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Galerkin Methods for Coupled Integral Equations

by

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Abstract

This thesis extends Burton's work [1] on the re-iterated Galerkin (Porter & Stirling [5]) and re-iterated Kantorovich methods by outlining both methods[1] for the first time to coupled integral equations. A wave scattering example (Chamberlain [2, 3]) is transformed into a coupled system and the new re-iterated methods are tested. Results show, as they did with Burton's work on single integral equations, that the re-iterated Kantorovich method fare better against the re-iterated Galerkin method, particularly for 1,2 or 3 dimensional subspaces. With a 1 dimensional subspace and after 8 and 5 iterations using the re-iterated Galerkin and re-iterated Kantorovich methods respectively, solutions differ by no more than 1.0×10^{-7}. With a 6 dimensional subspace only 1 iteration for both the re-iterated methods is needed under the same tolerance. Results are tablulated as are estimates for the spectral radii of operators that govern convergence rates. Other possiblities concerning choice of subspace and technique are explored.

[1]These re-iterated methods were built on Sloan's iterated Galerkin method [6] and allows one to 're-iterate' and obtain solutions to within a specified tolerance.

Contents

List of Symbols

f free term

$k(x,t)$ kernel

H Hilbert space

K kernel operator

P, P_1, P_2 orthogonal projection operators

E, E_{N_1}, E_{N_2} subspaces of H

$L_2^n(a,b)$ space of n-vectors of square integrable functions

$\{\chi_1^{(i)}, \chi_2^{(i)}, \ldots\}$ basis functions spanning E_{N_i}

$\mathbf{f}$ free term for coupled systems

$\mathbf{p}_0 = (p_1, p_2)^T$ initial Galerkin approximations

$\hat{\mathbf{p}}_0 = (\hat{p}_1, \hat{p}_2)^T$ initial iterated Galerkin approximations

$\langle \cdot, \cdot \rangle$ inner product

$\mathbf{r}_n$ non-iterated residual

$\hat{\mathbf{r}}_n$ residual or residual error

$\| \cdot \|$ norm

Chapter 1

Introduction

Throughout this entire thesis we will focus our attention to integral equations of the second kind. These equations take the following form

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)dt \qquad a \leq x \leq b \tag{1.1}$$

where f is the free term, λ a scalar, $k(x,t)$ is the kernel and $a, b \in \mathbf{R}$. We seek ϕ. Often integral equations of the second kind are shortened to

$$\phi = f + \lambda K\phi, \tag{1.2}$$

when they are regarded as equations in a Hilbert space H, ϕ and f being elements of the space and K an operator on the space.

Integral equations, for example

$$\phi(x) = 1 + \int_0^1 2\ \max(x,t)\phi(t)dt \qquad 0 \leq x \leq 1 \tag{1.3}$$

and

$$\phi(x) = \cos(\kappa_0 x) + \frac{1}{2\kappa_0}\int_0^1 \sin(\kappa_0 \mid x - t \mid)\phi(t)dt \qquad 0 \leq x \leq 1 \tag{1.4}$$

where κ_0 is a real parameter, were dealt with in Burton's thesis [1]. The first of these equations has a simple analytic solution. The second of these came from Chamberlain's work [2, 3] on wave scattering. One may be wondering just where

on earth did these equations come from, so to illustrate we will consider (1.3). On its expansion we have

$$\phi(x) = 1 + \int_0^x 2x\phi(t)dt + \int_x^1 2t\phi(t)dt. \tag{1.5}$$

Following swiftly with a differentiation with respect to x

$$\phi'(x) = \int_0^x 2\phi(t)dt + 2x\phi(x) - 2x\phi(x) \tag{1.6}$$

and on differentiating once more gives

$$\phi''(x) = 2\phi(x).$$

For boundary conditions, from (1.5) and (1.6) we find that

$$\begin{aligned} \phi'(0) &= 0 \\ \phi(1) - \phi'(1) &= 1. \end{aligned}$$

Hence (1.3) actually comes from the boundary value problem as described whose analytic solution is

$$\phi(x) = \frac{\cosh x\sqrt{2}}{\cosh\sqrt{2} - \sqrt{2}\sinh\sqrt{2}} \quad .$$

Integral equations offer an alternative way to tackle boundary value problems. The well-established Galerkin method and the Kantorovich method, a well-known adaptation of the Galerkin method, gave way to the iterated and re-iterated methods devised by Sloan (iterated Galerkin), Porter & Stirling (re-iterated Galerkin) and Burton[1] (iterated and re-iterated Kantorovich). In his thesis Burton primarily worked with the re-iterated methods comparing his results between methods. Analysis has shown that convergence of solution using the various methods (except the re-iterated Kantorovich method) would occur provided that the norm of the operator S, where

$$S = (I - KP)^{-1}(K - KP), \tag{1.7}$$

[1] Burton was supervised by Porter.

is less than 1. Here P denotes the orthogonal projection onto a given subspace E of our solution space H and convergence is with respect to the norm. Initially, the norm of S may not be less than one. In this case we would increase the size of our subspace, the reason being that $\| K - KP \| \rightarrow 0$ as $n \rightarrow \infty$, where n is the dimension of the subspace, and hope that this increase is sufficient to make $\| S \|$ less than 1. With the re-iterated Kantorovich method a different condition has to be satisfied namely that $\| SK \| < 1$ where S is as above though, as we know that

$$\| SK \| \leq \| S \| \| K \|,$$

this results once again in the need to expand our subspace[2]. The work by Burton confirmed much of the analysis on the re-iterated Galerkin and Kantorovich methods for single integral equations. The prospect of larger systems of integral equations and the corresponding analysis looms ahead. It may seem that this will just be a case of writing a program for decoupled systems of integral equations. In other words the program would simply be a extension to a program for independent "layered" integral equations. Not quite so, as a system of integral equations is not always necessarily decoupled as will be seen in the next chapter.

[2] $\| K \|$ is fixed from the beginning.

Chapter 2

Coupled Integral Equations

Before describing the Galerkin methods for coupled integral equations we first set the scene. Let $L_2^n(a,b)$ be our Hilbert space[1] and suppose we have a coupled system

$$\mathbf{\Phi} - K\mathbf{\Phi} = \mathbf{f} \tag{2.1}$$

where $\mathbf{\Phi} = (\phi_1, \phi_2)^T$, $\mathrm{f} = (f_1, f_2)^T$, $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$.
Expanding this gives

$$\begin{aligned} \phi_1 - K_{11}\phi_1 - K_{12}\phi_2 &= f_1, \\ \phi_2 - K_{21}\phi_1 - K_{22}\phi_2 &= f_2. \end{aligned}$$

Immediately we should spot that here our system is decoupled *if* K_{12} and K_{21} are zero operators for then we would have a pair of integral equations of the form $\phi_i - K_{ii}\phi_i = f_i$ for $i = 1, 2$. Still, it might seem strange how a coupled system could arise (i.e. where K_{12} and K_{21} are not zero operators) so here is an example on which every subsequent method in this thesis will be tested upon

$$\phi(x) = e^{i\kappa_0 x} - \frac{i}{2\kappa_0}\int_0^L e^{i\kappa_0|x-t|}\varrho(t)\phi(t)dt. \tag{2.2}$$

[1] $L_2^n(a,b)$ is the space of n-vectors of square integrable functions.

This integral equation is taken from [2] (or [3]) where the upper limit has been altered to a general value L, ϱ is real and known and κ_0 is a given real parameter (see Chapter 7 for more details). This equation arises from a study by Chamberlain of the use of integral equations to solve the mild-slope equation (due to Berkhoff) in 1 dimension. The mild-slope equation has its origins in water wave theory. Coastal engineers have been interested in predicting the behaviour of water waves. Advantages of the mild-slope equation are that it takes into account variable still water depth and it treats refraction and diffraction as dependent entities. Thus the mild slope equation is also known as the 'combined refraction-diffraction equation'. In his work Chamberlain converts the associated boundary value problem of the mild-slope equation into an equation, of which (2.2) is a special case, and then converts this into a pair of real-valued integral equations together with a finite-rank problem. In our case we will take (2.2) and derive a different coupled system as follows.

We let $\mathbf{\Phi}(x) = (\phi_1, \phi_2)^T$ where $\phi_1 = \mathbf{Re}\{\phi(x)\}$ and $\phi_2 = \mathbf{Im}\{\phi(x)\}$. With this in mind

$$\begin{aligned}
\phi(x) &= \cos\kappa_0 x + i\sin\kappa_0 x \\
&\quad -\frac{i}{2\kappa_0}\int_0^L (\cos\kappa_0(x-t) + i\sin\kappa_0 \mid x-t \mid)\varrho(t)(\phi_1(t)+i\phi_2(t))dt \\
&= \cos\kappa_0 x + \frac{1}{2\kappa_0}\int_0^L \sin\kappa_0 \mid x-t \mid \varrho(t)\phi_1(t)dt \\
&\quad +\frac{1}{2\kappa_0}\int_0^L \cos\kappa_0(x-t)\varrho(t)\phi_2(t)dt \\
&\quad +i\{\sin\kappa_0 x - \frac{1}{2\kappa_0}\int_0^L \cos\kappa_0(x-t)\varrho(t)\phi_1(t)dt \\
&\quad +\frac{1}{2\kappa_0}\int_0^L \sin\kappa_0 \mid x-t \mid \varrho(t)\phi_2(t)dt\}.
\end{aligned}$$

We can then write

$$\mathbf{\Phi}(x) = \begin{pmatrix} \cos\kappa_0 x \\ \sin\kappa_0 x \end{pmatrix} + \frac{1}{2\kappa_0}\int_0^L \begin{pmatrix} \sin\kappa_0 \mid x-t \mid & \cos\kappa_0(x-t) \\ -\cos\kappa_0(x-t) & \sin\kappa_0 \mid x-t \mid \end{pmatrix} \varrho(t)\mathbf{\Phi}(t)dt$$

where we recognise that $\mathbf{f} = \begin{pmatrix} \cos \kappa_0 x \\ \sin \kappa_0 \end{pmatrix}$ is our new free term

and $k(x,t) = \frac{\varrho(t)}{2\kappa_0} \begin{pmatrix} \sin \kappa_0 \mid x-t \mid & \cos \kappa_0 (x-t) \\ -\cos \kappa_0 (x-t) & \sin \kappa_0 \mid x-t \mid \end{pmatrix}$ is our new kernel[2].

Lo and behold a coupled system of integral equations.

[2]Note that the kernel is real and orthogonal.

Chapter 3

Galerkin Methods

3.1 Galerkin Method

Let $L_2^n(a,b)$ be the Hilbert space and E_{N_i}, $i = 1,2$ be subspaces of $L_2^n(a,b)$. We need to approximate the solutions ϕ_1 and ϕ_2 and we do so by letting $p_1 \in E_{N_1}$ and $p_2 \in E_{N_2}$ be our respective approximations. Then we look for solutions of the form

$$\phi_i \approx p_i = \sum_{n=1}^{N_i} a_n \chi_n^{(i)} \tag{3.1}$$

where a_n's are coefficients to be determined and $\chi_n^{(i)}$'s are our chosen basis functions spanning E_{N_i}.

By this construction we do not assume that the subspaces for p_1 and p_2 are the same. They may contain different basis functions and be of different dimensions. We now aim to satisfy the components lying in the subspaces such that

$$\begin{aligned} \langle p_1 - K_{11}p_1 - K_{12}p_2 - f_1, \chi_j^{(1)} \rangle &= 0 \qquad \text{for } j = 1, ..., N_1 \\ \langle p_2 - K_{21}p_1 - K_{22}p_2 - f_2, \chi_k^{(2)} \rangle &= 0 \qquad \text{for } k = 1, ..., N_2. \end{aligned}$$

Substitution of p_1 and p_2 from (3.1) then gives rise to $N_1 + N_2$ equations with the same number of unknowns, these being the coefficients we are after.

Inner products tend to hide what they really stand for with their $\langle \cdot, \cdot \rangle$ symbols. Throughout the analysis the following inner product will be taken

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}^T \overline{\mathbf{y}} dt$$

where $a, b \in \mathbf{R}$ and $\mathbf{x}, \mathbf{y}$ are finite n-vectors of square integrable functions. This means that $\mathbf{x}, \mathbf{y} \in L_2^n(a, b)$. For the proof that this definition satisfies our usual norm axioms see appendix. The space of all n-vectors of square integrable functions $L_2^n(a, b)$ guarantees that $\int_a^b \mathbf{x}^T \overline{\mathbf{x}} dt < z$ where $z < \infty$.

Acquisition of the coefficients mean of course that the Galerkin approximations p_1 and p_2 are in our hands. However, before taking these on through the iterated Galerkin method we should sort out the business of orthogonal projections.

Orthogonal Projections

Let P_i be the orthogonal projections of H onto E_{N_i} for $i = 1, 2$.

Then Galerkin's equations may be written in the form

$$\begin{aligned} P_1(p_1 - K_{11}p_1 - K_{12}p_2) &= P_1 f_1 \\ P_2(p_2 - K_{21}p_1 - K_{22}p_2) &= P_2 f_2 \end{aligned}$$

and these can be rewritten:

$$(I - P_1 K_{11})p_1 - P_1 K_{12} p_2 = P_1 f_1 \tag{3.2}$$

$$-P_2 K_{21} p_1 + (I - P_2 K_{22})p_2 = P_2 f_2. \tag{3.3}$$

Now let $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ and $\mathbf{p}_0 = (p_1, p_2)^T$. Then

$$PK = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} P_1 K_{11} & P_1 K_{12} \\ P_2 K_{21} & P_2 K_{22} \end{pmatrix} \tag{3.4}$$

and

$$\begin{aligned}(I-PK)\mathbf{p}_0 &= \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} P_1K_{11}p_1 + P_1K_{12}p_2 \\ P_2K_{21}p_1 + P_2K_{22}p_2 \end{pmatrix} \\ &= \begin{pmatrix} (I-P_1K_{11})p_1 - P_1K_{12}p_2 \\ -P_2K_{21}p_1 + (I-P_2K_{22})p_2 \end{pmatrix} \\ &= \begin{pmatrix} P_1f_1 \\ P_2f_2 \end{pmatrix} \\ &= P\mathbf{f}.\end{aligned}$$

Hence $(I-PK)\mathbf{p}_0 = P\mathbf{f}$ and so $\mathbf{p}_0 = (I-PK)^{-1}P\mathbf{f}$ is the Galerkin solution of the system provided $(I-PK)^{-1}$ is invertible. Though when exactly is $(I-PK)^{-1}$ invertible?

