

SHOCK MODELLING ON IRREGULAR GRIDS

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ABSTRACT

Two irregular grid schemes, based on Roe's linearised Riemann Solver, are presented for the solution of the Euler equations of gas dynamics in one-dimension. One of these schemes gives particularly good results for strongly shocked flows. The schemes are applied to some standard test problems including infinite shock reflection.

1. INTRODUCTION

The (linearised) approximate Riemann solver of Roe [1] reduces the solution of the Euler equations of gas dynamics to that of three scalar problems. We seek here to extend the scalar algorithm proposed by Roe [2] to grids for which the mesh spacing is not constant, whilst retaining an oscillation free scheme. In addition, we suggest a modification of this scheme for strongly shocked flows on a grid where the mesh spacings are in constant ratio. Results for both schemes are presented for a variety of test problems.

In §2 we state some desirable properties of an irregular grid scheme for the solution of a scalar conservation law and in §3 derive a central scheme and an upwind scheme for an irregular grid. In §4 we combine the schemes given in §3 to yield a scheme that is free from spurious oscillations. In §5 we present a different irregular grid scheme from that given in §4 for use with strongly shocked flows while in §6 we describe some specific test problems that can be used to test such schemes. Finally, in §7 we display the numerical results achieved for these test problems and compare the two different schemes proposed here. In the Appendix we show that the scheme given in §4 for a non-linear scalar conservation law is Total Variation Diminishing (TVD) under suitable restrictions.

2. DESIRABLE PROPERTIES OF AN IRREGULAR GRID SCHEME

In this section we define conservation and accuracy for a finite difference algorithm on an irregular grid. In addition we give sufficient conditions for a finite difference algorithm to be total variation diminishing (TVD).

We consider the scalar equation

$$u_t + au_x = 0 \quad (x,t) \in (-\infty, \infty) \times [0, T] \quad (2.1)$$

and define an irregular grid x_j in the x direction where

$x_j = x_{j-1} + \Delta x_{j-\frac{1}{2}}$. In addition we consider $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, where $x_{j+\frac{1}{2}} = \frac{1}{2}(x_j + x_{j+1})$, as the neighbourhood of the point x_j with length $\Delta x_j = \frac{1}{2}(\Delta x_{j-\frac{1}{2}} + \Delta x_{j+\frac{1}{2}})$ (see Fig. 1).

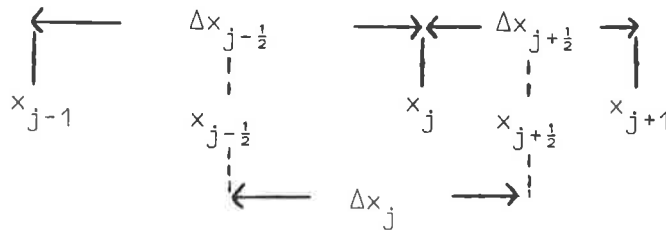


FIG. 1

2.1 Conservation

Using the differential equation (2.1) we have

$$\frac{\partial}{\partial t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u \, dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t \, dx = - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} au_x \, dx$$

i.e.
$$\frac{\partial}{\partial t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u \, dx = - a[u(x_{j+\frac{1}{2}}, t) - u(x_{j-\frac{1}{2}}, t)]$$

Thus

$$\begin{aligned} \sum_{\text{all } j} \frac{\partial}{\partial t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u \, dx &= -a \sum_{\text{all } j} (u(x_{j+\frac{1}{2}}, t) - u(x_{j-\frac{1}{2}}, t)) \\ &= -a [u_R - u_L] \end{aligned} \tag{2.2}$$

by cancellation, where u_L, u_R denote the values of u at the left end and right end of the region of consideration.

A discrete analogue of equation (2.2) is

$$\frac{1}{\Delta t} \sum_{\text{all } j} (u^j - u_j) \Delta x_j = -a [u_R - u_L] \tag{2.3}$$

(i.e. boundary terms) where u_j denotes the approximation to $u(x_j, n\Delta t)$ and u^j denotes the approximation at x_j at the next time level, i.e. $u(x_j, \overline{n+1} \Delta t)$.

We say that a scheme of the form

$$u^j = u_j + \sum_{\text{all } j} \gamma_{j-\frac{1}{2}} (u_j - u_{j-1}) \tag{2.4}$$

for solving equation (2.1) is conservative if equation (2.3) holds.

A standard interpretation is to consider the approximate solution of equation (2.1) as consisting of a set of piecewise constants (see Fig. 2).

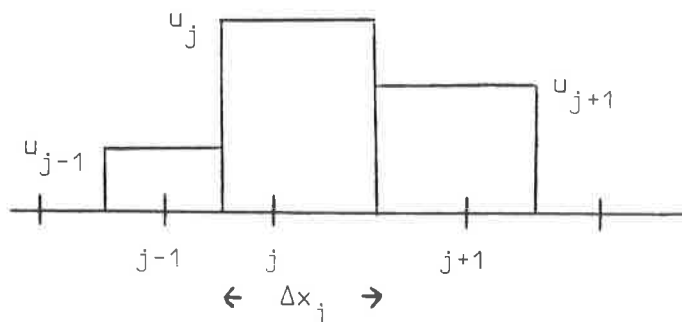


FIG. 2

2.2 Accuracy

We say that a scheme of the form given by equation (2.4) is p th order accurate if it is exactly satisfied for all polynomials of degree up to and including p ; i.e if equation (2.4) is exact for $u_j = x_j^0, x_j^1, x_j^2, \dots, x_j^p$. If the exact solution is a polynomial of degree k then at node j , without loss of generality, $u_j = x_j^k$, $u_j^j = (x_j - a\Delta t)^k$. By choosing x_j to be the origin of x then for p th order accuracy we require

$$(-a\Delta t)^k = - \sum_{\text{all } j} \gamma_{j-\frac{1}{2}} (-\Delta x_{j-\frac{1}{2}})^k \quad (2.5)$$

for $k = 0, 1, \dots, p$.

