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A Note on Damped Algebraic Riccati Equations*

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Abstract

In a recent paper, an algorithm that produces dampening controllers based on a periodic Hamiltonian was proposed. Central to this algorithm is the formulation of symmetric and skew-symmetric damped algebraic Riccati equations. It was shown that solutions to these two Riccati equations lead to a dampening feedback, *i.e.*, a stable closed-loop system for which the real parts of the eigenvalues are larger in modulus than the imaginary parts.

In this paper, we extend these results to include a broader class of Hermitian and skew-Hermitian solutions and show that every convex combination of these solutions produces a dampening feedback. This property can be used to vary the feedback with two parameters and thus obtain more flexibility in the controller design process.

Keywords: dampening feedback, damped system dynamics, periodic Schur decomposition, periodic Riccati equation, Hamiltonian systems, linear quadratic control.

1 Introduction

This paper is concerned with the problem of finding a feedback for a linear time-invariant system for which the closed-loop system poles are constrained to cones in the left half-plane. At first glance, it seems that this problem may be trivially solved by pole placement; however, this may not be desirable, especially for large systems, as pole placement has been shown to be an inherently ill-conditioned problem [2, 9]. While this problem may be satisfactorily solved in a polynomial framework [4], the formulation in a state-space framework had until just recently eluded simple formulation. The additional effort to formulate the problem in state-space was motivated by the desire to take advantage of the inherent numerical robustness of state-space methods over polynomial methods [8].

The first steps towards a simple state-space solution to this problem was the introduction of the *Damped Riccati Algorithm* (DRA) in a recent paper [3]. This algorithm functions in the following way: Given the standard linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{1}$$

with $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, and $C \in \mathbf{R}^{n,\ell}$, the DRA, under most circumstances, will compute a matrix $F \in \mathbf{R}^{m,n}$ such that the eigenvalues of the closed-loop matrix $A + BF$ are *damped*, *i.e.*, the eigenvalues of the closed-loop system have real parts greater in magnitude than their imaginary parts. Integral to this algorithm is the

solution of the *Symmetric Damped Algebraic Riccati Equation (SDARE)*

$$X(A^2 - RS) + (A^{2T} - SR)X - X(AR + RA^T)X + (A^T S + SA) = 0, \quad (2)$$

and the *Skew-Symmetric Damped Algebraic Riccati Equation (SSDARE)*

$$Y(A^2 + RS) + (A^{2T} + SR)Y - Y(AR - RA^T)Y + (A^T S - SA) = 0, \quad (3)$$

as well as the more standard [1] *Shifted Algebraic Riccati Equation (SHARE)*

$$N(A + \sigma I) + (A^T + \sigma I)N - NRN + S = 0. \quad (4)$$

Here, as for the rest of the paper $R := BB^T$ and $S := C^T C$. It was shown in [3] that if the SDARE and SSDARE have stabilising solutions, then the two feedbacks $F_X = -B^T X$ and $F_Y = -B^T Y$ will produce closed-loop systems that are damped. Since the SDARE is a standard Riccati equation, standard solvability theory [6] can be applied to characterise when a stabilising solution exists. The close relationship between the solvability of the SSDARE and the SDARE described in [3] also provides the theoretical basis for the solvability of the SSDARE. Once a system is dampened, it may be stabilised by computing an additional feedback which symmetrically reflects the poles across the imaginary axis. This is accomplished by computing the stabilising solution to a degenerate SHARE with $S = 0$ and $\sigma = 0$. If $\sigma > 0$, then the closed-loop eigenvalues will not only be stable and damped, but have guaranteed a degree of stability σ , in effect blunting the cone in the left half-plane which contains the system poles.

In this paper, we introduce two parameters: one for the modification of the SDARE and SSDARE and one for the convex combination of the solutions of these two equations, and show that we again obtain dampening feedbacks.