In order to answer this suppose that $(I-PK)^{-1}$ does exist. What would it look like? Consider (3.2) and (3.3). From (3.2)

$$p_1 = (I-P_1K_{11})^{-1}(P_1f_1 + P_1K_{12}p_2) \tag{3.5}$$

and from (3.3)

$$p_2 = (I-P_2K_{22})^{-1}(P_2f_2 + P_2K_{21}p_1). \tag{3.6}$$

Notice the necessity for $(I-P_1K_{11})$ and $(I-P_2K_{22})$ to be invertible to bring on the existence of p_1 and p_2. Substitution of (3.6) into (3.2) gives

$$(I-P_1K_{11})p_1 - P_1K_{12}(I-P_2K_{22})^{-1}(P_2f_2 + P_2K_{21}p_1) = P_1f_1 \tag{3.7}$$

and similarly substituting (3.5) into (3.3)

$$-P_2K_{21}(I-P_1K_{11})^{-1}(P_1f_1 + P_1K_{12}p_2) + (I-P_2K_{22})p_2 = P_2f_2. \tag{3.8}$$

These can be rewritten so that

$$\begin{aligned}p_1 &= [(I-P_1K_{11}) - P_1K_{12}(I-P_2K_{22})^{-1}P_2K_{21}]^{-1} \\ &\quad [P_1f_1 + P_1K_{12}(I-P_2K_{22})^{-1}P_2f_2] \\ p_2 &= [(I-P_2K_{22}) - P_2K_{21}(I-P_1K_{11})^{-1}P_1K_{12}]^{-1} \\ &\quad [P_2f_2 + P_2K_{21}(I-P_1K_{11})^{-1}P_1f_1]\end{aligned}$$

Hence, we have that

$$\mathbf{p}_0 = \begin{pmatrix} \alpha & \alpha\beta \\ \delta\gamma & \delta \end{pmatrix} P\mathbf{f} \tag{3.9}$$

where

$$\begin{aligned} \alpha &= [(I - P_1K_{11}) - P_1K_{12}(I - P_2K_{22})^{-1}P_2K_{21}]^{-1}, \\ \beta &= P_1K_{12}(I - P_2K_{22})^{-1}, \\ \gamma &= P_2K_{21}(I - P_1K_{11})^{-1}, \\ \delta &= [(I - P_2K_{22}) - P_2K_{21}(I - P_1K_{11})^{-1}P_1K_{12}]^{-1}. \end{aligned}$$

The 2×2 matrix is precisely $(I - PK)^{-1}$ and exists when $(I - P_1K_{11})$ and $(I - P_2K_{22})$ are invertible.

3.2 Iterated Galerkin Method

Let us define our iterated Galerkin approximations to be

$$\begin{aligned} \hat{p}_1 &= f_1 + K_{11}p_1 + K_{12}p_2 \\ \hat{p}_2 &= f_2 + K_{21}p_1 + K_{22}p_2 \end{aligned}$$

where p_1, p_2 are our previously found Galerkin approximations. Written in vector form this is $\hat{\mathbf{p}}_0 = \mathbf{f} + K\mathbf{p}_0$ where $\hat{\mathbf{p}}_0 = (\hat{p}_1, \hat{p}_2)^T$. Now

$$\begin{aligned} \hat{\mathbf{p}}_0 &= \mathbf{f} + K\mathbf{p}_0 \\ &= \mathbf{f} + K(I - PK)^{-1}P\mathbf{f} \end{aligned}$$

and if we can show that $(I - PK)^{-1}P = P(I - KP)^{-1}$ we can write

$$\begin{aligned} \hat{\mathbf{p}}_0 &= \{(I - KP) + KP\}(I - KP)^{-1}\mathbf{f} \\ &= (I - KP)^{-1}\mathbf{f}. \end{aligned}$$

It will soon become apparent why the relation $\hat{\mathbf{p}}_0 = (I - KP)^{-1}\mathbf{f}$ is useful.

To show that

$$(I - PK)^{-1}P = P(I - KP)^{-1}$$

consider that

$$(I - KP) = \begin{pmatrix} I - K_{11}P_1 & K_{12}P_2 \\ K_{21}P_1 & I - K_{22}P_2 \end{pmatrix}$$

and so therefore

$$P(I - KP) = \begin{pmatrix} P_1 - P_1K_{11}P_1 & P_1K_{12}P_2 \\ P_2K_{21}P_1 & P_2 - P_2K_{22}P_2 \end{pmatrix}.$$

Similarly,

$$I - PK = \begin{pmatrix} I - P_1K_{11} & P_1K_{12} \\ P_2K_{21} & I - P_2K_{22} \end{pmatrix}$$

but again

$$(I - PK)P = \begin{pmatrix} P_1 - P_1K_{11}P_1 & P_1K_{12}P_2 \\ P_2K_{21}P_1 & P_2 - P_2K_{22}P_2 \end{pmatrix}.$$

Therefore $P(I - KP) = (I - PK)P$. We have already shown that $(I - PK)^{-1}$ exists via the assumption that $(I - P_1K_{11})$ and $(I - P_2K_{22})$ are invertible. Also the existence of $(I - PK)^{-1}$ implies the existence of $(I - KP)^{-1}$ (a proof can be found in [4] p87) and so $(I - PK)^{-1}P = P(I - KP)^{-1}$.

The use of the relation found earlier now comes into play. We have to check to see if the iterated Galerkin approximation $\hat{\mathbf{p}}_0$ is better than the original Galerkin approximation $\mathbf{p}_0$. Now

$$\begin{aligned} \hat{\mathbf{p}}_0 - \mathbf{\Phi} &= (I - KP)^{-1}\mathbf{f} - (I - K)^{-1}\mathbf{f} \\ &= (I - KP)^{-1}\{(I - K) - (I - KP)\}(I - K)^{-1}\mathbf{f} \\ &= -(I - KP)^{-1}(K - KP)\mathbf{\Phi} \\ &= (I - KP)^{-1}(K - KP)(\mathbf{p}_0 - \mathbf{\Phi}) \end{aligned}$$

since $(K - KP)\mathbf{p}_0 = K\mathbf{p}_0 - KP\mathbf{p}_0 = K\mathbf{p}_0 - K\mathbf{p}_0 = 0$. Therefore

$$\hat{\mathbf{p}}_0 - \mathbf{\Phi} = S(\mathbf{p}_0 - \mathbf{\Phi})$$

where $S = (I - KP)^{-1}(K - KP)$ and by taking norms

$$\begin{aligned} \| \hat{\mathbf{p}}_0 - \mathbf{\Phi} \| &= \| S(\mathbf{p}_0 - \mathbf{\Phi}) \| \\ &\leq \| S \| \| \mathbf{p}_0 - \mathbf{\Phi} \| \\ &\leq \| \mathbf{p}_0 - \mathbf{\Phi} \| \end{aligned}$$

provided $\| S \| < 1$. However S is not just a single composition of operators. It is a matrix of them. Written out explicitly we find that by symmetry (c.f. $(I - PK)^{-1}$ in (3.9))

$$(I - KP)^{-1} = \begin{pmatrix} \lambda & \lambda\sigma \\ \mu\eta & \mu \end{pmatrix}$$

where

$$\begin{aligned} \lambda &= [(I - K_{11}P_1) - K_{12}P_2(I - K_{22}P_2)^{-1}K_{21}P_1]^{-1}, \\ \sigma &= K_{12}P_2(I - K_{22}P_2)^{-1}, \\ \eta &= K_{21}P_1(I - K_{11}P_1)^{-1}, \\ \mu &= [(I - K_{22}P_2) - K_{21}P_1(I - K_{11}P_1)^{-1}K_{12}P_2]^{-1} \end{aligned}$$

and so

$$\begin{aligned} S &= (I - KP)^{-1}(K - KP) \\ &= \begin{pmatrix} \lambda & \lambda\sigma \\ \mu\eta & \mu \end{pmatrix} \begin{pmatrix} K_{11} - K_{11}P_1 & K_{12} - K_{12}P_2 \\ K_{21} - K_{21}P_1 & K_{22} - K_{22}P_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda K_{11}(I - P_1) + \lambda\sigma K_{21}(I - P_1) & \lambda K_{12}(I - P_2) + \lambda\sigma K_{22}(I - P_2) \\ \mu\eta K_{11}(I - P_1) + \mu K_{21}(I - P_1) & \mu\eta K_{12}(I - P_2) + \mu K_{22}(I - P_2) \end{pmatrix}. \end{aligned}$$

3.3 Re-iterated Galerkin Method

Currently we have $\mathbf{\Phi} \approx \hat{\mathbf{p}}_0 = \mathbf{f} + K\mathbf{p}_0 = (I - KP)^{-1}\mathbf{f}$ from the iterated scheme. Introducing a current error $\mathbf{\Phi}_1$ we can write

$$\mathbf{\Phi} = \hat{\mathbf{p}}_0 + \mathbf{\Phi}_1.$$

Substitution of this into our original system $\mathbf{\Phi} = \mathbf{f} + K\mathbf{\Phi}$ gives

$$\hat{\mathbf{p}}_0 + \mathbf{\Phi}_1 = \mathbf{f} + K\hat{\mathbf{p}}_0 + K\mathbf{\Phi}_1$$

which when rearranged becomes

$$\begin{aligned} \mathbf{\Phi}_1 &= \mathbf{f} - (I - K)\hat{\mathbf{p}}_0 + K\mathbf{\Phi}_1 \\ &= \hat{\mathbf{r}}_0 + K\mathbf{\Phi}_1 \end{aligned}$$

where of course $\hat{\mathbf{r}}_0 = \mathbf{f} - (I - K)\hat{\mathbf{p}}_0$.

We have retrieved an equation of the same form as our original system the difference being that the free term has now become the residual $\hat{\mathbf{r}}_0$. Next we

(I) obtain the new Galerkin approximation $\mathbf{p}_1 \in (E_{N_1}, E_{N_2})^T$ where

$$\mathbf{\Phi}_1 \approx \mathbf{p}_1 = (I - PK)^{-1}P\hat{\mathbf{r}}_0,$$

(II) and then the corresponding iterated approximation

$$\hat{\mathbf{p}}_1 = \hat{\mathbf{r}}_0 + K\mathbf{p}_1 = (I - KP)^{-1}\hat{\mathbf{r}}_0$$

so that we end up with $\mathbf{\Phi} \approx \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1$.

Similar analysis shows that we do indeed have a better approximation to $\mathbf{\Phi}$. Consider

$$\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 - \mathbf{\Phi} = \hat{\mathbf{p}}_0 + (I - KP)^{-1}\hat{\mathbf{r}}_0 - \mathbf{\Phi}$$

$$
\begin{aligned}
&= \hat{\mathbf{p}}_0 - \mathbf{\Phi} + (I - KP)^{-1}(f - (I - K)\hat{\mathbf{p}}_0) \\
&= \hat{\mathbf{p}}_0 - \mathbf{\Phi} + (I - KP)^{-1}((I - K)(\hat{\mathbf{\Phi}} - \hat{\mathbf{p}}_0)) \\
&= \hat{\mathbf{p}}_0 - \mathbf{\Phi} - (I - KP)^{-1}(I - K)(\hat{\mathbf{p}}_0 - \mathbf{\Phi}) \\
&= (I - KP)^{-1}\{(I - KP) - (I - K)\}(\hat{\mathbf{p}}_0 - \mathbf{\Phi}) \\
&= (I - KP)^{-1}(K - KP)(\hat{\mathbf{p}}_0 - \mathbf{\Phi})
\end{aligned}
$$

Therefore

$$\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 - \mathbf{\Phi} = S(\hat{\mathbf{p}}_0 - \mathbf{\Phi}) = S^2(\mathbf{p}_0 - \mathbf{\Phi})$$

since $\hat{\mathbf{p}}_0 - \mathbf{\Phi} = S(\mathbf{p}_0 - \mathbf{\Phi})$.

This implies

$$\| \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 - \mathbf{\Phi} \| < \| \hat{\mathbf{p}}_0 - \mathbf{\Phi} \|$$

provided once again $\| S \| < 1$ which we assume would already have been satisfied.

More generally if we keep repeating the process,

$$\mathbf{\Phi} = \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + ... + \hat{\mathbf{p}}_m + \mathbf{\Phi}_{m+1}$$

where $\mathbf{\Phi}_{m+1}$ is the new correction term then

$$
\begin{aligned}
\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + ... + \hat{\mathbf{p}}_m + \hat{\mathbf{p}}_{m+1} - \mathbf{\Phi} &= S(\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + ... + \hat{\mathbf{p}}_m - \mathbf{\Phi}) \\
&= S^2(\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + ... + \hat{\mathbf{p}}_{m-1} - \mathbf{\Phi}) \\
&= ... \\
&= S^{m+1}(\hat{\mathbf{p}}_0 - \mathbf{\Phi}) \\
&= S^{m+2}(\mathbf{p}_0 - \mathbf{\Phi}).
\end{aligned}
$$

Taking norms we find that

$$
\begin{aligned}
\| \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + ... + \hat{\mathbf{p}}_m + \hat{\mathbf{p}}_{m+1} - \mathbf{\Phi} \| &\leq \| S \|^{m+2} \| (\mathbf{p}_0 - \mathbf{\Phi}) \| \\
&\leq \| (\mathbf{p}_0 - \mathbf{\Phi}) \|
\end{aligned}
$$

provided $\| S \| \leq 1$. Obviously the smaller $\| S \|$ is, the faster the rate of convergence.

Chapter 4

Choosing a Galerkin Subspace

4.1 The Neumann series and Subspaces

The **Neumann series** is a geometric series for operators. Formally our solution, *under certain conditions*, can be written

$$\begin{aligned} \mathbf{\Phi} &= (I-K)^{-1}\mathbf{f} \\ &= \sum_{n=0}^{\infty} K^n \mathbf{f} \\ &= \mathbf{f} + K\mathbf{f} + K^2\mathbf{f} + ... \end{aligned}$$

which is a geometric series with the operator K. The form of the solution naturally suggests we take $\{\mathbf{f}, K\mathbf{f}, K^2\mathbf{f}, ...\}$ as our subspace where K is a matrix of operators and $\mathbf{f}$ is a finite vector[1]. Porter & Stirling [4] showed that for single integral equations with the subspace $\{f, Kf, K^2f, ...\}$ that $\| P_n\phi - \phi \|^{\frac{1}{n}} \to 0$ as $n \to \infty$. (Notation: in their book Porter & Stirling used P_n as the orthogonal projection and not P). We would expect a similar result for coupled systems but will not prove it in this thesis.[2].

Question: So why not use the Neumann series to compute solutions numerically?

Well, those of you who recall the analogous series $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ where $z \in$

[1] A vector with a finite number of entries.

[2] The proof is likely to be the same but in the context of vectors.

C will probably also remember that series of this form converge when $\mid z \mid < 1$. It is also a common fact that $(I - L)^{-1}$, where L is a bounded linear operator from a Banach space X to itself, is invertible if $\| L \| < 1$. The two ideas combined make up the geometric series for operators. Usually the operator K is given at the start and so there is no guarantee that $\| K \| < 1$. The size of its norm cannot be manipulated as it is fixed from the beginning. The solution therefore cannot be in the form of a Neumann series if $\| K \| > 1$.

