 + \dfrac{\delta(\epsilon)}{2\beta}) M e^{\frac{M}{2\beta}}, \\ \\ |u_t| \leq M, \qquad |s_x| \leq M + M^2 T, \\ \\ |\rho^{\frac{\gamma-3}{2}} \rho_t| \leq \sqrt{\dfrac{1}{4k(\gamma - 1)}} (1 + \dfrac{\delta(\epsilon)}{2\beta} (1 + MT)) M e^{\frac{M}{2\beta}}. \end{cases}$$
$$\tag{18}$$

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Observing that $\delta(\epsilon)$ is a positive, bounded function of ϵ and $\gamma - 3 \le 0$, we constructed a sequence of the approximate solutions

$$\{\rho^\epsilon, u^\epsilon, s^\epsilon\} \in W^{1,\infty}((-\infty, \infty) \times [0, T])$$

for $0 < T < \infty$, which, by the embedding theorem, have a subsequence $\{\rho^{\epsilon_n}, u^{\epsilon_n}, s^{\epsilon_n}\}$ on any bounded regions Ω of $(-\infty, \infty) \times [0, \infty)$, converging uniformly to a triplet of Holder continuous functions $(\rho(x,t), u(x,t), s(x,t))$. We are going to prove that the limit functions $(\rho(x,t), \rho(x,t)u(x,t), s(x,t))$ is indeed the solution of the Cauchy problem $(1), (2)$.

We can rewrite (6) as follows:

$$\begin{cases} \rho_t + (\rho u)_x = \dfrac{\epsilon \rho^{\frac{3-\gamma}{2}} e^{-\frac{s}{2\beta}}}{\sqrt{4k(\gamma - 1)}} (w - z)_{xx}, \\[3mm] (\rho u)_t + (\rho u^2)_x + \rho(k e^{\frac{s}{\beta}} \rho^{\gamma-1})_x = \dfrac{\epsilon \rho}{2}(w + z)_{xx} \\[3mm] \qquad\qquad\qquad\qquad + \dfrac{\epsilon u \rho^{\frac{3-\gamma}{2}} e^{\frac{-s}{2\beta}}}{\sqrt{4k(\gamma - 1)}} (w - z)_{xx}, \\[3mm] s_t + u s_x = \epsilon s_{xx}. \end{cases} \qquad (19)$$

For the case of $1 < \gamma \le 2$, taking $\delta(\epsilon) \equiv 0$, we have

$$|(\rho^{\frac{\gamma-3}{2}})_x| = |\frac{\gamma - 3}{2} \rho^{2-\gamma} \rho^{\frac{\gamma-3}{2}} \rho_x| \le M.$$

In the case of $2 < \gamma < 3$, we choose $\delta(\epsilon) = \epsilon^l$, where $0 < l < \frac{\gamma-1}{2(\gamma-2)}$, then

$$|(\rho^{\frac{\gamma-3}{2}})_x| \le M \rho^{2-\gamma} \le \delta(\epsilon)^{\frac{2(2-\gamma)}{\gamma-1}} \le M \epsilon^{l \frac{2(2-\gamma)}{\gamma-1}}.$$

and so

$$\epsilon(\rho^{\frac{\gamma-3}{2}})_x \to 0 \text{ as } \epsilon \to 0.$$

Noticing that $\{\rho^{\epsilon_n}, u^{\epsilon_n}, s^{\epsilon_n}\}$ converges to $(\rho, \rho u, s)$ in $W^{1,\infty}$ on any bounded region $\Omega \subset (-\infty, \infty) \times [0, \infty)$, we get immediately from (19)