2.3 Total variation diminishing, (TVD)

The total variation of the solution of the solution u_j^n at time $n\Delta t$ is defined to be

$$TV(u^n) = \sum_j |u_{j+1}^n - u_j^n|$$

and a scheme that is total variation diminishing is one where

$$TV(u^{n+1}) \leq TV(u^n).$$

An important consequence of the TVD property is the prevention of spurious oscillations in the solution.

For a general scheme written in the form

$$u_j^j = u_j - C_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}} + D_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} \quad (2.6)$$

where $C_{j-\frac{1}{2}}, D_{j+\frac{1}{2}}$ are functions of (u_j) (i.e data dependent), it can be shown [3] that sufficient conditions for it to be TVD are

$$0 \leq C_{j+\frac{1}{2}}, \quad 0 \leq D_{j+\frac{1}{2}}, \quad 0 \leq C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1. \quad (2.7)$$

(We denote $\Delta u_{j+\frac{1}{2}} = u_{j+1} - u_j$).

In the next section we derive two schemes for an irregular grid.

3. A CENTRAL AND AN UPWIND SCHEME ON AN IRREGULAR GRID

In this section we develop two schemes for the solution of equation (2.1) on a general irregular grid.

3.1 Central scheme

Consider the centrally based scheme

$$u^j = u_j - \alpha_{j-\frac{1}{2}} \frac{a\Delta t}{\Delta x_j} (u_j - u_{j-1}) - \beta_{j+\frac{1}{2}} \frac{a\Delta t}{\Delta x_j} (u_{j+1} - u_j) \quad (3.1)$$

where $u_j \equiv u(x_j, n\Delta t)$ and $u^j \equiv u(x_j, \overline{n+1}\Delta t)$. We consider conditions on $\alpha_{j-\frac{1}{2}}, \beta_{j+\frac{1}{2}}$.

For conservation as defined by equation (2.3) we require

$$\frac{1}{\Delta t} \sum_{\text{all } j} (u^j - u_j) \Delta x_j = -a[u_R - u_L] \quad .$$

But

$$\frac{1}{\Delta t} \sum_{\text{all } j} (u^j - u_j) \Delta x_j = -a \sum_{\text{all } j} \alpha_{j-\frac{1}{2}} (u_j - u_{j-1}) + \beta_{j+\frac{1}{2}} (u_{j+1} - u_j) \quad (3.2)$$

and hence by comparison of equations (2.3) and (3.2) the scheme given by equation (3.1) is conservative if

$$\alpha_{j+\frac{1}{2}} + \beta_{j+\frac{1}{2}} = 1 \quad . \quad (3.3)$$

If we now require in addition that the scheme is first order accurate as defined by equation (2.5) then we must have

$$-a\Delta t = -\alpha_{j-\frac{1}{2}} \frac{a\Delta t}{\Delta x_j} (\Delta x_{j-\frac{1}{2}}) - \beta_{j+\frac{1}{2}} \frac{a\Delta t}{\Delta x_j} (\Delta x_{j+\frac{1}{2}}) \quad .$$

On substitution of $\beta_{j+\frac{1}{2}}$ using equation (3.3) and rearranging we obtain

$$(\alpha_{j+\frac{1}{2}} - \frac{1}{2})\Delta x_{j+\frac{1}{2}} = (\alpha_{j-\frac{1}{2}} - \frac{1}{2})\Delta x_{j-\frac{1}{2}}$$

Thus

$$(\alpha_{j+\frac{1}{2}} - \frac{1}{2})\Delta x_{j+\frac{1}{2}} = K \quad (3.4)$$

a constant, independent of j . We determine the constant K by making the scheme second order accurate on a regular grid, i.e. on a grid where $\Delta x_{j-\frac{1}{2}} = \Delta x_j = \Delta x$ for all j .

Putting $\alpha_{j+\frac{1}{2}} = \alpha, \beta_{j+\frac{1}{2}} = \beta$ for all j , equations (3.3) and (3.4) become

$$\alpha + \beta = 1 \quad (3.5a)$$

$$(\alpha - \frac{1}{2})\Delta x = K \quad (3.5b)$$

For second order accuracy (on a regular grid) we require

$$(-a\Delta t)^2 = -\alpha \frac{a\Delta t}{\Delta x} (0 - (-\Delta x)^2) = \beta \frac{a\Delta t}{\Delta x} ((\Delta x)^2 - 0)$$

i.e. $\alpha = \beta = \frac{a\Delta t}{\Delta x} \quad (3.6)$

Combining equations (3.5a-b) and (3.6) we find

$$\alpha = \frac{1}{2} \left[1 + \frac{a\Delta t}{\Delta x} \right], \quad \beta = \frac{1}{2} \left[1 - \frac{a\Delta t}{\Delta x} \right]$$

and

$$K = (\alpha - \frac{1}{2})\Delta x = \frac{a\Delta t}{2}$$

Therefore, from (3.4)

$$(\alpha_{j+\frac{1}{2}} - \frac{1}{2})\Delta x_{j+\frac{1}{2}} = K = \frac{a\Delta t}{2}$$

i.e. $\alpha_{j+\frac{1}{2}} = \frac{1}{2} \left[1 + \frac{a\Delta t}{\Delta x_{j+\frac{1}{2}}} \right] = \frac{1}{2}(1 + v_{j+\frac{1}{2}})$

where $v_{j+\frac{1}{2}} = a\Delta t/\Delta x_{j+\frac{1}{2}}$ is the C.F.L. number for the cell $[x_j, x_{j+1}]$.
Also, using equation (3.3),

$$\beta_{j+\frac{1}{2}} = \frac{1}{2} \left(1 - \frac{a\Delta t}{\Delta x_{j+\frac{1}{2}}} \right) = \frac{1}{2}(1 - v_{j+\frac{1}{2}}) .$$