It turns out that the question of whether the norm of K is less than 1 does not hinder the re-iterated Galerkin and Kantorovich methods as they do not require this condition to be satisfied. They only need $\| S \| < 1$ to achieve convergence of solutions. To illustrate the importance of this requirement consider once again our original system

$$(I - K)\mathbf{\Phi} = \mathbf{f}$$

and suppose $\| K \| > 1$ so that $(I - K)$ is not invertible. Now apply $(I - KP)^{-1}$ to both sides[3] so that

$$(I - KP)^{-1}(I - K)\mathbf{\Phi} = (I - KP)^{-1}\mathbf{f}.$$

Consider the left hand side. We have

$$\begin{aligned}(I - KP)^{-1}(I - K) &= (I - KP)^{-1}\{(I - KP) - (K - KP)\} \\ &= I - (I - KP)^{-1}(K - KP) \\ &= I - S.\end{aligned}$$

So in fact

$$(I - S)\mathbf{\Phi} = (I - KP)^{-1}\mathbf{f}$$

[3]remember that this is possible as long as $(I - P_1K_{11})$ and $(I - P_2K_{22})$ are invertible. It would not be interesting if they were not since the Galerkin method would immediately fail.

and assuming that our subspace is big enough so that $\| S \| < 1$ we can invert $(I - S)$ and write

$$\mathbf{\Phi} = (I - S)^{-1}(I - KP)^{-1}\mathbf{f}.$$

Hence we have stumbled upon a preconditioner that acts on $(I - K)$, namely $(I - KP)^{-1}$. Moreover, it has given us a glimpse of the existence of *a* solution[4] when $\| K \| \geq 1$. It would be quite a task to write down this inverse operator in an analytical manner but it can be seen to work numerically. In other words, it can be approximated at, for example, Gauss points on the computer.

The Neumann subspace $\{\mathbf{f}, K\mathbf{f}, ...\}$ for Chamberlain's example (2.2) is a reasonable one to take since, for this wave scattering example, the norm of K is less than one (estimates are given later and solutions from using the Neumann series are given in the appendix). However for problems where this is not the case we could opt for $\{\mathbf{f}, K\mathbf{f}, ...\}$ or use any other reasonable subspace - perhaps orthogonal types such as a sine or cosine series, Legendre polynomials etc. A subspace that has some affinity to the problem at hand is usually a successful one. What would be imperative would be the *size* of the subspace: the need for $\| S \| < 1$ would still linger.

4.2 An Extended Subspace

Despite the freedom of choice we shall still take $\{\mathbf{f}, K\mathbf{f}, K^2\mathbf{f}, ...\}$ as our subspace. Suppose we take $\{\mathbf{f}, K\mathbf{f}\}$ as our 2-dimensional trial subspace. Then our Galerkin approximations would take the following form

$$\mathbf{\Phi} \approx \mathbf{p}_0 = a_1\mathbf{f} + a_2K\mathbf{f} \tag{4.1}$$

[4] one has not proved the uniqueness of it!

where $\mathbf{\Phi} = (\phi_1, \phi_2)^T$, $\mathbf{p_0} = (p_1, p_2)^T$, $\mathbf{f} = (f_1, f_2)^T$, $K\mathbf{f} = (K_{11}f_1 + K_{12}f_2, K_{21}f_1 + K_{22}f_2)^T$ and a_1, a_2 are coefficients to be found.

Splitting (4.1) open we have

$$\begin{aligned} \phi_1 \approx p_1 &= a_1 f_1 + a_2(K_{11}f_1 + K_{12}f_2) \\ \phi_2 \approx p_2 &= a_1 f_2 + a_2(K_{21}f_1 + K_{22}f_2). \end{aligned}$$

However, there is one thing to notice here. We can instead write

$$\phi_1 \approx p_1 = a_1 f_1 + a_2 K_{11} f_1 + a_3 K_{12} f_2 \tag{4.2}$$

$$\phi_2 \approx p_2 = b_1 f_2 + b_2 K_{21} f_1 + b_3 K_{22} f_2 \tag{4.3}$$

by introducing **four** new coefficients a_3, b_1, b_2, b_3. We can rewrite (4.2) and (4.3) in matrix form

$$\mathbf{\Phi} \approx \mathbf{p} = A_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + A_2 \begin{pmatrix} K_{11}f_1 \\ K_{21}f_1 \end{pmatrix} + A_3 \begin{pmatrix} K_{12}f_2 \\ K_{22}f_2 \end{pmatrix}$$

where $A_i = \begin{pmatrix} a_i & 0 \\ 0 & bi \end{pmatrix}$ for $i = 1, 2, 3$.

Notice we also have two trial subspaces spanned by three elements not two as we would have initially suspected would be possible. It is easy to see that this splitting can be done more generally though we must remember that as A_i are matrices and not simple scalars the use of this extended subspace does not constitute to a Galerkin method. Nevertheless, intuitively we would imagine that these extra degrees of freedom would improve our initial approximations. Are these alterations any good numerically though?

Chapter 5

Kantorovich Methods

This chapter on Kantorovich methods will be fairly brief given that Burton has already written out the bulk of the theory in [1] and the theory for coupled systems is very similar to it, the differences being that f is now $\mathbf{f}$ and ϕ is now $\mathbf{\Phi}$. Other differences follow in similar fashion.

Notation: $\mathbf{p}_i$, $\hat{\mathbf{p}}_i$ and $\hat{\mathbf{r}}_i$ should be distinguished between the Galerkin and Kantorovich methods but it was decided against to clutter the notation in this thesis.

5.1 Kantorovich Method

Here is yet another adaptation of the Kantorovich method this time for coupled systems. There are many different methods one could try besides the ones already written about. If we are lucky we will see another one later. For now we shall continue with the Kantorovich methods which are analogous to the ones in Burton's thesis [1]. We begin with

$$\mathbf{\Phi} = \mathbf{f} + K\mathbf{\Phi}$$

and apply the operator K to both sides so that

$$\mathbf{\Psi} = K\mathbf{f} + K\mathbf{\Psi}$$

where $\mathbf{\Psi} = K\mathbf{\Phi}$. The solution $\mathbf{\Phi}$ can be recovered from $\mathbf{\Phi} = \mathbf{f} + \mathbf{\Psi}$.

This application of K is called the Kantorovich 'regularization'. Glancing back at this new system we see that we can tackle it as we did with the original one. That is we look for the Galerkin approximation to $\mathbf{\Psi}$ given by

$$\mathbf{\Psi} \approx \mathbf{q}_0 = \sum_{n=1}^{N} a_n \boldsymbol{\chi}_n,$$

where $\boldsymbol{\chi}_n = (\chi_n^{(1)}, \chi_n^{(2)})^T$ (so we note the use of the same subspaces E_{N_1} and E_{N_2} but keep in mind that the coefficients are likely to be different). Then we can let

$$\mathbf{\Phi} \approx \mathbf{p}_0 = \mathbf{f} + \mathbf{q}_0$$

where it has to be stressed that $\mathbf{p}_0$ is unlikely to be the same Galerkin approximation that would have been found from the original system since $\mathbf{q}_0$ was obtained through the Kantorovich regularization.

Why might this new approximation to $\mathbf{\Phi}$ be better?

Our starting point with the straight Galerkin approach is $\mathbf{\Phi} = \mathbf{f} + K\mathbf{\Phi}$ but now consider what happens when we substitute $\mathbf{\Phi} = \mathbf{f} + \mathbf{\Psi}$ into $\mathbf{\Phi} = \mathbf{f} + K\mathbf{\Phi}$ which is permissible from the Kantorovich approach. We find that $\mathbf{\Phi} = \mathbf{f} + K\mathbf{f} + K\mathbf{\Psi}$ where we note that $\mathbf{f}$ and $K\mathbf{f}$ are the first two terms of the Neumann series which make up part of the formal solution of $\mathbf{\Phi}$ when $\| K \| < 1$. Appealing to the fact that two terms of the Neumann series occur and not one as with the Galerkin approach we suspect that the Kantorovich method possesses a faster convergence rate particularly if $\| K \| < 1$. However if $\| K \| > 1$ (or possibly $\| K \| \geq 1$) then the Neumann series will diverge and we cannot expect a great improvement. Yet the Kantorovich method still has one more chance to get the upper hand. Burton [1] makes a remark that the Kantorovich regularization removes $\mathbf{f}$ from the calculations and replaces it with $K\mathbf{f}$ - an advantage if $\mathbf{f}$ is not particularly smoooth.

5.2 Iterated Kantorovich Method

The next step of course is to apply the process of iteration to the previously found Galerkin approximation $\mathbf{q}_0$ to $\mathbf{\Phi}$. So far we have

$$\mathbf{\Phi} \approx \mathbf{p}_0 = \mathbf{f} + \mathbf{q}_0.$$

Applying the iteration process to $\mathbf{q}_0$ gives

$$\hat{\mathbf{q}}_0 = K\mathbf{f} + K\mathbf{q}_0$$

so that now

$$\mathbf{\Phi} \approx \hat{\mathbf{p}}_0 = \mathbf{f} + \hat{\mathbf{q}}_0$$

and through similar error analysis (see Burton [1])

$$\| \hat{\mathbf{p}}_0 - \mathbf{\Phi} \| \leq \| S \| \| \mathbf{p}_0 - \mathbf{\Phi} \|$$

where S is precisely the operator found before whose norm has to be less than 1 to guarantee the superiority of the iterate $\hat{\mathbf{p}}_0$ to $\mathbf{p}_0$.

5.3 Re-iterated Kantorovich Method

We now start with $\mathbf{\Phi} \approx \hat{\mathbf{p}}_0$. Introducing a correction term as we did for the re-iterated Galerkin method

$$\mathbf{\Phi} = \hat{\mathbf{p}}_0 + \mathbf{\Phi}_1$$

leads us to

$$\mathbf{\Phi}_1 = \hat{\mathbf{r}}_0 + K\mathbf{\Phi}_1 \tag{5.1}$$

where $\hat{\mathbf{r}}_0 = \mathbf{f} - (I - K)\hat{\mathbf{p}}_0$ after substitution into the original system.
Reverting to a second Kantorovich regularization by letting $\mathbf{\Psi}_1 = K\mathbf{\Phi}_1$ we have

$$\mathbf{\Phi}_1 = \hat{\mathbf{r}}_0 + \mathbf{\Psi}_1$$

and so by substitution into (5.1)

$$\mathbf{\Psi}_1 = K\hat{\mathbf{r}}_0 + K\mathbf{\Psi}_1.$$

As with the re-iterated Galerkin method we now

(I) obtain the Galerkin approximation $\mathbf{q}_1$ to $\mathbf{\Psi}_1$ where $\mathbf{q}_1 \in (E_{N_1}, E_{N_2})^T$,

(II) and then obtain the iterated Galerkin approximation $\hat{\mathbf{q}}_1 = K\hat{\mathbf{r}}_0 + K\mathbf{q}_1$ to $\mathbf{\Psi}_1$.

Thus we will then have

$$\mathbf{\Phi}_1 \approx \hat{\mathbf{p}}_1 = \hat{\mathbf{r}}_0 + \hat{\mathbf{q}}_1$$

and hence possess an even better approximation given by

$$\phi \approx \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1.$$

This time however,

$$\begin{aligned} \| (\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1) - \mathbf{\Phi} \| &\leq \| SK \| \| \hat{\mathbf{p}}_0 - \mathbf{\Phi} \| \\ &= \| SK \| \| S \| \| \mathbf{p}_0 - \mathbf{\Phi} \| \end{aligned}$$

which is an improvement if $\| SK \| < 1$ given that $\| S \|$ is already less than 1.

If this condition is satisfied we can continue the process and set

$$\mathbf{\Phi} = \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + \mathbf{\Phi}_2$$

where $\mathbf{\Phi}_2$ is our new correction term.

More generally,

$$\| \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + \ldots + \hat{\mathbf{p}}_{m+1} - \mathbf{\Phi} \| \leq \| SK \|^{m+1} \| S \| \| \mathbf{p}_0 - \mathbf{\Phi} \| .$$

Comparing methods so far

The re-iterated Galerkin and Kantorovich methods were compared with one another analytically in Burton's thesis [1]. He found that after n iterations the re-iterated Galerkin solution contained the first $n+1$ terms of the Neumann series but that the re-iterated Kantorovich solution contained $2n+2$ terms thereby suggesting that the re-iterated Kantorovich method would have a much faster rate of convergence especially if $\| K \| < 1$. It is obvious from Burton's work [1] that the same holds for coupled systems.

Chapter 6

Construction

As always, one has to *know* what to tell the computer to do before passing commands to it and as yet we have described how the Galerkin and Kantorovich methods work in theory but have omitted the computational details.
This chapter addresses some of the missing aspects starting with the calculation of Galerkin approximations. First one has to say that, for the results that follow, ϕ_1 and ϕ_2 are approximated using subspaces of the same size. Furthermore, consecutive terms of the Neumann series beginning with $\mathbf{f}$ are used for the subspaces. This is a reasonable choice given that it seems $\| K \| < 1$ for our chosen example (2.2) and so $\mathbf{f}$, being the first term of the Neumann series, would make the greatest contribution to the solutions.

6.1 Constructing Approximations

Galerkin approximations

Recall the aim to satisfy the components of the solution lying in the same subspaces

$$\langle p_1 - K_{11}p_1 - K_{12}p_2 - f_1, \chi_j^{(1)} \rangle = 0 \qquad \text{for } j = 1, ..., N_1 \tag{6.1}$$

and

$$\langle p_2 - K_{21}p_1 - K_{22}p_2 - f_2, \chi_k^{(2)} \rangle = 0 \qquad \text{for } k = 1, ..., N_2 \tag{6.2}$$

where $\chi^{(i)}$ are our chosen basis functions and N_1, N_2 are the dimensions of the subspaces for our Galerkin approximations, p_1 and p_2, respectively.