$$\begin{cases} \iint_{t\geq 0} \rho\phi_t + \rho u\phi_x \, dx\,dt + \int_{t=0} \rho_0\phi\,dx = 0, \\[2mm] \iint_{t\geq 0} \rho u\phi_t + \rho u^2\phi_x + \rho(ke^{\frac{s}{\beta}}\rho^{\gamma-1})_x\phi\,dx\,dt + \int_{t=0} \rho_0 u_0\phi\,dx = 0, \\[2mm] \iint_{t\geq 0} s\phi_t + us_x\phi\,dx\,dt + \int_{t=0} s_0\phi\,dx = 0, \end{cases} \qquad (20)$$

for all $\phi \in C_0^1((-\infty,\infty) \times [0,\infty))$. Therefore $(\rho, \rho u, s)$ is the Holder continuous solution of the Cauchy problem $(1),(2)$. If we smooth the data by a mollifier, the following theorem is obtained.

Theorem 6 *Let* $(\rho_0(x), u_0(x), s_0(x)) \in C^1$ *satisfy*

$$c_2 \leq w_0 \leq c_3, \qquad\qquad c_1 \leq z_0 \leq c_2, \qquad\qquad |s_0(x)| \leq M,$$

$$0 \leq w_{0x} \leq M, \qquad\qquad 0 \leq z_{0x} \leq M, \qquad\qquad |s_{0x}(x)| \leq M.$$

Then the Cauchy problem $(1),(2)$ *has a global Holder continuous solution* $(\rho, \rho u, s)$ *defined by* (20).

Part *II*. Error estimates

In this part of the paper, we consider the error estimates between the exact solution (ρ, u, s) and the viscosity solutions $(\rho^\epsilon, u^\epsilon, s^\epsilon)$.

Lemma 7 *Let the conditions in Theorem 6 be satisfied and in addition,*

$$|w_{0xx}| \leq M, \qquad\qquad |z_{0xx}| \leq M, \qquad\qquad |s_{0xx}| \leq M,$$

then the solutions $(w^\epsilon, z^\epsilon, s^\epsilon)$ *of the Cauchy problem* $(6),(7)$ *satisfy* $(8),(9)$ *and*

$$|w_{0xx}^\epsilon| \leq M, \qquad\qquad |z_{xx}^\epsilon| \leq M, \qquad\qquad |s_{xx}^\epsilon| \leq M + M^2 T. \qquad (21)$$

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Proof. To prove (21), we differentiate (6) twice with respect to x, and let $A = w_{xx}, B = z_{xx}, C = s_{xx}$, then

$$
\begin{cases}
A_t + \dfrac{(\gamma+1)w + (3-\gamma)z}{4}A_x \\
\quad + \dfrac{3(\gamma+1)w_x + 2(3-\gamma)z_x}{4}A + \dfrac{(3-\gamma)w_x}{4}B = \epsilon A_{xx}, \\[2mm]
B_t + \dfrac{(\gamma+1)z + (3-\gamma)w}{4}B_x \\
\quad + \dfrac{3(\gamma+1)z_x + 2(3-\gamma)w_x}{4}B + \dfrac{(3-\gamma)z_x}{4}A = \epsilon B_{xx}, \\[2mm]
C_t + \dfrac{w+z}{2}C_x + (w_x + z_x)C + \dfrac{A+B}{2}s_x = \epsilon C_{xx}.
\end{cases}
$$

In a similar way to the proof of Lemma 4, we can obtain the estimates (21). The details are omitted.

Let $(w^{\epsilon_2}, z^{\epsilon_2}, s^{\epsilon_2})$ and $(w^{\epsilon_1}, z^{\epsilon_1}, s^{\epsilon_1})$ be the solutions of the Cauchy problem (6), (7) with respect to $\epsilon = \epsilon_2, \epsilon = \epsilon_1$, we have

Lemma 8 *Let the conditions in Lemma 7 be satisfied, then*

$$
\begin{cases}
|w^{\epsilon_2} - w^{\epsilon_1}| \le \delta(\epsilon_2) - \delta(\epsilon_1) + \epsilon_1 MT, \\[2mm]
|z^{\epsilon_2} - z^{\epsilon_1}| \le \delta(\epsilon_2) - \delta(\epsilon_1) + \epsilon_1 MT, \\[2mm]
|s^{\epsilon_2} - s^{\epsilon_1}| \le lT,
\end{cases}
\tag{22}
$$

where $l = (\delta(\epsilon_2) - \delta(\epsilon_1) + \epsilon_1 MT)M + \epsilon_1(M + M^2 T)$.