2 Parameterised Damped Riccati Equations

In [3], it was shown that the solutions to the SDARE and the SSDARE are related to the stable invariant subspaces of the matrix product $H_X = H_2 H_1$ and $H_Y = H_1 H_2$, respectively, where

$$H_1 := \begin{bmatrix} A & -R \\ S & A^T \end{bmatrix}, \quad H_2 := \begin{bmatrix} -A & R \\ S & A^T \end{bmatrix}. \quad (5)$$

If the columns of the $2n \times n$ matrix

$$T_s = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}, \quad (6)$$

span the invariant subspace corresponding to the stable eigenvalues of the $2n \times 2n$ Hamiltonian matrix H_X , then the symmetric stabilizing solution X of the SDARE (2) (if it exists) is given by

$$X = T_{21}T_{11}^{-1}. \quad (7)$$

An analogous result holds for Y .

We will now introduce two parameters, one which scales the two equations and one which defines a convex combination of the solutions. Let

$$D := \begin{bmatrix} I & \\ & \alpha I \end{bmatrix}$$

where $\alpha = e^{i\phi}$, $\phi \in [-\pi/4, \pi/4]$, and consider the scaled matrices

$$H_1(\alpha) = DH_1D^{-1}, \quad H_2(\alpha) = D^{-1}H_2D.$$

as well as

$$H_Z(\alpha) = H_2(\alpha)H_1(\alpha), \quad H_W(\alpha) = H_1(\alpha)H_2(\alpha). \quad (8)$$

Associated with these products are the *Hermitian Damped Algebraic Riccati Equation (HDARE)*

$$\begin{aligned} Z(A^2 - \alpha^2RS) + (A^{2T} - \bar{\alpha}^2SR)Z \\ - Z(\bar{\alpha}AR + \alpha RA^T)Z + (\alpha A^T S + \bar{\alpha}SA) = 0, \end{aligned} \quad (9)$$

and the *Skew-Hermitian Damped Algebraic Riccati Equation (SHDARE)*

$$\begin{aligned} W(A^2 + \bar{\alpha}^2RS) + (A^{2T} + \alpha^2SR)W \\ - W(\alpha AR - \bar{\alpha}RA^T)W + (\bar{\alpha}A^T S - \alpha SA) = 0, \end{aligned} \quad (10)$$

respectively.

The HDARE and the SHDARE have a very close relationship with the SDARE and the SSDARE; namely, the matrices $H_1(\alpha)$ and $H_1(1)$ are similar for all α and analogously the matrices $H_2(\alpha)$ and $H_2(1)$ are similar for all α . This allows the relations between Z and W to be derived analogously to those between X and Y . In fact, the solution Z of (9) and W of (10) for the same α on the unit circle satisfy

$$A^T Z - WA + \bar{\alpha}WRZ + \alpha S = 0. \quad (11)$$

This equation is central to proving that both $-(A - \alpha RW)(A - \bar{\alpha}RZ)$ and $-(A - \bar{\alpha}RZ)(A - \alpha RW)$ are stable and that $(A - \alpha RW)$ and $(A - \bar{\alpha}RZ)$ are damped.

As it happens, α is not the only parameter that may be varied without affecting system dampedness. Any convex combination of the two complementary solutions $\bar{\alpha}Z$ and αW produces a dampening controller as well. This is shown in the following theorem.

Theorem 1 *Let*

$$K = \beta\bar{\alpha}Z + (1 - \beta)\alpha W \quad (12)$$

with Z, W being stabilising solutions of (9) and (10), respectively. For $\alpha = e^{i\phi}$, $\phi \in (-\pi/4, \pi/4)$ and $\beta \in [0, 1]$ the eigenvalues of $A - RK$ are within the closure of the damped region of the complex plane (excluding the point 0) (see Figure 1).