Suppose we use a 1D subspace, $\{\mathbf{f}\}$, so that

$$\begin{aligned} \phi_1 &\approx p_1 = a_1 f_1 \\ \phi_2 &\approx p_2 = a_1 f_2. \end{aligned}$$

Then substitution into (6.1) and (6.2) gives

$$\begin{aligned} a_1\{\langle f_1, f_1\rangle - \langle K_{11}f_1, f_1\rangle\} - a_1\langle K_{12}f_2, f_1\rangle &= \langle f_1, f_1\rangle \\ a_1\{\langle f_2, f_2\rangle - \langle K_{22}f_2, f_2\rangle\} - a_1\langle K_{21}f_1, f_2\rangle &= \langle f_2, f_2\rangle \end{aligned}$$

and so

$$\begin{aligned} &(\langle (I - K_{11})f_1, f_1\rangle - \langle K_{12}f_2, f_1\rangle \\ &\quad -\langle K_{21}f_1, f_2\rangle + \langle (I - K_{22})f_2, f_2\rangle)a_1 = (\langle f_1, f_1\rangle + \langle f_2, f_2\rangle). \end{aligned}$$

Now suppose we take a 2D subspace e.g. $\{\mathbf{f}, K\mathbf{f}\}$ so that this time

$$\begin{aligned} \phi_1 &\approx p_1 = a_1 f_1 + a_2(K_{11}f_1 + K_{12}f_2) \\ \phi_2 &\approx p_2 = a_1 f_2 + a_2(K_{21}f_1 + K_{22}f_2). \end{aligned}$$

In matrix form, say $A\mathbf{x} = \mathbf{y}$,

$$\begin{pmatrix} \langle (I - K_{11})f_1, f_1\rangle - \langle K_{12}f_2, f_1\rangle & \langle (I - K_{11})K_1^\alpha, f_1\rangle - \langle K_{12}K_1^\beta, f_1\rangle \\ \quad -\langle K_{21}f_1, f_2\rangle + \langle (I - K_{22})f_2, f_2\rangle & \quad -\langle K_{21}K_1^\alpha, f_2\rangle + \langle (I - K_{22})K_1^\beta, f_2\rangle \\ \\ \langle (I - K_{11})f_1, K_1^\alpha\rangle - \langle K_{12}f_2, K_1^\alpha\rangle & \langle (I - K_{11})K_1^\alpha, K_1^\alpha\rangle - \langle K_{12}K_1^\beta, K_1^\alpha\rangle \\ \quad -\langle K_{21}f_1, K_1^\beta\rangle + \langle (I - K_{22})f_2, K_1^\beta\rangle & \quad -\langle K_{21}K_1^\alpha, K_1^\beta\rangle + \langle (I - K_{22})K_1^\beta, K_1^\beta\rangle \end{pmatrix}$$

where $K\mathbf{f}$ is split into two components: $K_1^\alpha = K_{11}f_1 + K_{12}f_2$ and $K_1^\beta = K_{21}f_1 + K_{22}f_2$. In this case

$$x = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad y = \begin{pmatrix} \langle f_1, f_1\rangle + \langle f_2, f_2\rangle \\ \langle f_1, K_1^\alpha\rangle + \langle f_2, K_1^\beta\rangle \end{pmatrix}.$$

In general, for a $(N+1)$ dimensional subspace $\{\mathbf{f}, K\mathbf{f}, ..., K^N\mathbf{f}\}$, the matrix A would look like this:

$$\begin{pmatrix} \langle (I-K_{11})f_1, f_1\rangle - \langle k_{12}f_2, f_1\rangle & \cdots & \langle (I-K_{11})K_N^\alpha, f_1\rangle - \langle K_{12}K_N^\beta, f_1\rangle \\ \quad -\langle K_{21}f_1, f_2\rangle + \langle (I-K_{22})f_2, f_2\rangle & & \quad -\langle K_{21}K_N^\alpha, f_2\rangle + \langle (I-K_{22})K_N^\beta, f_2\rangle \\ \vdots & & \vdots \\ \langle (I-K_{11})f_1, K_N^\alpha\rangle - \langle K_{12}f_2, K_N^\alpha\rangle & \cdots & \langle (I-K_{11})K_N^\alpha, K_N^\alpha\rangle - \langle K_{12}K_N^\beta, K_N^\alpha\rangle \\ \quad -\langle K_{21}f_1, K_N^\beta\rangle + \langle (I-K_{22})f_2, K_N^\beta\rangle & & \quad -\langle K_{21}K_N^\alpha, K_N^\beta\rangle + \langle (I-K_{22})K_N^\beta, K_N^\beta\rangle \end{pmatrix}$$

and

$$x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N+1} \end{pmatrix}, \qquad y = \begin{pmatrix} \langle f_1, f_1\rangle + \langle f_2, f_2\rangle \\ \vdots \\ \langle f_1, K_N^\alpha\rangle + \langle f_2, K_N^\beta\rangle \end{pmatrix}.$$

Clearly, once the matrix system is set up, computing the coefficients is an easy task with a good mathematics package. Substituting the coefficients back into

$$\begin{aligned} p_1 &= a_1 f_1 + a_2 K_1^\alpha + ... + a_{N+1} K_N^\alpha \\ p_2 &= a_1 f_2 + a_2 K_1^\beta + ... + a_{N+1} K_N^\beta \end{aligned}$$

where $K^r\mathbf{f} = \begin{pmatrix} K_r^\alpha \\ K_r^\beta \end{pmatrix}$, returns the Galerkin approximations.

Iterated Galerkin approximations

Here we will simply give an indication of the form of the iterated Galerkin approximations. With an $(N+1)$ dimensional subspace the first iterated Galerkin approximation would be like this

$$\begin{aligned} \hat{\mathbf{p}}_0 &= \mathbf{f} + K\mathbf{p}_0 \\ &= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} a_1 f_1 + a_2 K_1^\alpha + \cdots + a_{N+1} K_N^\alpha \\ a_1 f_2 + a_2 K_1^\beta + \cdots + a_{N+1} K_N^\beta \end{pmatrix} \\ &= \begin{pmatrix} f_1 + K_{11}(a_1 K_0^\alpha + a_2 K_1^\alpha + \cdots + a_{N+1} K_N^\alpha) \\ \quad + K_{12}(a_1 K_0^\beta + a_2 K_1^\beta + \cdots + \_N+1 K_N^\beta) \\ \\ f_2 + K_{21}(a_1 K_0^\alpha + a_2 K_1^\alpha + \cdots + a_{N+1} K_N^\alpha) \\ \quad + K_{22}(a_1 K_0^\beta + a_2 K_1^\beta + \cdots + a_{N+1} K_N^\beta) \end{pmatrix} \end{aligned}$$

again where $K^r\mathbf{f} = \begin{pmatrix} K_r^\alpha \\ K_r^\beta \end{pmatrix}$. The re-iterated Galerkin approximation has a similar form. Different suffixes would be in place but the construct of the approximations would be the same.

Kantorovich approximations

One of the most defining features of the Kantorovich scheme is that the free term is no longer $\mathbf{f}$ but $K\mathbf{f}$. The coefficient matrix takes a very similar form and once the current Galerkin approximations q_1, q_2 have been found we simply let $p_1 = f_1 + q_1$ and $p_2 = f_2 + q_2$ be the current approximations for ϕ_1 and ϕ_2.

Iterated Kantorovich approximations

The iteration process is applied to $\mathbf{q}_0$ giving $\hat{\mathbf{q}}_0 = K\mathbf{f} + K\mathbf{q}_0$ and so we set $\mathbf{\Phi} \approx \hat{\mathbf{p}}_0 = \mathbf{f} + \hat{\mathbf{q}}_0$. The same type of construct as with the iterated Galerkin approximation.

Re-iterated Kantorovich approximations

The other defining feature of the Kantorovich approach is the regularization step. Once we have $\mathbf{\Phi} \approx \hat{\mathbf{p}}_0 = \mathbf{f} + \hat{\mathbf{q}}_0$ we then introduce the correction term $\mathbf{\Phi}_1$. This we know leads us to an equation in $\mathbf{\Phi}_1$. We perform a second Kantorovich regularization, turn out an equation in $\mathbf{\Psi}_1$ and start again with the Kantorovich and iterated Kantorovich method. The important thing to note is that the regularization is repeated for each iteration.

6.2 Smoothing

The 10 point Gauss-Legendre quadrature was used to evaluate integrals on the computer. By choice the interval was always broken up into 20 subintervals and

so this meant approximating $\mathbf{\Phi}$ at Gauss points. Throughout the analysis, we took $\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}^T \overline{\mathbf{y}} dt$ where $\mathbf{x}, \mathbf{y} \in L_2^n(a,b)$. Computationally $\mathbf{x}, \mathbf{y}$ will be 1-vectors. In other words we only ever compute one integral at a time. However, there was one particular integral which arose when approximating $K\mathbf{f}, K^2\mathbf{f}, ...,$ or more precisely $K_{11}f_1$ and $K_{22}f_2$ for the example (2.2), and that was

$$\int_0^L \sin(\kappa_0 \mid x - t \mid)\varrho(t)f_i(t)dt \qquad i = 1,2 \tag{6.3}$$

where the slope discontinuity, when $x = t$, meant that Gaussian quadrature was unsuitable. Gaussian quadrature rules work well for smooth functions especially polynomials. In fact, N point quadrature rules are exact for polynomials of degree $2N-1$ but due to the slope discontinuity, approximations to (6.3) would be poor. So how was this resolved?

A trick in Chamberlain's work [2, 3] solved this little nightmare. We simply rewrite (6.3) as

$$\int_0^L \sin(\kappa_0 \mid x - t \mid)\varrho(t)f_i(t)dt =$$
$$\int_0^L \sin(\kappa_0 \mid x - t \mid)(\varrho(t)f_i(t) - \varrho(x)f_i(x))dt + \varrho(x)f_i(x)\int_0^L \sin(\kappa_0 \mid x - t \mid)dt.$$

where $i = 1,2$. Why is this better? Well Gaussian quadrature works for the first integral on the right hand side because it has a continuous derivative. The second integral on the right can be evaluated analytically (*without* the need for Gaussian quadrature).

The trick can be written more generally

$$\int_a^b k(x,t)f(t)dt = \int_a^b k(x,t)(f(t) - f(x))dt + f(x)\int_a^b k(x,t)dt \tag{6.4}$$

where k is some function of x and t with a discontinuity at $x = t$ and $a, b \in \mathbf{R}$.

Chapter 7

Results

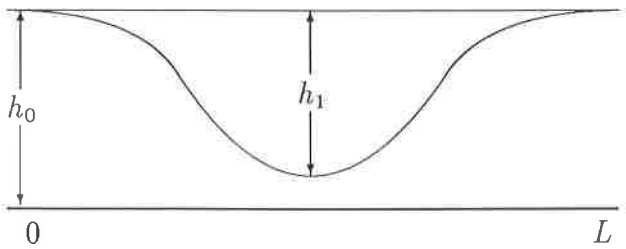
Before we see the results it might be beneficial to take a look at what Chamberlain's wave scattering equation is actually describing. For the results that follow for the wave scattering equation (2.2) the depth profile was given by

$$h(x) = h_0 + h_1 - (h_1 - h_0)\cos\left(\frac{2\pi x}{L}\right)$$

for $0 < x < L$ where h_0 is the depth for $x < 0$ and $x > L$ and h_1 is the minimum depth. Parameters chosen were $L = 3$, $\nu = 1$, $h_0 = 0.6$ and $h_1 = 0.45$ and so our particular depth profile was

$$h(x) = \frac{1}{20}\left\{21 - 3\cos\left(\frac{2\pi x}{3}\right)\right\}.$$

Diagrammatically we had the following situation



Depth profile

The curve is the profile of the sea bed, h_0 is the depth and h_1 is the minimum depth. Here is the equation again to save one having to flick back

$$\phi(x) = e^{i\kappa_0 x} - \frac{i}{2\kappa_0}\int_0^L e^{i\kappa_0|x-t|}\varrho(t)\phi(t)dt \tag{7.1}$$

where $x \in [0, L]$ and ϱ is real and known. The real parameter κ_0 is the positive real root of

$$\nu = \kappa_0 \tanh(\kappa_0 h_0)$$

where ν is the deep water wavenumber and h_0 is the depth in $x < 0$ and $x > L$. Chamberlain's wave scattering equation arises out of considering situations of this type. In his work Chamberlain considered many other depth profiles (see [2, 3]). Although this thesis is primarily concerned with coupled systems it was possible to treat Chamberlain's example as a single equation (and let Matlab deal with complex numbers which it did so with great effect). Hence there are four sets of results to present: those from the coupled re-iterated Galerkin method, those from the coupled re-iterated Kantorovich method and those from treating the wave scattering equation as a single equation using both the Galerkin and Kantorovich re-iteration methods. There will in fact be more data sets shown later concerning approximations using the Neumann series (more as a reference than anything else and so these are simply held in the appendix), concerning the use of the extended subspace that was mentioned in Chapter 4 and the 'residual technique' which will be described later.

Re-iteration continued until figures converged to within a tolerance of 1.0×10^{-7}. The top row of some of the tables have $x \approx i$ for $i = 0, 1, 2, 3$. What is meant by this is that x is the closest Gauss point to i for $i = 0, 1, 2, 3$.

As mentioned previously a 10 point Gauss-Legendre quadrature was used and 20 subintervals was chosen throughout. Hence there were 200 Gauss points in total spread over $[0, 3]$, our interval for x.

For reference a table for the Gauss points closest to $i = 0, 1, 2$, and 3 is given here.

	Closest Gauss point
$x \approx 0$	1.9570103612121196e-03
$x \approx 1$	1.0075046545596935e+00
$x \approx 2$	1.9924953454403060e+00
$x \approx 3$	2.9980429896387877e+00

7.1 Underestimates of $\rho(S)$ and $\rho(SK)$ and $\| K \|$

The **spectral radius**, denoted $\rho(T)$, is defined to be the limit of $\| T \|^{\frac{1}{n}}$ as $n \to \infty$ where T is a bounded linear operator. From this we can see that

$$\begin{aligned} \rho(T) &\leq \lim_{n\to\infty} (\| T \|^n)^{\frac{1}{n}} \\ &= \| T \| . \end{aligned}$$

Underestimates of $\rho(S)$ (or $\rho(SK)$ for the re-iterated Kantorovich results) are given following the first main set of tables. These were found using the observation in [1] that

$$\frac{\| \hat{\mathbf{r}}_n \|}{\| \hat{\mathbf{r}}_{n-1} \|} \longrightarrow \begin{cases} \rho(S), & \text{for the re-iterated Galerkin method} \\ \rho(SK), & \text{for the re-iterated Kantorovich method} \end{cases} \tag{7.2}$$

as $n \to \infty$. We will not prove this but simply observe that the residual $\hat{\mathbf{r}}_n$ (or 'residual error' as it is sometimes known) is given by

$$\begin{aligned} \hat{\mathbf{r}}_n &= \mathbf{f} - (I - K)(\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + \cdots + \hat{\mathbf{p}}_n) \\ &= (I - K)(\mathbf{\Phi} - \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + \cdots + \hat{\mathbf{p}}_n) \end{aligned}$$

and we expect sucessive residuals to decrease in magnitude for convergence for if $\hat{\mathbf{r}}_n \to \underline{0}$ as $n \to \infty$ then $\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + \cdots + \hat{\mathbf{p}}_n \to \mathbf{\Phi}$ as $n \to \infty$.

In particular, for convergence we should find that

$$\frac{\| \hat{\mathbf{r}}_n \|}{\| \hat{\mathbf{r}}_{n-1} \|} < 1$$

for each fixed $n = 0, 1, 2, \ldots$, where from now on $\hat{\mathbf{r}}_{-1}$ is defined as $\mathbf{f}$.

Simply recalling that

$$\| \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + \cdots + \hat{\mathbf{p}}_{m+1} - \mathbf{\Phi} \| \leq \| S \| \| \hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1 + \cdots + \hat{\mathbf{p}}_m - \mathbf{\Phi} \|$$

reveals that sucessive approximations should improve in the norm by as much as a factor of $\| S \|$.

We have to stop here. The convergence property of (7.2) should be shown but this would prove difficult. For instance, how can we *explicitly* define the spectral radius of a matrix of operators such as S or SK? Does it even exist?

We will not begin another thesis on such matters at this moment in time but will simply observe the figures given to us by the computer.

Next, underestimates for $\| K \|$ are given a little later using another observation in [1] that

$$\frac{\| \hat{\mathbf{r}}_n \|}{\| \mathbf{r}_n \|} \leq \| K \|, \qquad n = 0, 1, 2, \ldots,$$

where $\mathbf{r}_n = \hat{\mathbf{r}}_{n-1} - (I - K)\mathbf{p}_n$, the non-iterated residual which comes from $\mathbf{\Phi}_n = \hat{\mathbf{r}}_{n-1} + K\mathbf{\Phi}_n$.

This should be easier to prove. We begin with the following expressions,

$$\hat{\mathbf{r}}_n = \mathbf{f} - (I - K)(\hat{\mathbf{p}}_0 + \ldots + \hat{\mathbf{p}}_n)$$

and

$$\mathbf{r}_n = \hat{\mathbf{r}}_{n-1} - (I - K)\mathbf{p}_n$$

where $n = 0, 1, 2, \ldots$.