Thus the desired scheme becomes

$$\begin{aligned} u^j &= u_j - \frac{1}{2}(1 + v_{j-\frac{1}{2}})(u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \\ &= \frac{1}{2}(1 - v_{j+\frac{1}{2}})(u_{j+1} - u_j) \frac{a\Delta t}{\Delta x_j} \end{aligned} \quad (3.7)$$

We can rearrange equation (3.7) to give, for $a > 0$,

$$\begin{aligned} u^j &= u_j - (u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \\ &\quad + \frac{1}{2}(1 - v_{j-\frac{1}{2}})(u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \\ &= \frac{1}{2}(1 - v_{j+\frac{1}{2}})(u_{j+1} - u_j) \frac{a\Delta t}{\Delta x_j} \end{aligned} \quad (3.8)$$

and, for $a < 0$,

$$\begin{aligned} u^j &= u_j - (u_{j+1} - u_j) \frac{a\Delta t}{\Delta x_j} \\ &\quad + \frac{1}{2}(1 + v_{j+\frac{1}{2}})(u_{j+1} - u_j) \frac{a\Delta t}{\Delta x_j} \\ &= \frac{1}{2}(1 + v_{j-\frac{1}{2}})(u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \end{aligned} \quad (3.9)$$

It is now possible to identify the following Lax-Wendroff type two stage algorithm for solving equation (2.1): for $a > 0$

$$u^{j-1} = u_{j-1} + \frac{g_{j-\frac{1}{2}}}{\Delta x_{j-1}} \quad (3.10a)$$

$$u^j = u^j + \frac{\phi_{j-\frac{1}{2}}}{\Delta x_j} - \frac{g_{j-\frac{1}{2}}}{\Delta x_j} \quad (3.10b)$$

and, for $a < 0$,

$$u^{j-1} = u_{j-1} + \frac{\phi_{j-\frac{1}{2}}}{\Delta x_{j-1}} = \frac{g_{j-\frac{1}{2}}}{\Delta x_{j-1}} \quad (3.10c)$$

$$u^j = u_j + \frac{g_{j-\frac{1}{2}}}{\Delta x_j} \quad (3.10d)$$

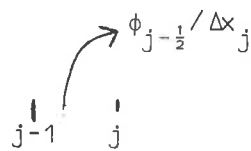
where

$$\phi_{j-\frac{1}{2}} = -a\Delta t(u_j - u_{j-1}) \quad (3.10e)$$

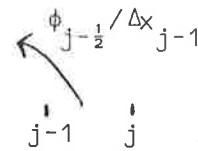
and

$$g_{j-\frac{1}{2}} = \frac{1}{2}(1 - |v_{j-\frac{1}{2}}|)\phi_{j-\frac{1}{2}} \quad (3.10f)$$

Schematically we have for each cell an increment stage, in the form

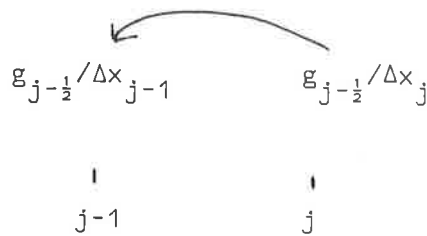


$a > 0$

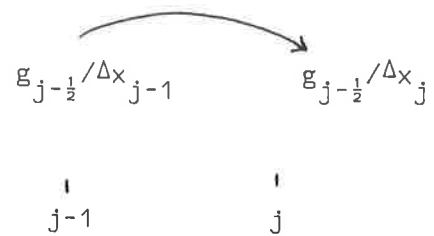


$a < 0$

together with a transfer stage of the form



$a > 0$



$a < 0$

3.1 Upwind scheme

We now repeat the analysis of the previous section for the case of an upwind scheme. Here, we must distinguish between the two cases

(a) $a > 0$ and (b) $a < 0$ at the outset.

(a) $a > 0$.

Consider the upwind scheme

$$u^j = u_j - \alpha_{j-\frac{1}{2}} \frac{a\Delta t}{\Delta x_j} (u_j - u_{j-1}) - \beta_{j-\frac{3}{2}} \frac{a\Delta t}{\Delta x_j} (u_{j-1} - u_{j-2}) \quad (3.11)$$

For conservation as defined by equation (2.3) we find that the scheme given by equation (3.11) is conservative if

$$\alpha_{j-\frac{1}{2}} + \beta_{j-\frac{1}{2}} = 1. \quad (3.12)$$

In addition, for first order accuracy as defined by equation (2.5) we obtain

$$-a\Delta t = -\alpha_{j-\frac{1}{2}} \frac{a\Delta t}{\Delta x_j} (\Delta x_{j-\frac{1}{2}}) - \beta_{j-\frac{3}{2}} \frac{a\Delta t}{\Delta x_j} (\Delta x_{j-\frac{3}{2}}).$$

On substitution of $\alpha_{j-\frac{1}{2}}$ using equation (3.12) and rearranging we find that

$$\beta_{j-\frac{1}{2}} \Delta x_{j-\frac{1}{2}} + \frac{1}{2} \Delta x_{j+\frac{1}{2}} = \beta_{j-\frac{3}{2}} \Delta x_{j-\frac{3}{2}} + \frac{1}{2} \Delta x_{j-\frac{1}{2}}$$

Thus

$$\beta_{j-\frac{1}{2}} \Delta x_{j-\frac{1}{2}} + \frac{1}{2} \Delta x_{j+\frac{1}{2}} = K, \quad (3.13)$$

a constant, independent of j . As before, we can make the scheme second order accurate on a regular grid to determine K .