Proof. Consider the Hermitian matrices

$$M_Z = -(\alpha Z + \bar{\alpha}W)(A - \bar{\alpha}RZ) = ZRZ - \bar{\alpha}A^T Z - \alpha ZA - S$$

and

$$M_W = -(\alpha Z + \bar{\alpha}W)(A - \alpha RW) = WRW + \alpha A^T W - \bar{\alpha}WA + S.$$

Lyapunov's Theorem [7] shows (by analogy from Theorem 1 of [3]) that the stability of $-(A - \bar{\alpha}RZ)(A - \alpha RW)$ implies that M_Z and M_W are positive definite whenever S is positive definite. Introduce

$$\begin{aligned} M_K &:= \beta M_Z + (1 - \beta)M_W \\ &= -(\alpha Z + \bar{\alpha}W)(A - RK) \\ &= -(A - RK)^H(\bar{\alpha}Z - \alpha W), \end{aligned} \quad (13)$$

where K is defined as in (12). Following the proof of Theorem 1 in [3], dampedness is proved if the following two conditions hold:

$$\begin{aligned} P_Z &= -(A - RK)^{2H}M_K - M_K(A - RK)^2 < 0 \\ M_K &> 0. \end{aligned} \quad (14)$$

Attending to the latter condition first, we note that since M_Z and M_W are positive definite then M_K will be positive definite for $\beta \in [0, 1]$. With the substitution of the expressions in (13) for M_K in the first equation of (14) and $\bar{\alpha}^2 + \alpha^2 = 2 \cos 2\phi > 0$ for ϕ in $(-\pi/4, \pi/4)$, we have that

$$\begin{aligned} P_Z &= -(A - RK)^{2H}M_K - M_K(A - RK)^2 \\ &= (A - RK)^H \left((A - RK)^H(\alpha Z + \bar{\alpha}W) + \right. \\ &\quad \left. (\bar{\alpha}Z - \alpha W)(A - RK) \right) (A - RK) \\ &= -(\bar{\alpha}^2 + \alpha^2)(A - RK)^H \left(S + \beta ZRZ + (1 - \beta)W^H RW \right) (A - RK) \\ &< 0 \end{aligned} \quad (15)$$

whenever S is positive definite. As in Theorem 1 of [3], it follows that the eigenvalues of $A - RK$ are in the interior of the damped region of the complex plane. If S is only positive semidefinite then by continuity it follows that the eigenvalues are in the closed damped region of the complex plane. \square

Remark 1 The two nominal solutions with regard to the parameterisation α , $Z(1)$ and $W(1)$, are real and correspond to X and Y in [3].

Remark 2 Consider equation (11) again. We can identify a simple relationship between two solutions of the HDARE and the SHDARE corresponding to the same parameter α ; namely that

$$Z(\alpha) = iW(i\bar{\alpha}). \quad (16)$$

Thus one family of solutions determines the other. It is noteworthy that for $\alpha_0 = e^{i\phi}$ with $\phi = \pm\pi/4$, stability may not be proved by Theorem 1. Observe, however, that for $\alpha_0 = e^{i\phi}$ with $\phi = \pi/4$, we have $i\bar{\alpha}_0 = \alpha_0$; thus by (16), $Z(\alpha_0) = iW(\alpha_0)$. Equation (11) can therefore be written as the “rotated” Riccati equation

$$\bar{\alpha}_0 A^T Z + \alpha_0 Z A - Z R Z + S = 0. \quad (17)$$

Similarly, for $\alpha_0 = e^{i\phi}$ with $\phi = -\pi/4$, we have $i\bar{\alpha}_0 = -\alpha_0$. Utilising the relationship between $Z(\alpha)$ and $W(\alpha)$ in (16), we have $Z(\alpha_0) = -iW(\alpha_0)$, obtaining the same “rotated” Riccati equation.

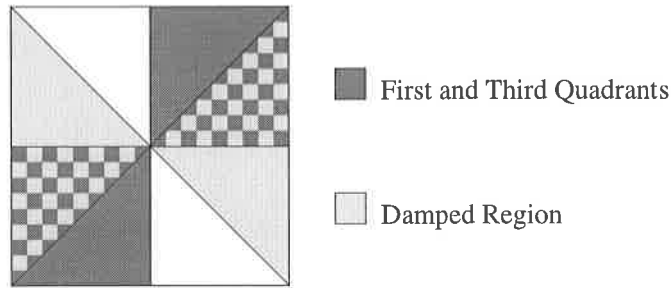


Figure 1: Regions in the Complex Plane

This “rotated” Riccati equation (17) functions as follows: Let

$$H_r = \begin{bmatrix} \alpha_0 A & -R \\ -S & -\bar{\alpha}_0 A^T \end{bmatrix}$$