Working on the first of these

$$\begin{aligned} \hat{\mathbf{r}}_n &= \mathbf{f} - (I - K)(\hat{\mathbf{p}}_0 + \ldots + \hat{\mathbf{p}}_n) \\ &= \hat{\mathbf{r}}_{n-1} - (I - K)\hat{\mathbf{p}}_n \end{aligned}$$

$$
\begin{aligned}
&= \hat{\mathbf{r}}_{n-1} + K\hat{\mathbf{p}}_n - \mathbf{p}_n \\
&= \hat{\mathbf{r}}_{n-1} + K(\hat{\mathbf{r}}_{n-1} + K\mathbf{p}_n) - (\hat{\mathbf{r}}_{n-1} + K\mathbf{p}_n) \\
&= K(\hat{\mathbf{r}}_{n-1} + K\mathbf{p}_n - \mathbf{p}_n) \\
&= K(\hat{\mathbf{r}}_{n-1} - (I-K)\mathbf{p}_n) \\
&= K\hat{\mathbf{r}}_n, \qquad\qquad n = 0, 1, 2, \ldots .
\end{aligned}
$$

Taking norms and rearranging gives

$$
\frac{\| \hat{\mathbf{r}}_n \|}{\| \mathbf{r}_n \|} \leq \| K \|, \qquad n = 0, 1, 2, \ldots .
$$

7.2 Coupled Integral Results

We begin with the results for the re-iterated Galerkin and re-iterated Kantorovich methods using the coupled and single equation approaches.

First there are tables for the solutions at Gauss points. Next there are estimates for the spectral radii and the norm of $\| K \|$. Then there are bar charts chronicling the number of iterations each method takes against the size of the subspace.

Notes: (1) The figures in the following tables have been truncated and *not* rounded. (2a) The first entry for the tables of the spectral radii is $\frac{\|\hat{\mathbf{r}}_0\|}{\|\mathbf{f}\|}$ and for $\| K \|$ is $\frac{\|\hat{\mathbf{r}}_0\|}{\|\mathbf{r}_0\|}$, each corresponding to either the re-iterated Galerkin or re-iterated Kantorovich method. (2b) The norm that was used for these underestimates was $\left(\int_0^3 \mathbf{x}^T \overline{\mathbf{x}} dt\right)^{1/2}$, which was easy to do on the computer since solutions were already evaluated at Gauss points covering the interval $[0, 3]$. (3) Bar charts were obtained using **re-iterated** methods and so where the title states 'Galerkin' or 'Kantorovich', it is the corresponding **re-iterated** method that is meant.

Re-iterated Galerkin Results Using Coupled Approach									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.953967609	0.022805639	0.061079931	0.993996881	-0.955722336	0.094682016	-0.172635532	-1.015295809
	1	0.944356256	0.013144880	0.053688941	0.998435071	-0.937281759	0.106403016	-0.162471817	-0.984357045
	2	0.946716378	0.011125497	0.052219224	0.995666759	-0.939280742	0.104957136	-0.162831755	-0.985277323
	3	0.946829581	0.011629432	0.052625120	0.995877148	-0.939059187	0.105020678	-0.162777045	-0.985167601
	4	0.946780209	0.011600766	0.052565898	0.995885472	-0.939100122	0.105014625	-0.162785722	-0.985191843
	5	0.946786154	0.011600296	0.052572688	0.995880847	-0.939092130	0.105014823	-0.162783839	-0.985185359
	6	0.946785562	0.011600543	0.052571906	0.995881527	-0.939093280	0.105014767	-0.162784116	-0.985186351
	7	0.946785621	0.011600518	0.052572002	0.995881451	-0.939093130	0.105014778	-0.162784080	-0.985186221
	8	0.946785614	0.011600519	0.052571989	0.995881459	-0.939093148	0.105014776	-0.162784084	-0.985186237
2	0	0.944731901	0.013522456	0.053977806	0.998261612	-0.938002481	0.105944919	-0.162869050	-0.985566239
	1	0.946606652	0.011186847	0.052263221	0.995783028	-0.939169070	0.105034968	-0.162799199	-0.985185075
	2	0.946830733	0.011607353	0.052607570	0.995863297	-0.939073944	0.105013999	-0.162781195	-0.985177694
	3	0.946781484	0.011602524	0.052568727	0.995886151	-0.939096805	0.105015836	-0.162784658	-0.985188815
	4	0.946786012	0.011600142	0.052572224	0.995880829	-0.939092777	0.105014639	-0.162784021	-0.985185921
	5	0.946785588	0.011600572	0.052571984	0.995881523	-0.939093178	0.105014791	-0.162784089	-0.985186266
	6	0.946785615	0.011600514	0.052571990	0.995881453	-0.939093145	0.105014776	-0.162784083	-0.985186233
3	0	0.946504132	0.011314569	0.052356983	0.995918569	-0.939105275	0.105084728	-0.162803019	-0.985205919
	1	0.946809872	0.011595080	0.052596611	0.995870287	-0.939073790	0.105019324	-0.162782500	-0.985179847
	2	0.946783731	0.011600132	0.052568738	0.995883763	-0.939096354	0.105015550	-0.162784529	-0.985188122
	3	0.946785954	0.011600557	0.052572415	0.995881181	-0.939092797	0.105014741	-0.162784038	-0.985186048
	4	0.946785576	0.011600517	0.052571937	0.995881494	-0.939093196	0.105014783	-0.162784090	-0.985186262
	5	0.946785620	0.011600520	0.052571998	0.995881453	-0.939093140	0.105014776	-0.162784083	-0.985186232
4	0	0.946800230	0.011586234	0.052589054	0.995871809	-0.939074783	0.105021386	-0.162783147	-0.985180669
	1	0.946784927	0.011600314	0.052569908	0.995883279	-0.939095604	0.105015589	-0.162784440	-0.985187830
	2	0.946785746	0.011600453	0.052572183	0.995881295	-0.939092948	0.105014785	-0.162784063	-0.985186132
	3	0.946785605	0.011600522	0.052571970	0.995881475	-0.939093170	0.105014781	-0.162784087	-0.985186249
	4	0.946785616	0.011600519	0.052571993	0.995881456	-0.939093144	0.105014777	-0.162784083	-0.985186234
5	0	0.946785662	0.011599638	0.052570827	0.995882729	-0.939094605	0.105015867	-0.162784378	-0.985187486
	1	0.946785636	0.011600512	0.052571984	0.995881444	-0.939093172	0.105014796	-0.162784087	-0.985186247
	2	0.946785615	0.011600519	0.052571991	0.995881457	-0.939093147	0.105014777	-0.162784084	-0.985186236
6	0	0.946785636	0.011600518	0.052571992	0.995881435	-0.939093162	0.105014789	-0.162784085	-0.985186239
	1	0.946785615	0.011600519	0.052571991	0.995881457	-0.939093147	0.105014777	-0.162784084	-0.985186235

Re-iterated Galerkin Results Using Single Equation Approach									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.947309769	0.010841239	0.058899980	0.976800892	-0.905710395	0.091067587	-0.139968848	-0.956592714
	3	0.946781250	0.011620309	0.052444417	0.996094139	-0.939662076	0.105226874	-0.163278651	-0.985624276
	5	0.946785016	0.011601881	0.052559681	0.995895530	-0.939137641	0.105022344	-0.162813561	-0.985225226
	7	0.946785553	0.011600610	0.052570910	0.995882319	-0.939096477	0.105014736	-0.162785701	-0.985189458
	9	0.946785609	0.011600525	0.052571901	0.995881506	-0.939093387	0.105014730	-0.162784158	-0.985186490
	11	0.946785614	0.011600519	0.052571984	0.995881460	-0.939093163	0.105014770	-0.162784086	-0.985186255
2	0	0.946961887	0.011607595	0.053873549	0.992579481	-0.934296910	0.104863983	-0.162815872	-0.979496117
	2	0.946790282	0.011600392	0.052607845	0.995788304	-0.938955454	0.105008819	-0.162785386	-0.985020663
	4	0.946785748	0.011600516	0.052573030	0.995878799	-0.939089204	0.105014626	-0.162784148	-0.985181493
	6	0.946785618	0.011600519	0.052572021	0.995881382	-0.939093033	0.105014773	-0.162784086	-0.985186100
	8	0.094678561	0.011600519	0.052571992	0.995881456	-0.939093143	0.105014777	-0.162784084	-0.985186231
3	0	0.946783473	0.011598416	0.052561201	0.995886182	-0.939087980	0.105000498	-0.162781030	-0.985166773
	1	0.946785542	0.011600445	0.052571666	0.995881654	-0.939092975	0.105014289	-0.162784001	-0.985185566
	2	0.946785612	0.011600516	0.052571980	0.995881465	-0.939093140	0.105014760	-0.162784082	-0.985186212
	3	0.946785614	0.011600519	0.052571990	0.995881458	-0.939093146	0.105014776	-0.162784084	-0.985186234
4	0	0.946785600	0.011600510	0.052571987	0.995881539	-0.939093150	0.105014691	-0.162784053	-0.985186133
	1	0.946785614	0.011600519	0.052571991	0.995881459	-0.939093146	0.105014775	-0.162784083	-0.985186233

Re-iterated Kantorovich Results Using Coupled Approach									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.953967609	0.022805639	0.061079931	0.993996881	-0.955722336	0.094682016	-0.172635532	-1.015295809
	1	0.946600113	0.011008633	0.052129818	0.995720447	-0.939057673	0.105098921	-0.162708808	-0.984903067
	2	0.946783635	0.011608929	0.052572390	0.995887068	-0.939101388	0.105012886	-0.162787235	-0.985197136
	3	0.946785539	0.011600358	0.052571846	0.995881459	-0.939093094	0.105014843	-0.162784030	-0.985186068
	4	0.946785615	0.011600523	0.052571992	0.995881460	-0.939093148	0.105014776	-0.162784084	-0.985186238
	5	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
2	0	0.944731901	0.013522456	0.053977806	0.998261612	-0.938002481	0.105944919	-0.162869050	-0.985566239
	1	0.946827184	0.011611774	0.052610815	0.995867989	-0.939071736	0.105015722	-0.162781327	-0.985178415
	2	0.094678598	0.011600298	0.052572406	0.995880942	-0.939092661	0.105014693	-0.162784019	-0.985185924
	3	0.946785617	0.011600515	0.052571994	0.995881452	-0.939093141	0.105014776	-0.162784083	-0.985186232
	4	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
3	0	0.946504132	0.011314569	0.052356983	0.995918569	-0.939105275	0.105084728	-0.162803019	-0.985205919
	1	0.946785215	0.011601494	0.052569902	0.995883529	-0.939096201	0.105015232	-0.162784430	-0.985187996
	2	0.946785598	0.011600529	0.052571963	0.995881479	-0.939093177	0.105014777	-0.162784087	-0.985186251
	3	0.946785614	0.011600519	0.052571990	0.995881458	-0.939093147	0.105014777	-0.162784084	-0.985186235
4	0	0.946800230	0.011586234	0.052589054	0.995871809	-0.939074783	0.105021386	-0.162783147	-0.985180669
	1	0.946785765	0.011600435	0.052572207	0.995881281	-0.939092921	0.105014792	-0.162784061	-0.985186123
	2	0.946785616	0.011600518	0.052571993	0.995881456	-0.939093143	0.105014777	-0.162784083	-0.985186234
	3	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
5	0	0.946785662	0.011599638	0.052570827	0.995882729	-0.939094605	0.105015867	-0.162784378	-0.985187486
	1	0.946785615	0.011600515	0.052571985	0.995881464	-0.939093154	0.105014783	-0.162784085	-0.985186242
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
6	0	0.946785636	0.011600518	0.052571992	0.995881435	-0.939093162	0.105014789	-0.162784085	-0.985186239
	1	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235

Re-iterated Kantorovich Results Using Single Equation Approach									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.947309769	0.010841239	0.058899980	0.976800892	-0.905710395	0.091067587	-0.139968848	-0.956592714
	1	0.946772166	0.011524051	0.052442457	0.994950912	-0.937674635	0.103273688	-0.160365333	-0.984518121
	2	0.946782576	0.011595643	0.052533392	0.995827613	-0.939053050	0.104859144	-0.162601509	-0.985214077
	3	0.946785281	0.011600284	0.052567896	0.995879012	-0.939094907	0.105003424	-0.162772849	-0.985193166
	4	0.946785586	0.011600513	0.052571654	0.995881412	-0.939093580	0.105014083	-0.162783543	-0.985187003
	5	0.946785613	0.011600519	0.052571967	0.995881464	-0.939093194	0.105014744	-0.162784070	-0.985186301
	6	0.946785614	0.011600519	0.052571989	0.995881459	-0.939093150	0.105014776	-0.162784085	-0.985186240
2	0	0.946961887	0.011607595	0.053873549	0.992579481	-0.934296910	0.104863983	-0.162815872	-0.979496117
	1	0.946784622	0.011600068	0.052565601	0.995892995	-0.939107511	0.105012281	-0.162783251	-0.985200600
	2	0.946785619	0.011600519	0.052572011	0.995881385	-0.939093051	0.105014763	-0.162784067	-0.985186127
	3	0.946785615	0.011600519	0.0525719 91	0.995881458	-0.939093147	0.105014777	-0.162784084	-0.985186236
	4	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186236
3	0	0.946783473	0.011598416	0.052561201	0.995886182	-0.939087980	0.105000498	-0.162781030	-0.985166773
	1	0.946785612	0.011600516	0.052571979	0.995881465	-0.939093139	0.105014759	-0.162784081	-0.985186210
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
4	0	0.946785600	0.011600510	0.052571987	0.995881539	-0.939093150	0.105014691	-0.162784053	-0.985186133
	1	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
5	0	0.946785614	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
	1	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235

Re-iterated Galerkin/Coupled Approach/$\rho(S)$			
No. of dimensions	1	2	3
No. of iterations	8	6	5
	0.01972453672757	0.00231160298728	0.00022707736917
	0.13467133337891	0.12639801749762	0.10958055308234
	0.13959812482407	0.12247850816997	0.14274823497170
	0.13236529411318	0.14060357233738	0.12322316490550
	0.15235695256579	0.11382022540380	0.12997518689376
	0.13488220387719	0.09726715523733	0.12872143854321
	0.12474648704686	0.08087100912276	
	0.12413789064624		
	0.12953947792903		
No. of dimensions	4	5	6
No. of iterations	4	2	1
	0.00002022896025	0.00000167492878	0.00000001944983
	0.12527162296129	0.01410065248910	0.03210416478565
	0.08777114215125	0.03587043465389	
	0.11025866728479		
	0.09808653333133		

Re-iterated Galerkin/Single Equation Approach/$\rho(S)$		
No. of dimensions	1	2
No. of iterations	11	8
	0.03033312808798	0.00432373230730
	0.21734388238320	0.16942671935119
	0.29288226641174	0.16929004455630
	0.26267682490394	0.16920674411420
	0.27496514602763	0.16918460965480
	0.27012376033304	0.16917941710301
	0.27200496551546	0.16917824994819
	0.27127720196393	0.16917799276970
	0.27155648194311	0.16917793696830
	0.27144981533752	
	0.27149033356792	
	0.27147501192256	
No. of dimensions	3	4
No. of iterations	3	1
	0.00001272612168	0.00000006358617
	0.03227006047818	0.01839193519326
	0.03401462680316	
	0.03456444105728	