Thus set $\Delta x_{j-\frac{1}{2}} = \Delta x_j = \Delta x$, $\alpha_{j+\frac{1}{2}} = \alpha$ and $\beta_{j+\frac{1}{2}} = \beta$ for all j so equations (3.12) and (3.13) become

$$\alpha + \beta = 1 \quad (3.14a)$$

$$(\beta + \frac{1}{2})\Delta x = K \quad (3.14b)$$

For second order accuracy (on the regular grid) we require

$$(-a\Delta t)^2 \equiv -\alpha \frac{a\Delta t}{\Delta x} (0 - (-\Delta x)^2) - \beta \frac{a\Delta t}{\Delta x} ((-\Delta x)^2 - (-2\Delta x)^2)$$

i.e. $\alpha + 3\beta = \frac{a\Delta t}{\Delta x}$ (3.15)

Combining equations (3.14a-b) and (3.15) we find that

$$\alpha = \frac{1}{2} \left(3 - \frac{a\Delta t}{\Delta x} \right), \quad \beta = -\frac{1}{2} \left(1 - \frac{a\Delta t}{\Delta x} \right)$$

and

$$K = (\beta + \frac{1}{2})\Delta x = \frac{a\Delta t}{2}$$

Therefore

$$\beta_{j-\frac{1}{2}}\Delta x_{j-\frac{1}{2}} + \frac{1}{2}\Delta x_{j+\frac{1}{2}} = K = \frac{a\Delta t}{2}$$

i.e.

$$\beta_{j-\frac{1}{2}} = -\frac{1}{2} \left(\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}} - \frac{a\Delta t}{\Delta x_{j-\frac{1}{2}}} \right) = -\frac{1}{2} (r_{j-\frac{1}{2}}^+ - v_{j-\frac{1}{2}})$$

and using equation (3.12)

$$\alpha_{j-\frac{1}{2}} = \frac{1}{2} (2 + r_{j-\frac{1}{2}}^+ - v_{j-\frac{1}{2}})$$

where $r_{j-\frac{1}{2}}^+ = \Delta x_{j+\frac{1}{2}}/\Delta x_{j-\frac{1}{2}}$, the ratio of successive mesh spacings.

Thus the upwind scheme when $a > 0$ becomes

$$\begin{aligned} u^j = u_j &= \frac{1}{2} (2 + r_{j-\frac{1}{2}}^+ - v_{j-\frac{1}{2}}) (u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \\ &+ \frac{1}{2} (r_{j-\frac{3}{2}}^+ - v_{j-\frac{3}{2}}) (u_{j-1} - u_{j-2}) \frac{a\Delta t}{\Delta x_j} \end{aligned} \quad (3.16)$$

We can rearrange equation (3.16) to give

$$\begin{aligned} u^j = u_j &- (u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \\ &+ \frac{1}{2} (r_{j-\frac{3}{2}}^+ - v_{j-\frac{3}{2}}) (u_{j-1} - u_{j-2}) \frac{a\Delta t}{\Delta x_j} \\ &- \frac{1}{2} (r_{j-\frac{1}{2}}^+ - v_{j-\frac{1}{2}}) (u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \end{aligned} \quad (3.17)$$

for $a > 0$.

Before we interpret equation (3.17) as an algorithm involving an 'increment' and 'transfer' stage we derive briefly the corresponding upwind scheme when $a < 0$.

(b) $a < 0$.

Consider the upwind scheme

$$u^j = u_j - \alpha_{j+\frac{1}{2}} \frac{a \Delta t}{\Delta x_j} (u_{j+1} - u_j) = \beta_{j+\frac{3}{2}} \frac{a \Delta t}{\Delta x_j} (u_{j+2} - u_{j+1}) \quad (3.18)$$

If we now apply the same procedure as before we arrive at the following expression for $\alpha_{j+\frac{1}{2}}$, $\beta_{j+\frac{1}{2}}$

$$\alpha_{j+\frac{1}{2}} = \frac{1}{2} \left(2 + \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} + \frac{a \Delta t}{\Delta x_{j+\frac{1}{2}}} \right) = \frac{1}{2} (2 + r_{j+\frac{1}{2}}^- + v_{j+\frac{1}{2}})$$

and

$$\beta_{j+\frac{1}{2}} = -\frac{1}{2} (r_{j+\frac{1}{2}}^- + v_{j+\frac{1}{2}})$$

where

$$r_{j+\frac{1}{2}}^- = \Delta x_{j-\frac{1}{2}} / \Delta x_{j+\frac{1}{2}}.$$

Thus the upwind scheme when $a < 0$ becomes

$$\begin{aligned} u^j = u_j - \frac{1}{2} (2 + r_{j+\frac{1}{2}}^- + v_{j+\frac{1}{2}}) (u_{j+1} - u_j) \frac{a \Delta t}{\Delta x_j} \\ + \frac{1}{2} (r_{j+\frac{3}{2}}^- + v_{j+\frac{3}{2}}) (u_{j+2} - u_{j+1}) \frac{a \Delta t}{\Delta x_j} \end{aligned} \quad (3.19)$$

which, on rearranging, gives

$$\begin{aligned} u^j = u_j - (u_{j+1} - u_j) \frac{a \Delta t}{\Delta x_j} \\ + \frac{1}{2} (r_{j+\frac{3}{2}}^- + v_{j+\frac{3}{2}}) (u_{j+2} - u_{j+1}) \frac{a \Delta t}{\Delta x_j} \\ - \frac{1}{2} (r_{j+\frac{1}{2}}^- + v_{j+\frac{1}{2}}) (u_{j+1} - u_j) \frac{a \Delta t}{\Delta x_j} \end{aligned} \quad (3.20)$$