where $\alpha_0 = e^{i\pi/4}$. Due to the Hamiltonian structure of iH_r^2 , half of the eigenvalues of H_r are in the first and third quadrants of the complex plane and the other half are in the second and fourth quadrants. Suppose that $(Z_1^T, Z_2^T)^T$ is a basis of the invariant subspace of H_r with respect to all of the eigenvalues in the first and third quadrants of the complex plane, that is,

$$H_r \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} T$$

where T is a $n \times n$ matrix with eigenvalues in the first and third quadrants. If Z_1 is nonsingular, then $Z = Z_2 Z_1^{-1}$ is a solution of (17). Moreover, the eigenvalues of the closed-loop matrix $\alpha_0 A - RZ$ are in the first and third quadrants. Since the transformation $\lambda \rightarrow \lambda/\alpha_0$ maps the first and third quadrants complex plane to the damped region (v. Figure 1), it follows that the eigenvalues of $A - \bar{\alpha}_0 RZ$ are in the damped region of the complex plane.

Remark 3 Despite the fact that a complex feedback is unrealisable for real systems, numerical experiments show for the vast majority of randomly generated cases (999 out of 1000) $A - R * \text{real}(\bar{\alpha}Z)$ is damped. Indeed, for the two-by-two SISO case the dampedness of $A - R * \text{real}(\bar{\alpha}Z)$ may be shown analytically.

3 Examples

In this section, we illustrate the effect of the parameters α and β on closed-loop eigenvalues. The system matrix A is constructed such that its eigenvalues are in the undamped region. We show how these eigenvalues are transferred to the damped region via solving the damped Riccati equation.

Example 1 In this example, we demonstrate the properties of the Hermitian and Skew-Hermitian DAREs when performed under complex arithmetic for scalar inputs, as was done in Example 1 in [3]. Let

$$A = a_r + ia_i \quad , \quad B = b_r + ib_i \quad , \quad \alpha = e^{i\phi}.$$

In complex arithmetic, we replace the transposition operation with the complex-conjugation operation, and we seek to find the complex scalars Z and W which form the optimal closed-loop scalars

$$\tilde{A}_Z = A - BB^H(\bar{\alpha}Z) \quad , \quad \tilde{A}_W = A - BB^H(\alpha W).$$

It may be easily confirmed, with $S = 0$ that

$$Z = \frac{(a_r^2 - a_i^2)}{(a_r \cos \phi + a_i \sin \phi)(b_r^2 + b_i^2)} \quad , \quad W = \frac{-i(a_r^2 - a_i^2)}{(a_i \cos \phi + a_r \sin \phi)(b_r^2 + b_i^2)},$$

$$\tilde{A}_Z = (a_i + ia_r)\rho \quad , \quad \tilde{A}_W = (a_i + ia_r)/\rho,$$

where

$$\rho = \frac{a_i \cos \phi + a_r \sin \phi}{a_r \cos \phi + a_i \sin \phi}.$$

Thus we obtain

$$\left(\beta\tilde{A}_Z + (1 - \beta)\tilde{A}_W\right) = (a_i + ia_r)(\beta\rho + (1 - \beta)/\rho)$$

implying that

$$|\operatorname{real}\left(\beta\tilde{A}_Z + (1 - \beta)\tilde{A}_W\right)| > |\operatorname{imag}\left(\beta\tilde{A}_Z + (1 - \beta)\tilde{A}_W\right)|$$

for $|a_i| > |a_r|$, $\beta \in [0, 1]$, $\phi \in [-\pi/4, \pi/4]$.