Re-iterated Kantorovich/Coupled Approach/$\rho(SK)$			
No. of dimensions	1	2	3
No. of iterations	5	4	3
	0.01806225878335	0.00202747925896	0.00018090282418
	0.02105127418584	0.01740708491188	0.01611315469522
	0.02133824253584	0.01660464662182	0.01136497212850
	0.01672271610614	0.01036570138630	0.01291630350035
	0.02047842505371	0.00767859344530	
	0.01843812971885		
No. of dimensions	4	5	6
No. of iterations	3	2	1
	0.00002186550165	0.00000142283716	0.00000001582483
	0.01183470408028	0.00509745659218	0.00120035195828
	0.01235973116278	0.00485680835716	
	0.01270350232230		

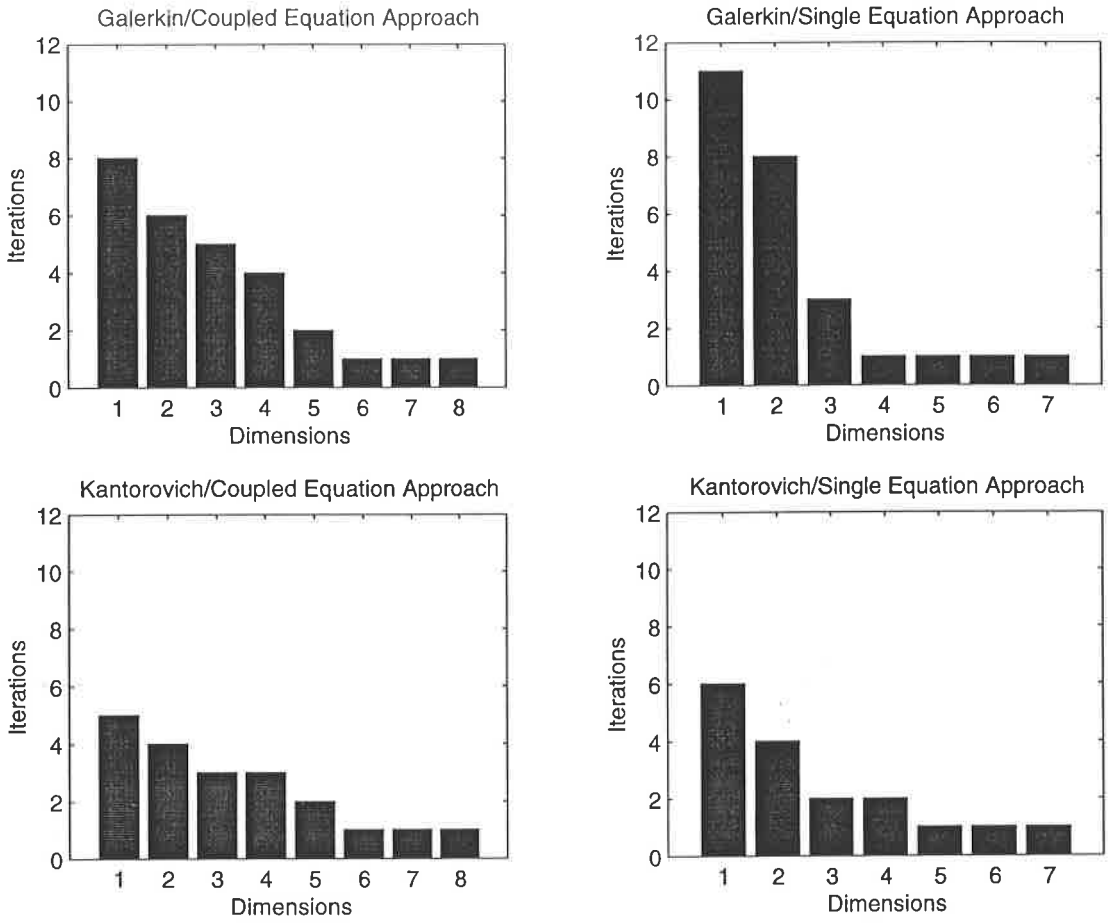
Re-iterated Kantorovich/Single Equation Approach/$\rho(SK)$			
No. of dimensions	1	2	3
No. of iterations	6	4	2
	0.02672778050730	0.00358338577976	0.00000857598713
	0.06145675494207	0.00332760545439	0.00110327996124
	0.07132332552170	0.00623598635800	0.00119574181234
	0.07145134906443	0.00439722670313	
	0.07121546067534	0.00528974194214	
	0.07118838480550		
	0.07119123100288		
No. of dimensions	4	5	-
No. of iterations	2	1	-
	0.0000002203499216	0.0000000010553718	-
	0.0047728165320711	0.0020242317303558	-
	0.0048821330239489		

Re-iterated Galerkin/Coupled Approach/$\| K \|$			
No. of dimensions	1	2	3
No. of iterations	8	6	5
	0.13073900774380	0.16068213013215	0.11832640333830
	0.16417287435047	0.13317028876590	0.11667004286585
	0.13993307391025	0.12866106042748	0.16272084785285
	0.13538259909492	0.14667666589915	0.15289289632315
	0.15291485850602	0.14474590851132	0.15351412978721
	0.15620714607805	0.13620528751002	0.15546282583155
	0.15893930449102	0.12539169804454	
	0.16044729548167		
	0.16051756773681		
No. of dimensions	4	5	6
No. of iterations	4	2	1
	0.10243149519602	0.14320310811857	0.06791021272024
	0.16023830226581	0.07998905303200	0.07634898517525
	0.13429630132786	0.08377470200756	
	0.15131978055025		
	0.14367885313021		

Re-iterated Galerkin/Single Equation Approach/$\| K \|$		
No. of dimensions	1	2
No. of iterations	11	8
	0.22698780997811	0.24299813541794
	0.19539861054483	0.24222219858622
	0.20775428599464	0.24210335462227
	0.20290626796297	0.24213773808814
	0.20558455985329	0.24220562535297
	0.20652651755075	0.24228024039011
	0.20920193816145	0.24235562215890
	0.21233301211711	0.24243044354433
	0.21638573629596	0.24250440611048
	0.22117510839114	
	0.22673050728455	
	0.23297033211150	
No. of dimensions	3	4
No. of iterations	3	1
	0.10440747920543	0.06405115689104
	0.10576465734599	0.06381244094100
	0.10639751193703	
	0.10681147283319	

Re-iterated Kantorovich/Coupled Approach/$\| K \|$			
No. of dimensions	1	2	3
No. of iterations	5	4	3
	0.00276608285187	0.00030698990223	0.00002739390424
	0.00302631793098	0.00235152965436	0.00201187150587
	0.00308359179560	0.00248207352072	0.00170823894974
	0.00230673213538	0.00153641431236	0.00197655840818
	0.00302320798950	0.00115272451016	
	0.00259097062154		
No. of dimensions	4	5	6
No. of iterations	3	2	1
	0.00000331107307	0.00000215458944	0.00000002396340
	0.00191087395178	0.00628433362492	0.00152202037627
	0.00190112261115	0.00598645633766	
	0.00193078894997		

Re-iterated Kantorovich/Single Equation Approach/$\| K \|$			
No. of dimensions	1	2	3
No. of iterations	6	4	2
	0.00419054874920	0.00538756427813	0.00001298615893
	0.01011547071352	0.00411274987120	0.00119051610216
	0.01139060416764	0.00819645494870	0.00118917423974
	0.01116891312391	0.00555041057216	
	0.01112108490083	0.00682143197632	
	0.01112138919881		
	0.01112258385629		
No. of dimensions	4	5	-
No. of iterations	2	1	-
	0.00000003336738	0.00000000015981	
	0.00026435127797	0.00013764741738	
	0.00027532474554		



It seems that the most notable advantage of the coupled approach is that less iterations are needed when a 1D subspace is opted. Also we can see that in general less iterations are required using the re-iterated Kantorovich method than using the re-iterated Galerkin method for 1,2 or 3 dimensional subspaces. In terms of cost the re-iterated Kantorovich is slightly more expensive to run due to the regularization step. However, this is a comparatively small sacrifice noting the lesser number of iterations that the re-iterated Kantorovich require. It was found that increasing the size of the subspace (to 7 or 8 dimensions) causes the coefficient matrix A (see previous chapter) to become near singular. There was the possibility of performing some normalisation (to keep the magnitude of the numbers within a range that the computer can cope with) before constructing

the matrix but this was deemed unecessary, as only one iteration was needed for 5 or 6 dimensional subspaces and good results were obtained with even smaller subspaces with a reasonable number of iterations. Therefore normalisation was never carried out. Underestimates for $\rho(S), \rho(SK)$ and $\| K \|$ should not be taken too literally. It is unclear as to how close these are to the true values[1] and, because the consecutive residuals decrease in size, iterating more times only allow rounding errors to seep in.

7.3 Extended Subspace Results

Here we present the results by using an extended subspace as outlined previously in Chapter 4. The idea was tried with the re-iterated Galerkin and re-iterated Kantorovich methods.

Note: The number of dimensions running down the first column may be a little misleading. This is because, for example, the 3D subspace refers to the **two** sets of subspaces (as shown in (4.2) and (4.3)) so where

$$\begin{aligned} p_1, q_1 &\in \text{span}\{f_1, K_{11}f_1, K_{12}f_2\} \\ p_2, q_2 &\in \text{span}\{f_2, K_{21}f_1, K_{22}f_2\}, \end{aligned}$$

the results for which are expected to be more or less equivalent to the previous set of results for the 2D subspace where

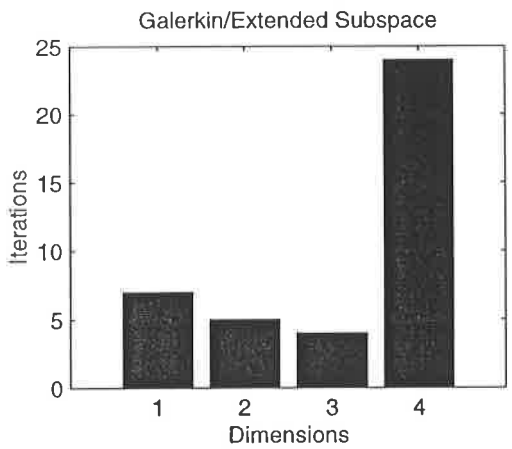
$$\begin{aligned} p_1, q_1 &\in \text{span}\{f_1, K_{11}f_1 + K_{12}f_2\} \\ p_2, q_2 &\in \text{span}\{f_2, K_{21}f_1 + K_{22}f_2\}. \end{aligned}$$

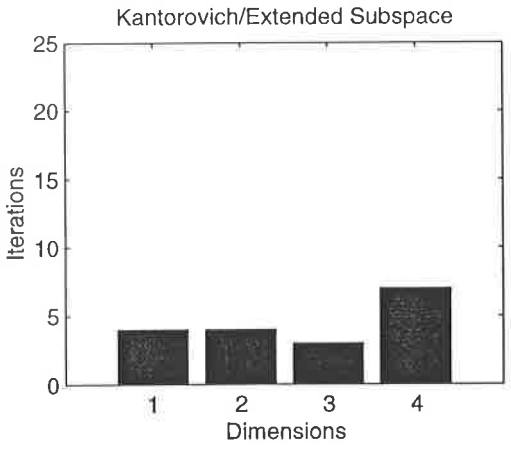
Intuitively we would think that the extended subspace approach is slightly better due primarily to the extra degree of freedom endowed upon the Galerkin coefficients.

[1] although in Burton's thesis ([1] p32-p33) there was mention that the closer $\mathbf{r}_n$ is to a multiple of the eigenvector corresponding to the largest positive eigenvalue of K, the closer the ratio $\frac{\|\tilde{\mathbf{r}}_n\|}{\|\mathbf{r}_n\|}$ is to $\| K \|$.

Re-iterated Galerkin Results Using Extended Subspace									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.953885808	0.013046233	0.054376427	0.990936143	-0.956650420	0.096810281	-0.172363317	-1.013417749
	3	0.946772498	0.011598073	0.052575099	0.995895248	-0.939071628	0.105032471	-0.162777778	-0.985166348
	5	0.946785475	0.011600491	0.052572031	0.995881601	-0.939092880	0.105014975	-0.162784003	-0.985185979
	7	0.946785613	0.011600519	0.052571991	0.995881459	-0.939093143	0.105014779	-0.162784083	-0.985186232
2	0	0.934545899	0.010687864	0.048986734	0.999882056	-0.942537277	0.102909898	-0.167068455	-0.996160212
	3	0.946785597	0.011600620	0.052573455	0.995881896	-0.939092968	0.105015030	-0.162783933	-0.985186311
	5	0.946785614	0.011600519	0.052571995	0.995881459	-0.939093146	0.105014777	-0.162784083	-0.985186236
3	0	0.946820800	0.011688259	0.053015909	0.995777644	-0.939126533	0.105500379	-0.162831764	-0.985319114
	2	0.946785652	0.011600562	0.052572155	0.995881354	-0.939093194	0.105015102	-0.162784118	-0.985186302
	4	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
4	0	1.316417806	1.295166257	1.248625301	1.368604110	-1.228980203	0.061712854	-1.471242689	-1.193361401
	6	0.948258026	0.000232417	0.059217150	1.002352665	-0.947480478	0.011296512	-0.180052921	-0.098937429
	12	0.946820543	0.000117879	0.052704072	0.099596720	-0.939193823	0.010512304	-0.163029895	-0.098523993
	18	0.946785685	0.011600912	0.052572261	0.995881642	-0.939093368	0.105015013	-0.162784608	-0.985186352
	24	0.946785621	0.011600547	0.052572013	0.995881469	-0.939093157	0.105014791	-0.162784117	-0.985186242

Re-iterated Kantorovich Results Using Extended Subspace									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.953885808	0.013046233	0.054376427	0.990936143	-0.956650420	0.096810281	-0.172363317	-1.013417749
	2	0.946785066	0.011600449	0.052570211	0.995882640	-0.939095379	0.105014920	-0.162784561	-0.985187976
	4	0.946785614	0.011600519	0.052571990	0.995881458	-0.939093147	0.105014777	-0.162784084	-0.985186235
2	0	0.934545899	0.010687864	0.048986734	0.999882056	-0.942537277	0.102909898	-0.167068455	-0.996160212
	2	0.946785763	0.011600495	0.052571829	0.995881635	-0.939093138	0.105014994	-0.162784035	-0.985186111
	4	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
3	0	0.946820800	0.011688259	0.053015909	0.995777644	-0.939126533	0.105500379	-0.162831764	-0.985319114
	1	0.946785551	0.011600815	0.052574210	0.995881333	-0.939093124	0.105015872	-0.162784126	-0.985186704
	2	0.946785614	0.011600520	0.052572001	0.995881458	-0.939093146	0.105014779	-0.162784083	-0.985186237
	3	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
	4	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
4	0	0.916980446	-0.095982998	-0.045104365	0.962664197	-0.911354589	0.060415597	-0.049854228	-0.966400053
	3	0.946804640	0.011685123	0.052638428	0.995914599	-0.939129532	0.105060022	-0.162886369	-0.985207029
	5	0.946786043	0.011601329	0.052573195	0.995881374	-0.939092712	0.105014717	-0.162784333	-0.985186117
	7	0.946785613	0.011600511	0.052571985	0.995881454	-0.939093142	0.105014772	-0.162784073	-0.985186233





Before making observations on this set of data we will look at the matrix system $A\mathbf{x} = \mathbf{y}$ itself for the 2D case using the extended subspace. Here is the coefficient matrix A:

$$\begin{pmatrix} \langle (I-K_{11})f_1, f_1\rangle & -\langle K_{12}f_2, f_1\rangle & \langle (I-K_{11})K_{11}f_1, f_1\rangle & -\langle K_{12}K_{21}f_1, f_1\rangle \\ -\langle K_{21}f_1, f_2\rangle & \langle (I-K_{22})f_2, f_2\rangle & -\langle K_{21}K_{11}f_1, f_2\rangle & \langle (I-K_{22})K_{21}f_1, f_2\rangle \\ \langle (I-K_{11})f_1, K_{11}f_1\rangle & -\langle (K_{12}f_2, K_{11}f_1\rangle & \langle (I-K_{11})K_{11}f_1, K_{11}f_1\rangle & -\langle K_{12}K_{21}f_1, K_{11}f_1\rangle \\ -\langle K_{21}f_1, K_{21}f_1\rangle & \langle (I-K_{22})f_2, K_{21}f_1\rangle & -\langle K_{21}K_{11}f_1, K_{21}f_1\rangle & \langle (I-K_{22})K_{21}f_1, K_{21}f_1\rangle \end{pmatrix}$$

and

$$\mathbf{x} = \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} \langle f_1, f_1\rangle \\ \langle f_2, f_2\rangle \\ \langle f_1, K_{11}f_1\rangle \\ \langle f_2, K_{21}f_1\rangle \end{pmatrix}.$$

This is a 4×4 matrix and not a 2×2 matrix (c.f. Chapter 6) as we had before with the straightforward coupled system but notice that elements are the same in both but that they have been split up for the subspace extension.