Combining the schemes given by equations (3.17) and (3.20) we can identify the following Warming and Beam type two stage algorithm for solving equation (2.1): for $a > 0$

$$u^{j-1} = u_{j-1} + \frac{h_{j-\frac{1}{2}}^+}{\Delta x_{j-1}} \quad (3.21a)$$

$$u^j = u_j + \frac{\phi_{j-\frac{1}{2}}}{\Delta x_j} - \frac{h_{j-\frac{1}{2}}^+}{\Delta x_j} \quad (3.21b)$$

and, for $a < 0$,

$$u^{j-1} = u_{j-1} + \frac{\phi_{j-\frac{1}{2}}}{\Delta x_{j-1}} - \frac{h_{j+\frac{1}{2}}^-}{\Delta x_{j-1}} \quad (3.21c)$$

$$u^j = u_j + \frac{h_{j+\frac{1}{2}}^-}{\Delta x_j} \quad (3.21d)$$

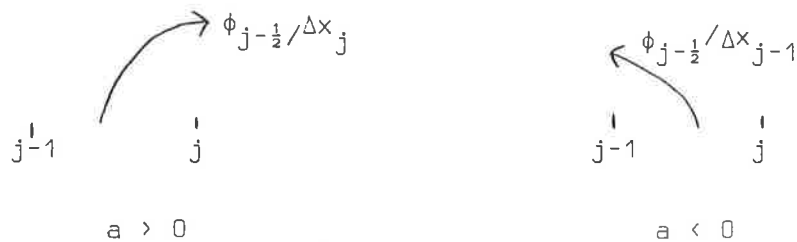
where

$$\phi_{j-\frac{1}{2}} = -a\Delta t(u_j - u_{j-1}) \quad (3.21e)$$

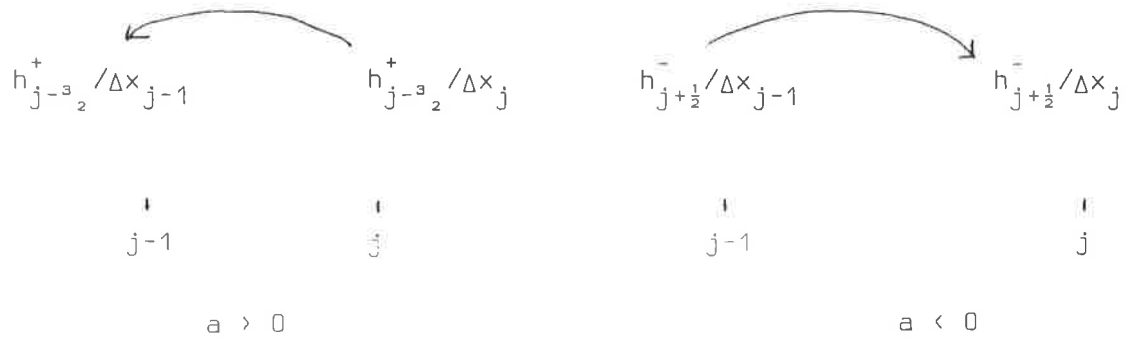
as before, and

$$h_{j-\frac{1}{2}}^\pm = \frac{1}{2}(r_{j-\frac{1}{2}}^\pm - |v_{j-\frac{1}{2}}|)\phi_{j-\frac{1}{2}} \quad (3.21f)$$

Schematically (for each case) we have for each cell an increment stage of the form



as before, together with a transfer stage of the form



We now have two finite difference schemes for the numerical solution of equation (2.1), unfortunately, neither scheme on its own can be a total variation diminishing scheme.

In the next section we show how to combine the schemes developed here to obtain a single total variation diminishing scheme guaranteed to have no spurious oscillations in the solution.

4. A TOTAL VARIATION DIMINISHING SCHEME ON AN IRREGULAR GRID

In this section we combine the two schemes given in the last section to obtain a scheme which is total variation diminishing (TVD) and which, as a consequence, avoids spurious oscillations.

We again distinguish the two cases (a) $a > 0$ and (b) $a < 0$.

(a) $a > 0$.

The Lax-Wendroff type scheme given by equation (3.8) may be written

$$u^j = u_j - \frac{a \Delta t}{\Delta x_j} \Delta u_{j-\frac{1}{2}} - \frac{\Delta_-}{\Delta x_j} \left(\frac{1}{2} (1 - |v_{j+\frac{1}{2}}|) \Delta u_{j+\frac{1}{2}} \right) \quad (4.1)$$

where $\Delta_- v_j = v_j - v_{j-1}$.