Example 2 In this example, we validate the assertions in Remark 3. We start with a two by two matrix in controllable canonical form. Let us further write the system matrix in a form akin to that which describes a standard second order system in terms of its natural frequency ω and its damping factor ζ [5]:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\omega\zeta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ R &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{18}$$

Note that an undamped system as defined in this paper has $\zeta \in [0, 1/\sqrt{2}]$. Let us assume that ϕ and ζ are in the following interval $[-\pi/4, \pi/4]$ and

$$\frac{1}{2\cos\phi} < \zeta < \frac{1}{\sqrt{2}}, \tag{19}$$

so that the eigenvalues of A are in the undamped region.

As this system is relatively simple, it is possible to get closed-form expressions for Z and W

$$\begin{aligned} Z &= \frac{2\omega(2\zeta^2-1)}{1-\zeta^2(\alpha+\bar{\alpha})^2} \begin{bmatrix} (\alpha+\bar{\alpha})\omega^2\zeta & \bar{\alpha}\omega \\ \alpha\omega & (\alpha+\bar{\alpha})\zeta \end{bmatrix}, \\ W &= \frac{2\omega(2\zeta^2-1)}{1+\zeta^2(\alpha-\bar{\alpha})^2} \begin{bmatrix} -(\alpha-\bar{\alpha})\omega^2\zeta & -\alpha\omega \\ \bar{\alpha}\omega & -(\alpha-\bar{\alpha})\zeta \end{bmatrix}. \end{aligned} \tag{20}$$

As mentioned earlier, the aim of this example is to illustrate the effect of the feedback on the projection of the *complex-valued* feedback matrix onto the field of *real numbers*. The paradigm of the parameterisation of a second order system in terms of its natural frequency and damping factor is again exploited; the closed-loop

matrix $A - R \text{ real}(\bar{\alpha}Z)$ will be described in terms of ω_Z and ζ_Z , whose definitions follow from those of (18). The closed-loop matrix $A - R \text{ real}(\alpha W)$ will likewise be described in terms of ω_W and ζ_W . These values are

$$\begin{aligned}\omega_Z &= \omega \sqrt{\frac{(1-4\zeta^2 \sin^2 \phi)}{(4\zeta^2 \cos^2 \phi - 1)}} , & \zeta_Z &= \frac{\zeta \cos 2\phi}{\sqrt{4\zeta^2 - 1 - 4\zeta^4 \sin^2 2\phi}}, \\ \omega_W &= \omega \sqrt{\frac{(4\zeta^2 \cos^2 \phi - 1)}{(1-4\zeta^2 \sin^2 \phi)}} , & \zeta_W &= \frac{\zeta \cos 2\phi}{\sqrt{4\zeta^2 - 1 - 4\zeta^4 \sin^2 2\phi}}.\end{aligned}\tag{21}$$

Under the condition (19), values of ω_Z , ω_W , ζ_Z and ζ_W are well defined. Moreover

$$\zeta_Z^2 = \frac{\zeta^2 (\cos 2\phi)^2}{4\zeta^2 - 1 - 4\zeta^4 \sin^2 2\phi}$$

is a decreasing function on ζ , since its derivative

$$\frac{-2\zeta(1 - 4\zeta^4(\sin 2\phi)^2)}{(4\zeta^2 - 1 - 4\zeta^4 \sin^2 2\phi)^2}$$

is less than or equal to zero. Thus $\zeta = 1/\sqrt{2}$ minimises the function of ζ_Z^2 , i.e.

$$\zeta_Z \geq \zeta_Z\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

This proves that the eigenvalues of \tilde{A}_Z and \tilde{A}_W are in the damped region. When ζ is in the interval defined by (19), the eigenvalues of \tilde{A}_Z and \tilde{A}_W are not only in the damped region, but also stable. When $0 \leq \zeta < 1/(2 \cos \phi)$, however, the eigenvalues \tilde{A}_Z and \tilde{A}_W are still in the damped region but no longer stable. In this case the form defined by (18) is no longer valid.

4 Concluding Remarks

In this paper we have extended results of [3] on damped Riccati equations to include a family of dampening feedbacks that may be parameterised over two variables. This parameterisation may be of significance as it allows additional degrees of freedom with which other control criteria in addition to dampedness may be achieved.

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