Looking at the data the subspace extension worked quite well to begin with. Comparison of the 'Galerkin/Coupled Equation Approach' and 'Galerkin/Extended Subspace Approach' shows that for a 1D subspace the latter takes 7 iterations whilst the former takes 8 iterations. Remembering that the 3D subspace of the latter is approximately equivalent to the 2D subspace of the former we also see that the latter performs better with only 4 iterations against 6. We can make

similar observations with the re-iterated Kantorovich. These improvements do not last though. The coefficient matrix A becomes singular as we increase the size of the subspace due to the splitting of the subspace elements. The entries to the bottom right hand side of the coefficient matrix get smaller and smaller; the splitting of the subspace only serves to worsen the situation. This is illustrated with the observation that the number of iterations increase quite dramatically when a 4D subspace is chosen.

Estimates for spectral radii and norms of K have been purposely omitted. Recall that with this extended subspace we do not strictly use a Galerkin method.

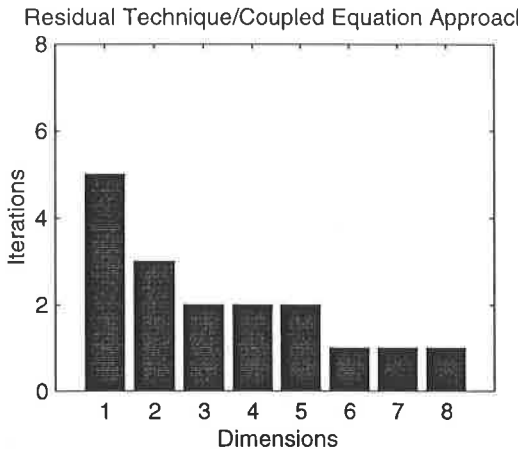
7.4 Residual Technique

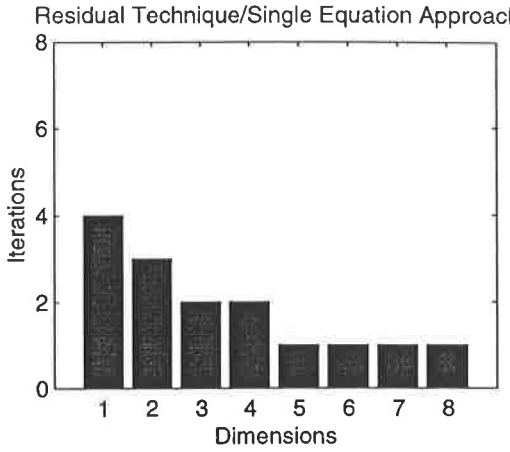
New ideas are often stumbled upon. The 'residual technique' is one such example. This section contains results for when the free term is replaced by the new residual after each iteration. Here are the results.

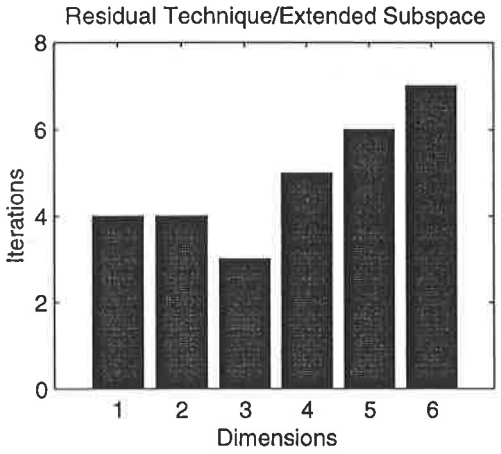
Residual Technique: Re-iterated Galerkin Results Using Coupled Equation Approach									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.953967609	0.022805639	0.061079931	0.993996881	-0.955722336	0.094682016	-0.172635532	-1.015295809
	1	0.946501062	0.011093886	0.052162375	0.995819808	-0.938991313	0.105076548	-0.162614417	-0.984920483
	2	0.946789009	0.011607781	0.052575448	0.995884179	-0.939100200	0.105015804	-0.162790964	-0.985193840
	3	0.946785482	0.011600399	0.052571890	0.995881490	-0.939093034	0.105014835	-0.162783997	-0.985186064
	4	0.946785617	0.011600523	0.052571994	0.995881459	-0.939093148	0.105014777	-0.162784086	-0.985186238
	5	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
2	0	0.944731901	0.013522456	0.053977806	0.998261612	-0.938002481	0.105944919	-0.162869050	-0.985566239
	1	0.946788183	0.011598839	0.052571542	0.995881115	-0.939096525	0.105017863	-0.162788980	-0.985187460
	2	0.946785614	0.011600511	0.052571986	0.995881453	-0.939093143	0.105014772	-0.162784075	-0.985186233
	3	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
3	0	0.946504132	0.011314569	0.052356983	0.995918569	-0.939105275	0.105084728	-0.162803019	-0.985205919
	1	0.946785589	0.011600523	0.052571990	0.995881505	-0.939093130	0.105014808	-0.162784065	-0.985186229
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
4	0	0.946800230	0.011586234	0.052589054	0.995871809	-0.939074783	0.105021386	-0.162783147	-0.985180669
	1	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
5	0	0.946785662	0.011599638	0.052570827	0.995882729	-0.939094605	0.105015867	-0.162784378	-0.985187486
	1	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
6	0	0.946785636	0.011600518	0.052571992	0.995881435	-0.939093162	0.105014789	-0.162784085	-0.985186239
	1	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235

Residual Technique Re-iterated Galerkin Results Using Single Equation Approach									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.947309769	0.010841239	0.058899980	0.976800892	-0.905710395	0.091067587	-0.139968848	-0.956592714
	1	0.947009244	0.011694520	0.052799256	0.995803725	-0.939047933	0.105402972	-0.163369329	-0.984821684
	2	0.946784005	0.011604453	0.052572914	0.995884915	-0.939095208	0.105019206	-0.162790679	-0.985184934
	3	0.946785548	0.011600540	0.052571979	0.995881479	-0.939093140	0.105014735	-0.162784054	-0.985186266
	4	0.946785614	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014776	-0.162784083	-0.985186236
2	0	0.946961887	0.011607595	0.053873549	0.992579481	-0.934296910	0.104863983	-0.162815872	-0.979496117
	1	0.946782919	0.011601021	0.052569585	0.995880666	-0.939092019	0.105011463	-0.162780702	-0.985184099
	2	0.946785613	0.011600519	0.052571981	0.995881452	-0.939093148	0.105014760	-0.162784063	-0.985186235
	3	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
3	0	0.946783473	0.011598416	0.052561201	0.995886182	-0.939087980	0.105000498	-0.162781030	-0.985166773
	1	0.946785614	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784083	-0.985186235
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
4	0	0.946785600	0.011600510	0.052571987	0.995881539	-0.939093150	0.105014691	-0.162784053	-0.985186133
	1	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
5	0	0.946785614	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
	1	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235

Residual Technique: Re-iterated Galerkin Results Using Extended Subspace									
No. of dimensions	No. of iterations	$x \approx 0$		$x \approx 1$		$x \approx 2$		$x \approx 3$	
		Re	Im	Re	Im	Re	Im	Re	Im
1	0	0.953885808	0.013046233	0.054376427	0.990936143	-0.956650420	0.096810281	-0.172363317	-1.013417749
	1	0.946495191	0.011366982	0.052286956	0.995901512	-0.939061996	0.104884448	-0.162536004	-0.985159944
	2	0.946788353	0.011601597	0.052572623	0.995882882	-0.939097318	0.105018153	-0.162790593	-0.985187976
	3	0.946785564	0.011600479	0.052571924	0.995881465	-0.939093151	0.105014766	-0.162784017	-0.985186230
	4	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093147	0.105014777	-0.162784085	-0.985186236
2	0	0.934545899	0.010687864	0.048986734	0.999882056	-0.942537277	0.102909898	-0.167068455	-0.996160212
	1	0.946872524	0.011507186	0.052556385	0.995818107	-0.939095280	0.105101279	-0.162867083	-0.985096581
	2	0.946784351	0.011601282	0.052571179	0.995882679	-0.939094358	0.105014613	-0.162783684	-0.985187615
	3	0.946785633	0.011600501	0.052571987	0.995881446	-0.939093147	0.105014795	-0.162784096	-0.985186213
	4	0.946785614	0.011600519	0.052571991	0.995881458	-0.939093147	0.105014777	-0.162784083	-0.985186235
3	0	0.946820800	0.011688259	0.053015909	0.995777644	-0.939126533	0.105500379	-0.162831764	-0.985319114
	1	0.946785618	0.011600576	0.052572051	0.995881282	-0.939093204	0.105014736	-0.162784025	-0.985186025
	2	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
	3	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
4	0	1.316417806	1.295166257	1.248625301	1.368604110	-1.228980203	0.617128549	-1.471242689	-1.193361401
	1	0.946495720	0.010851234	0.051715253	0.996121180	-0.940060683	0.104894721	-0.162722765	-0.986259845
	2	0.946784203	0.011565211	0.052531358	0.995872970	-0.939113694	0.105004287	-0.162763313	-0.985202017
	3	0.946785722	0.011600539	0.052572075	0.995881358	-0.939093059	0.105014687	-0.162783983	-0.985186208
	4	0.946785615	0.011600519	0.052571991	0.995881457	-0.939093146	0.105014776	-0.162784083	-0.985186235
	5	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235
5	0	1.316753416	1.432597140	1.351580213	1.433302979	-1.252278303	0.652507747	-1.566803875	-1.228103126
	1	0.925762422	0.022883022	0.050389691	1.019458417	-0.944020451	0.101026043	-0.157406498	-1.003771622
	2	0.946359087	0.010790554	0.051847215	0.995692027	-0.938635295	0.104716595	-0.161871478	-0.984778576
	3	0.946759249	0.011636010	0.052598810	0.995898846	-0.939072390	0.105026756	-0.162804771	-0.985168551
	4	0.946785469	0.011600164	0.052571623	0.995881222	-0.939093429	0.105014485	-0.162784069	-0.985186556
	5	0.946785615	0.011600518	0.052571991	0.995881457	-0.939093146	0.105014776	-0.162784083	-0.985186236
	6	0.946785615	0.011600519	0.052571991	0.995881458	-0.939093146	0.105014777	-0.162784084	-0.985186235







Again estimates for the spectral radii of S and SK and the norm of K have been omitted. This time because the subspace is altered with each iteration (recall that the free term is replaced with the new residual after each iteration).

By looking at the results it appears as though the technique works just as well as the re-iterated Kantorovich method. How can this be?

Let as try and ascertain what is happening when we run the program. Taking a 2D subspace our first Galerkin approximation is given by

$$\mathrm{p}_0 = a_1\mathrm{f} + a_2 K\mathrm{f}$$

where a_1, a_2 are our Galerkin coefficients. Our iterated approximation is then

$$\hat{\mathrm{p}}_0 = \mathrm{f} + K\mathrm{p}_0$$

and by introducing a current error term ϕ_1 we obtain

$$\mathbf{\Phi}_1 = \hat{\mathbf{r}}_0 + K\mathbf{\Phi}_1$$

where $\hat{\mathbf{r}}_0 = \mathbf{f} - (I - K)\hat{\mathbf{p}}_0$.

Now *instead* of finding a Galerkin approximation $\mathbf{p}_1$ to $\mathbf{\Phi}_1$ by taking

$$\mathbf{p}_1 = a_3\mathbf{f} + a_4K\mathbf{f}$$

we replace the free term by the residual $\hat{\mathbf{r}}_0$ so that

$$\mathbf{p}_1 = a_3\hat{\mathbf{r}}_0 + a_4K\hat{\mathbf{r}}_0.$$

The iterated approximation will then be

$$\hat{\mathbf{p}}_1 = \hat{\mathbf{r}}_0 + K\mathbf{p}_1$$

which leads us to

$$\mathbf{\Phi}_2 = \hat{\mathbf{r}}_1 + K\mathbf{\Phi}_2$$

where $\hat{\mathbf{r}}_1 = \mathbf{f} - (I - K)(\hat{\mathbf{p}}_0 + \mathbf{p}_1)$.

So what consequences does replacing the free term by successive residuals have?

Let us consider the residual $\hat{\mathbf{r}}_0$. On expansion

$$\begin{aligned}
\hat{\mathbf{r}}_0 &= \mathbf{f} - (I - K)\hat{\mathbf{p}}_0 \\
&= \mathbf{f} - (I - K)(\mathbf{f} + K\mathbf{p}_0) \\
&= \mathbf{f} - \mathbf{f} - K\mathbf{p}_0 + K\mathbf{f} + K^2\mathbf{p}_0 \\
&= K\mathbf{f} - (I - K)K\mathbf{p}_0 \\
&= K\mathbf{f} - (I - K)K(a_1\mathbf{f} + a_2K\mathbf{f}) \\
&= K\mathbf{f} - a_1K\mathbf{f} - a_2K^2\mathbf{f} + a_1K^2\mathbf{f} + a_2K^3\mathbf{f} \\
&= (1 - a_1)K\mathbf{f} + (a_1 - a_2)K^2\mathbf{f} + a_2K^3\mathbf{f}
\end{aligned}$$

where the absence of $\mathbf{f}$ is apparent. Instead of $\mathbf{p}_1$ being spanned by $\{\mathbf{f}, K\mathbf{f}\}$, it is now spanned by $\{\hat{\mathbf{r}}_0, K\hat{\mathbf{r}}_0\}$ which is made up of a linear combination of $\{K\mathbf{f}, K^2\mathbf{f}, K^3\mathbf{f}, K^4\mathbf{f}\}$.