The 'incremental flux' term

$$= \frac{a \Delta t}{\Delta x_j} \Delta u_{j-\frac{1}{2}}$$

of (4.1) by itself can never produce spurious oscillations at, for example, discontinuities of the solution. However, the 'anti-diffusive flux' of (4.1),

$$-\frac{\Delta_-}{\Delta x_j} \left(\frac{1}{2}(1 - |v_{j+\frac{1}{2}}|) \Delta u_{j+\frac{1}{2}} \right),$$

that arises from the transfer part of the algorithm may produce such spurious oscillations (see §7). To overcome this we limit the amount of antidiffusive flux, i.e. consider the algorithm

$$u^j = u_j - \frac{a\Delta t}{\Delta x_j} \Delta u_{j-\frac{1}{2}} - \frac{\Delta_-}{\Delta x_j} \left(\psi_{j+\frac{1}{2}}^+ \frac{a\Delta t}{2} (1 - |v_{j+\frac{1}{2}}|) \Delta u_{j+\frac{1}{2}} \right) \quad (4.2a)$$

where $\psi_{j+\frac{1}{2}}^+$ is the limiter assumed to be of the form

$$\psi_{j+\frac{1}{2}}^+ = \psi(m_{j+\frac{1}{2}}^+) \quad \text{with} \quad m_{j+\frac{1}{2}}^+ = \left(\frac{r_{j-\frac{1}{2}}^+ - |v_{j-\frac{1}{2}}|}{1 - |v_{j+\frac{1}{2}}|} \right) \frac{\Delta u_{j-\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}}. \quad (4.2b)$$

We note that $\psi_{j+\frac{1}{2}}^+ \equiv 1$, $\psi_{j+\frac{1}{2}}^+ \equiv m_{j+\frac{1}{2}}^+$ correspond to the Lax-Wendroff and Warming/Beam upwind schemes, respectively.

We can consider ψ to be one of a wide class of limiters, (see [4]), e.g. the 'Superbee' limiter. In particular, we shall specify that $\psi(m) \geq 0$ with $\psi(m) = 0$ for $m < 0$. Furthermore, we shall stipulate that $0 \leq \psi(m) \leq 2$ and that $0 \leq \frac{\psi(m)}{m} \leq 2$ (see [4]).

We now show that the scheme given by equations (4.2a-b) together with these restrictions is TVD if $a\Delta t/\Delta x_{j-\frac{1}{2}} \leq \frac{1}{2}$ for all j , i.e. the local C.F.L. number $v_{j-\frac{1}{2}} \leq \frac{1}{2}$.

The scheme given by equations (4.2a-b) can be written

$$u^j = u_j - C_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}} + D_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}}$$

where

$$C_{j-\frac{1}{2}} = \frac{a\Delta t}{\Delta x_j} \left\{ 1 + \frac{1}{2} \frac{\psi_{j+\frac{1}{2}}^+}{m_{j+\frac{1}{2}}^+} (r_{j-\frac{1}{2}}^+ - |v_{j-\frac{1}{2}}|) - \frac{1}{2} \psi_{j-\frac{1}{2}}^+ (1 - |v_{j-\frac{1}{2}}|) \right\}$$

and

$$D_{j+\frac{1}{2}} = 0.$$

Now, if we stipulate that $v_{j-\frac{1}{2}} = a\Delta t/\Delta x_{j-\frac{1}{2}} \leq \frac{1}{2}$ for all j , then

$$\begin{aligned} C_{j-\frac{1}{2}} &\leq \frac{a\Delta t}{\Delta x_j} (1 + \frac{1}{2} \cdot 2r_{j-\frac{1}{2}}^+) = \frac{a\Delta t}{\Delta x_j} (1 + r_{j-\frac{1}{2}}^+) \\ &= \frac{a\Delta t}{\Delta x_j} \left(\frac{\Delta x_{j-\frac{1}{2}} + \Delta x_{j+\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}} \right) = 2 \frac{a\Delta t}{\Delta x_{j-\frac{1}{2}}} \leq 1 \end{aligned} \quad (4.3)$$

since $\psi_{j-\frac{1}{2}}^+ \geq 0$, $2 \geq \psi_{j+\frac{1}{2}}^+/m_{j+\frac{1}{2}}^+ \geq 0$, $v_{j-\frac{1}{2}} \geq 0$ and $1 - |v_{j-\frac{1}{2}}| \geq 0$.

Also

$$C_{j-\frac{1}{2}} \geq \frac{a\Delta t}{\Delta x_j} (1 - \frac{1}{2} \cdot 2) \geq 0 \quad (4.4)$$

since $2 \geq \psi_{j-\frac{1}{2}}^+ \geq 0$, $v_{j-\frac{1}{2}} \geq 0$, $\psi_{j+\frac{1}{2}}^+/m_{j+\frac{1}{2}}^+ \geq 0$ and $r_{j-\frac{1}{2}}^+ - |v_{j-\frac{1}{2}}| = r_{j-\frac{1}{2}}^+ (1 - v_{j+\frac{1}{2}}) \geq 0$. It follows from equations (4.3), (4.4), (2.6) and (2.7) that the scheme given by equations (4.2a-b) is TVD if $v_{j-\frac{1}{2}} = a\Delta t/\Delta x_{j-\frac{1}{2}} \leq \frac{1}{2}$ for all j .

(b) $a < 0$.