Proof. Let $E = w^{\epsilon_2} - w^{\epsilon_1}, F = z^{\epsilon_2} - z^{\epsilon_1}, G = s^{\epsilon_2} - s^{\epsilon_1}$, then

$$
\begin{cases}
E_t + \dfrac{(\gamma+1)w^{\epsilon_2} + (3-\gamma)z^{\epsilon_2}}{4}E_x + \dfrac{(\gamma+1)E + (3-\gamma)F}{4}w_x^{\epsilon_1} + \epsilon_1 w_{xx}^{\epsilon_1} = \epsilon_2 E_{xx}, \\[2mm]
F_t + \dfrac{(\gamma+1)z^{\epsilon_2} + (3-\gamma)w^{\epsilon_2}}{4}F_x + \dfrac{(\gamma+1)F + (3-\gamma)E}{4}z_x^{\epsilon_1} + \epsilon_1 z_{xx}^{\epsilon_1} = \epsilon_2 F_{xx}, \\[2mm]
G_t + \dfrac{w^{\epsilon_2} + z^{\epsilon_2}}{2}G_x + \dfrac{E+F}{2}s_x^{\epsilon_1} + \epsilon_1 s_{xx}^{\epsilon_1} = \epsilon_2 G_{xx},
\end{cases}
\tag{23}
$$

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resulting from system (6) with respect to $\epsilon = \epsilon_2, \epsilon = \epsilon_1$, and with initial data

$$E(0,t) = \delta(\epsilon_2) - \delta(\epsilon_1), \qquad F(0,t) = \delta(\epsilon_2) - \delta(\epsilon_1), \qquad G(0,t) = 0.$$

In a similar way as the proof of Lemma 4, we can obtain the estimates of E, F by making the transformation

$$\begin{cases} \mp E = \bar{E} + \delta(\epsilon_2) - \delta(\epsilon_1) + \frac{N(x^2 + cLe^t)}{L^2} + \epsilon_1 MT, \\[2mm] \pm F = \bar{F} + \delta(\epsilon_2) - \delta(\epsilon_1) + \frac{N(x^2 + cLe^t)}{L^2} + \epsilon_1 MT, \end{cases} \tag{24}$$

and the estimates of G by making the transformation

$$\bar{G} = G \pm lt \pm \frac{N(x^2 + cLe^t)}{L^2}.$$

The details are omitted.

Since we can choose $\delta(\epsilon) = 0$ when $1 < \gamma \le 2$, or $\gamma = 3$ and $\delta(\epsilon) = \epsilon$ when $2 < \gamma < 3$, letting $\epsilon_1 \downarrow 0, \epsilon_2 = \epsilon$ in (22), we obtain the following main theorem.

<u>Theorem</u> 9 *Let the conditions in Lemma 7 be satisfied, then*

$$|u^\epsilon - u| \le M(T)\epsilon, \qquad |\rho^\epsilon - \rho| \le M(T)\epsilon, \qquad |s^\epsilon - s| \le M(T)\epsilon,$$

where $M(T)$ is a positive constant depending only on the time T, but independent of ϵ.

Remark. The result in this paper can be easily extended to the more general hyperbolic system(for some suitable $p(\rho)$)

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2)_x + \rho(e^{\frac{s}{\beta}} p(\rho))_x = 0 \\ s_t + u s_x = 0, \end{cases}$$

where the nonstrictly hyperbolic line $\rho = 0$ is also permitted. Further generalization to the non-isentropic equation (3) of polytropic gas is under investigation.

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