Let us take a look at the next residual $\hat{\mathbf{r}}_1$ for which $\mathbf{p}_2 = a_5\hat{\mathbf{r}}_1 + a_6K\hat{\mathbf{r}}_1$. Now

$$\begin{aligned}
\hat{\mathbf{r}}_1 &= \mathbf{f} - (I-K)(\hat{\mathbf{p}}_0 + \hat{\mathbf{p}}_1) \\
&= \{\mathbf{f} - (I-K)\hat{\mathbf{p}}_0\} - (I-K)\hat{\mathbf{p}}_1 \\
&= \hat{\mathbf{r}}_0 - (I-K)(\hat{\mathbf{r}}_0 + K\mathbf{p}_1) \\
&= \hat{\mathbf{r}}_0 - \hat{\mathbf{r}}_0 - K\mathbf{p}_1 + K\hat{\mathbf{r}}_0 + K^2\mathbf{p}_1 \\
&= K\hat{\mathbf{r}}_0 - (I-K)K\mathbf{p}_1 \\
&= K\hat{\mathbf{r}}_0 - (I-K)K(a_3\hat{\mathbf{r}}_0 + a_4K\hat{\mathbf{r}}_0) \\
&= K\hat{\mathbf{r}}_0 - a_3K\hat{\mathbf{r}}_0 - a_4K^2\hat{\mathbf{r}}_0 + a_3K^2\hat{\mathbf{r}}_0 + a_4K^3\hat{\mathbf{r}}_0 \\
&= (1-a_3)K\hat{\mathbf{r}}_0 + (a_3 - a_4)K^2\hat{\mathbf{r}}_0 + a_4K^3\hat{\mathbf{r}}_0.
\end{aligned}$$

So $\mathbf{p}_2$ is spanned by $\{\hat{\mathbf{r}}_1, K\hat{\mathbf{r}}_1\}$ which is made up of a linear combination of $\{K^2\mathbf{f}, K^3\mathbf{f}, ..., K^8\mathbf{f}\}$, a total of 7 elements!

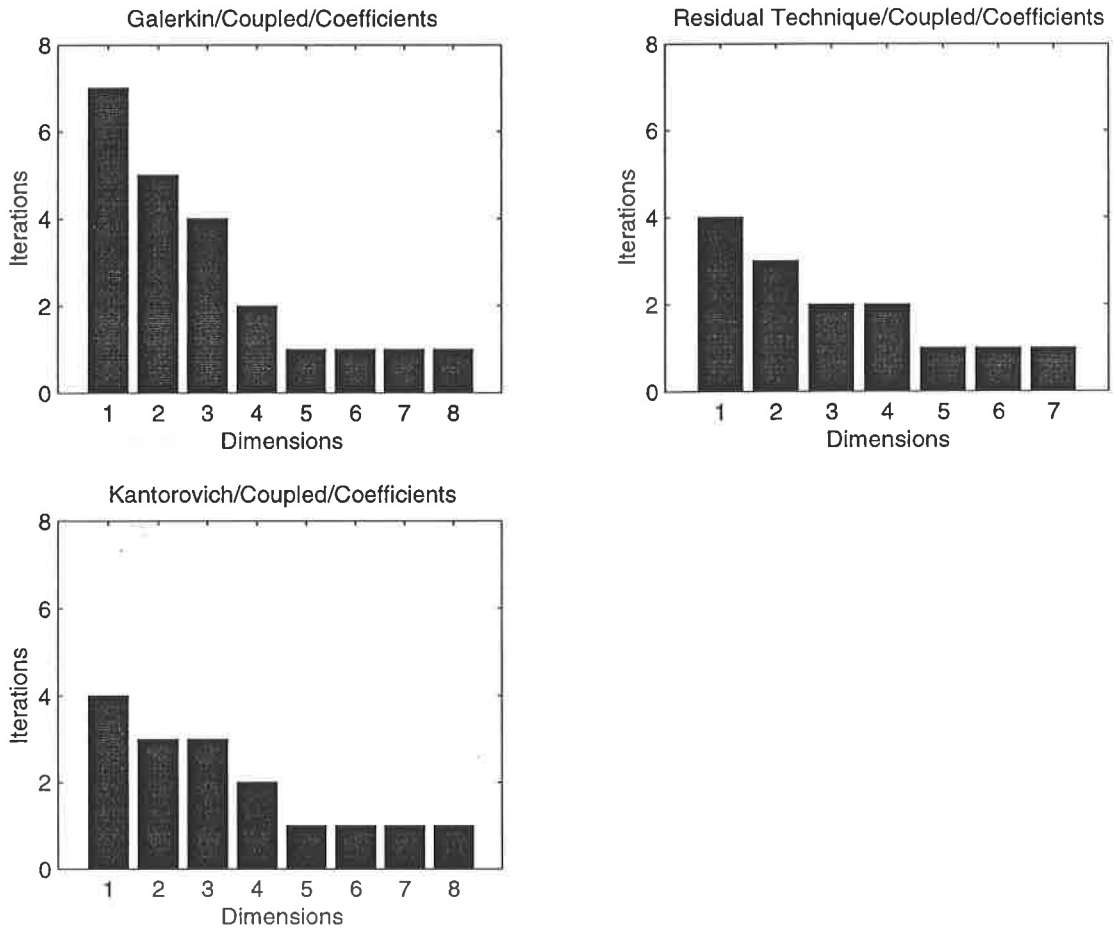
In general therefore,

$$\hat{\mathbf{r}}_n = (1-a_{2n+1})K\hat{\mathbf{r}}_{n-1} + (a_{2n+1} - a_{2n+2})K^2\hat{\mathbf{r}}_{n-1} + a_{2n+2}K^3\hat{\mathbf{r}}_{n-1}$$

for $n = 0, 1, 2, \ldots$, where $\hat{\mathbf{r}}_{-1}$ is defined to be $\mathbf{f}$. As with the Kantorovich methods $\mathbf{f}$ is taken out of the iterative proceedure. Unfortunately, there is a price to pay. The program takes longer to run because the matrix of coefficients, A, has to be computed again for every iteration. There is no need for this with the re-iterated Galerkin or re-iterated Kantorovich methods.

7.5 Galerkin Variant

Returning to Section 4.2, it may have been obvious that two leaps were made when one would have been more appropriate for comparison. What would happen if instead of both (1) allowing some leeway for the Galerkin coefficients, and (2) extending the subspace, we only permit (1) to occur? How would this compare to our usual coupled system approach?



Again looking at the corresponding system before analysing the data we have, for a 2D subspace (that refers to the two sets of basis functions $\{f_1, K_{11}f_1 + K_{12}f_2\}$ and$\{f_2, K_{21}f_1 + K_{22}f_2\}$), our matrix A as

$$\begin{pmatrix} \langle (I-K_{11})f_1, f_1\rangle & -\langle K_{12}f_2, f_1\rangle & \langle (I-K_{11})K_1^{\alpha}, f_1\rangle & -\langle K_{12}K_1^{\beta}, f_1\rangle \\ -\langle K_{21}f_1, f_2\rangle & \langle (I-K_{22})f_2, f_2\rangle & -\langle K_{21}K_1^{\alpha}, f_2\rangle & \langle (I-K_{22})K_1^{\beta}, f_2\rangle \\ \langle (I-K_{11})f_1, K_1^{\alpha}\rangle & -\langle (K_{12}f_2, K_1^{\alpha}\rangle & \langle (I-K_{11})K_1^{\alpha}, K_1^{\alpha}\rangle & -\langle K_{12}K_1^{\beta}, K_1^{\alpha}\rangle \\ -\langle K_{21}f_1, K_1^{\beta}\rangle & \langle (I-K_{22})f_2, K_1^{\beta}\rangle & -\langle K_{21}K_1^{\alpha}, K_1^{\beta}\rangle & \langle (I-K_{22})K_1^{\beta}, K_1^{\beta}\rangle \end{pmatrix}$$

and

$$\mathrm{x} = \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix}, \qquad \mathrm{y} = \begin{pmatrix} \langle f_1, f_1\rangle \\ \langle f_2, f_2\rangle \\ \langle f_1, K_1^{\alpha}\rangle \\ \langle f_2, K_1^{\beta}\rangle \end{pmatrix}.$$

Notice that the matrix A has more splitting than for the usual coupled system approach but less than for the extended subspace approach. Still if there was one expected but unwanted consequence it was that the coefficient matrix, A, would become close to singular just as it did when the extended subspace was in use. This simply did not become a problem. The non-singularity of the matrix A held out for as long as was necessary.

Comparing these bar graphs with their corresponding coupled system counterparts shows a marked improvement. In general one less iteration is needed when the sets of Galerkin coefficients are allowed to differ.

Obviously, due to the extra coefficients there were more calculations for the computer to perform but the difference in run time was negligible.

Chapter 8

Conclusions

Overall, for the wave scattering equation (2.2), the re-iterated Kantorovich method required fewer iterations at a slightly greater cost than for the re-iterated Galerkin method (the estimates for $\rho(SK)$ were generally smaller than the estimates for $\rho(S)$). In terms of the number of iterations required, the coupled approach appears to have worked better for smaller dimensional subspaces and the single equation approach appears to have worked better for bigger dimensional subspaces. Between the coupled and single equation approach neither is much better than the other because with a smaller subspace there is the need for more iterations and for a larger subspace, there is the need for more preliminary calculations due to the extra subspace elements.

There are good possibilities concerning the structure of the coefficient matrix A. Allowing the sets of coefficients to differ and *not* extending the subspace worked well. Another possibility is to use $\{K\mathbf{f}, K^2\mathbf{f}, ...\}$ and not $\{\mathbf{f}, K\mathbf{f}, ...\}$ as the subspace for the re-iterated Kantorovich method. This was hinted in Burton's thesis as possibly being the more appropriate given how $\mathbf{f}$ vanishes after a Kantorovich regularization.

The obvious way forward is to extend the idea of coupled systems to bigger systems. Before embarking on this, one should ask if using coupled systems is actually

better than treating integral equations in a singular manner. Judging by the results in this thesis there was no outright winner. Analysis may yet hold the key and turn over a few untouched stones.

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Appendix

Inner products

We require an inner product (or scalar product) where our arguments are finite vectors. Let our Hilbert space be $L_2^n(a,b)$ so that our inner product can be given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}^T \overline{\mathbf{y}} dt$$

where $\mathbf{x} = (x_1, ..., x_N)^T$ and $\mathbf{y} = (y_1, ..., y_N)^T$ and belong in $L_2^n(a,b)$.
It is simply a formality but we must check that this definition meets the requirements necessary for an inner product

(i) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \overline{\mathbf{y}, \mathbf{x}} \rangle$,

(ii) $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$,

(iii) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$,

(iv) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ when $\mathbf{x} \neq \underline{0}$

where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in H$ and $\lambda \in \mathbf{C}$.
Let the checking commence.
(i)

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}^T \overline{\mathbf{y}} dt$$

$$
\begin{aligned}
&= \int_a^b (x_1, ..., x_N) \overline{\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}} dt \\
&= \int_a^b (x_1\overline{y}_1 + x_2\overline{y}_2 + ... + x_N\overline{y}_N) dt \\
&= \int_a^b \overline{(\overline{x}_1 y_1 + \overline{x}_2 y_2 + ... + \overline{x}_N y_N)} dt \\
&= \int_a^b \overline{\overline{(y_1, y_2, ..., y_N)} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}} dt \\
&= \int_a^b \overline{\mathbf{y}^T \overline{\mathbf{x}}} dt \\
&= \overline{\langle \mathbf{y}, \mathbf{x} \rangle},
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\langle \lambda\mathbf{x}, \mathbf{y} \rangle &= \int_a^b \lambda (x_1, ..., x_N) \overline{\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}} dt \\
&= \lambda \int_a^b (x_1, ..., x_N) \overline{\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}} dt \\
&= \lambda \langle \mathbf{x}, \mathbf{y} \rangle,
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= \int_a^b (x_1 + y_1, ..., x_N + y_N) \overline{\begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}} dt \\
&= \int_a^b [(x_1 + y_1), \overline{z}_1 + (x_2 + y_2)\overline{z}_2 + ... + (x_N + y_N)\overline{z}_N)] dt \\
&= \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle,
\end{aligned}
$$

(iv)

$$
\langle \lambda\mathbf{x}, \mathbf{x} \rangle = \int_a^b (x_1, ..., x_N) \overline{\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}} dt
$$

$$
\begin{aligned}
&= \int_a^b (x_1\overline{x}_1, ..., x_N\overline{x}_N)dt \\
&= \int_a^b \mid x_1 \mid^2 + ... + \mid x_N \mid^2 dt \\
&> 0 \text{ if } \mathbf{x} \neq \underline{0}.
\end{aligned}
$$

Hence the conditions for an inner product are satisfied.

Results using the Neumann series

Results were obtained using consecutive terms of the Neumann series (beginning with $\mathbf{f}$). These are for reference only.

Approximations Using the Neumann Series				
No. of terms	$x \approx 0$		$x \approx 1$	
	Re	**Im**	**Re**	**Im**
1	0.999996056	0.002808531	0.124586418	0.992208760
2	0.952724920	0.023345526	0.059365364	0.994045157
3	0.943558493	0.013117909	0.052746729	0.998575806
4	0.946685171	0.010950409	0.052117218	0.995643243
5	0.946873485	0.011618919	0.052634941	0.995848397
10	0.946785612	0.011600525	0.052571993	0.995881462
15	0.946785615	0.011600519	0.052571991	0.995881458
No. of terms	$x \approx 2$		$x \approx 3$	
	Re	**Im**	**Re**	**Im**
1	-0.960464031	0.278404100	-0.398473262	-0.917179949
2	-0.955594318	0.089721832	-0.166538297	-1.017944770
3	-0.936728444	0.104570951	-0.159556386	-0.984669154
4	-0.939120861	0.105124909	-0.162808342	-0.984903116
5	-0.939084571	0.105039350	-0.162830733	-0.985166683
10	-0.939093146	0.105014777	-0.162784083	-0.985186237
15	-0.939093146	0.105014777	-0.162784084	-0.985186235

It is interesting to note that figures are correct to 7 decimal places after 10 to 15 terms have been used.

Miscellaneous

It is often useful to transform an interval (a, b) to (c, d). Chamberlain's example (2.2) only required the interval $(0, L)$ but perhaps it will be worth while to consider a more general interval.

There are two situations that may arise: either we wish to stretch the interval or contract it. In both cases one relation holds the key:

$$\frac{y-c}{d-c} = \frac{x-a}{b-a}$$

Any mathematician who has been through some basic Cartesian instruction should recognise this as a formula for a straight line. On rearrangement (for transforming from (a, b) to (c, d))

$$y = \frac{(x-a)(d-c)}{b-a} + c$$

or in terms of x (for transforming from (c, d) to (a, b))

$$x = \frac{(y-c)(b-a)}{d-c} + a.$$

The equation of the straight line, besides giving the equation of a straight line, describes the governing ratio for stretching or contracting of an interval.