Consider now the Lax-Wendroff scheme given by equation (3.9) where $a < 0$, and write the scheme with limited antidiffusive flux in the form

$$u^j = u_j - \frac{a\Delta t}{\Delta x_j} \Delta u_{j+\frac{1}{2}} + \frac{\Delta_+}{\Delta x_j} \left\{ \psi_{j-\frac{1}{2}}^- \frac{a\Delta t}{2} (1 - |v_{j-\frac{1}{2}}|) \Delta u_{j-\frac{1}{2}} \right\} \quad (4.5a)$$

where ψ is a limiter with the same restrictions as before, $\Delta_+ v_j = v_{j+1} - v_j$ and

$$\psi_{j-\frac{1}{2}}^- = \psi(m_{j-\frac{1}{2}}^-) \quad \text{with} \quad m_{j-\frac{1}{2}}^- = \left(\frac{r_{j+\frac{1}{2}}^- - |v_{j+\frac{1}{2}}|}{1 - |v_{j-\frac{1}{2}}|} \right) \frac{\Delta u_{j+\frac{1}{2}}}{\Delta u_{j-\frac{1}{2}}} \quad (4.5b)$$

This scheme can again be rewritten in the form

$$u^j = u_j + C_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}} + D_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}}$$

where

$$C_{j-\frac{1}{2}} = 0$$

and

$$D_{j+\frac{1}{2}} = \frac{a\Delta t}{\Delta x_j} \left\{ -1 + \frac{1}{2} \psi_{j+\frac{1}{2}}^- (1 - |v_{j+\frac{1}{2}}|) - \frac{1}{2} \frac{\psi_{j-\frac{1}{2}}^-}{m_{j-\frac{1}{2}}^-} (r_{j+\frac{1}{2}}^- - |v_{j+\frac{1}{2}}|) \right\}$$

If we similarly stipulate $v_{j-\frac{1}{2}} = a\Delta t/\Delta x_{j-\frac{1}{2}} \geq -\frac{1}{2}$ for all j , then

$$\begin{aligned} D_{j+\frac{1}{2}} &= \frac{a\Delta t}{\Delta x_j} \left\{ 1 + \frac{1}{2} \frac{\psi_{j-\frac{1}{2}}^-}{m_{j-\frac{1}{2}}^-} (r_{j+\frac{1}{2}}^- - |v_{j+\frac{1}{2}}|) - \frac{1}{2} \psi_{j+\frac{1}{2}}^- (1 - |v_{j+\frac{1}{2}}|) \right\} \\ &\leq -\frac{a\Delta t}{\Delta x_j} (1 + \frac{1}{2} \cdot 2r_{j+\frac{1}{2}}^-) = -\frac{a\Delta t}{\Delta x_j} (1 + r_{j+\frac{1}{2}}^-) \\ &= -\frac{a\Delta t}{\Delta x_j} \left(\frac{\Delta x_{j+\frac{1}{2}} + \Delta x_{j-\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} \right) = -\frac{2a\Delta t}{\Delta x_{j+\frac{1}{2}}} \leq 1 \quad (4.6) \end{aligned}$$

since $\psi_{j+\frac{1}{2}}^- \geq 0$, $2 \geq \psi_{j-\frac{1}{2}}^- / m_{j-\frac{1}{2}}^- \geq 0$ and $1 - |v_{j+\frac{1}{2}}| \geq 0$. Also

$$D_{j+\frac{1}{2}} \geq -\frac{a\Delta t}{\Delta x_j} (1 - \frac{1}{2} \cdot 2) \geq 0 \quad (4.7)$$

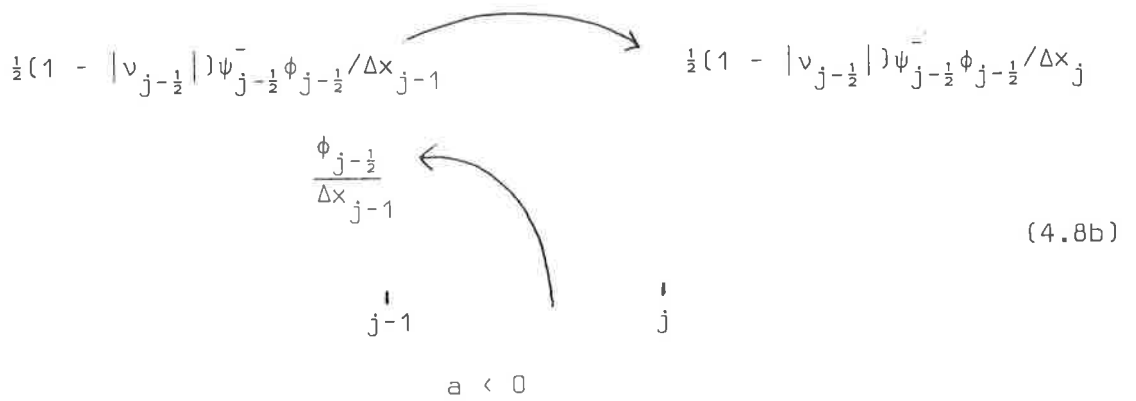
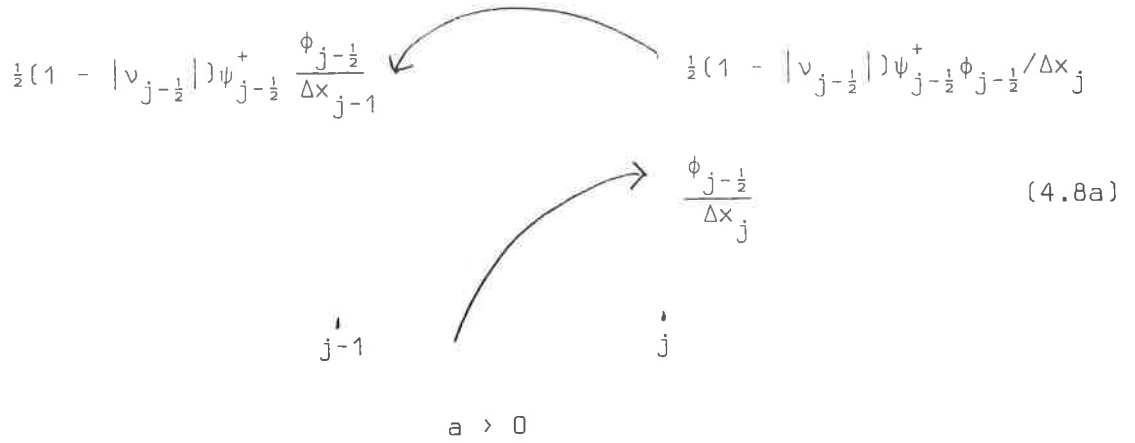
since $2 \geq \psi_{j+\frac{1}{2}}^- \geq 0$, $v_{j+\frac{1}{2}} \leq 0$, $\psi_{j-\frac{1}{2}}^- / m_{j-\frac{1}{2}}^- \geq 0$ and

$r_{j+\frac{1}{2}}^- - |v_{j+\frac{1}{2}}| = r_{j+\frac{1}{2}}^- (1 + v_{j-\frac{1}{2}}) \geq 0$. It follows from equations

(4.6), (4.7), (2.6) and (2.7) that the scheme given by equations (4.5a-b)

is TVD if $v_{j-\frac{1}{2}} = a\Delta t / \Delta x_{j-\frac{1}{2}} \geq -\frac{1}{2}$ for all j .

Schematically we have constructed the following algorithm



where

$$\phi_{j-\frac{1}{2}} = -a\Delta t \Delta u_{j-\frac{1}{2}} \quad (4.8c)$$

and

$$\psi_{j-\frac{1}{2}}^\pm = \psi(m_{j-\frac{1}{2}}^\pm) \quad \text{with} \quad m_{j-\frac{1}{2}}^\pm = \frac{(r_{j-\frac{1}{2}-s}^\pm - |v_{j-\frac{1}{2}-s}|) \Delta u_{j-\frac{1}{2}-s}}{(1 - |v_{j-\frac{1}{2}}|) \Delta u_{j-\frac{1}{2}}} \quad (4.8d)$$

and

$$s = \text{sign}(a) \quad (4.8e)$$

We can combine these two schemes to obtain approximate solutions with a TVD property (see Appendix) to the non-linear scalar conservation law

$$u_t + (f(u))_x = 0 \quad (4.9)$$

as follows

$$\begin{array}{ccc}
 b_{j-1/2}/\Delta x_{j-1} & \xleftarrow{\hspace{2cm}} & b_{j-1/2}/\Delta x_j \\
 | & & | \\
 j-1 & & j \\
 & \xrightarrow{\hspace{2cm}} & -\frac{\Delta t}{\Delta x_j} \Delta f_{j-1/2} \\
 & & | \\
 & & j
 \end{array} \quad (4.10a)$$

$s_{j-1/2} > 0$

$$\begin{array}{ccc}
 b_{j-1/2}/\Delta x_{j-1} & \xrightarrow{\hspace{2cm}} & b_{j-1/2}/\Delta x_j \\
 | & & | \\
 j-1 & & j \\
 & \xrightarrow{\hspace{2cm}} & -\frac{\Delta t}{\Delta x_{j-1}} \Delta f_{j-1/2} \\
 & & | \\
 & & j-1
 \end{array} \quad (4.10b)$$

$s_{j-1/2} < 0$

where

$$v_{j-1/2} = \begin{cases} \frac{\Delta t \Delta f_{j-1/2}}{\Delta x_{j-1/2} \Delta u_{j-1/2}} & u_j \neq u_{j-1} \\ \frac{\Delta t}{\Delta x_{j-1/2}} f'(u_j) & u_j = u_{j-1} \end{cases} \quad (4.10c)$$

$$v_{j-1/2} = \frac{\Delta t}{\Delta x_{j-1/2}} f'(u_j) \quad u_j = u_{j-1} \quad (4.10d)$$

$$s_{j-\frac{1}{2}} = \text{sign}(v_{j-\frac{1}{2}}) \quad (4.10e)$$

$$\Delta f_{j-\frac{1}{2}} = f_j - f_{j-1} = f(u_j) - f(u_{j-1}) \quad (4.10f)$$

$$b_{j-\frac{1}{2}} = -\frac{1}{2}(1 - |v_{j-\frac{1}{2}}|) \Delta t \Delta f_{j-\frac{1}{2}} \psi(M_{j-\frac{1}{2}}) \quad (4.10g)$$

and

$$M_{j-\frac{1}{2}} = \frac{\left(\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}} - \varepsilon_{j-\frac{1}{2}}} - |v_{j-\frac{1}{2}} - s_{j-\frac{1}{2}}| \right) \Delta f_{j-\frac{1}{2}}}{(1 - |v_{j-\frac{1}{2}}|) \Delta f_{j-\frac{1}{2}}} = s_{j-\frac{1}{2}} \quad (4.10h)$$

We can also extend our irregular grid scheme to include the solution of the Euler equations for an ideal compressible fluid in one dimension, namely

$$\begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}_t + \begin{pmatrix} \rho u \\ p + \rho u^2 \\ u(e+p) \end{pmatrix}_x = \underline{0} \quad (4.11a)$$

where

$$e = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \quad (4.11b)$$

and ρ , u , p , e and γ represent the density, velocity, pressure, energy and ratio of specific heat capacities of the fluid, respectively.

Using the flux difference splitting technique of Roe [1] the approximate solution of equations (4.11a-b) reduces to the solution of three scalar problems and this can be done on an irregular grid using the scheme given by equations (4.10a-h).

In addition, we can easily incorporate into the scheme a device to disperse entropy-violating solutions and treat expansion fans correctly. This is done by considering the one-sided scheme given by equations (4.10a-h)

as a two-sided scheme, sending increments to both ends of a cell, (see [5]).

In the next section we develop a slightly different irregular grid scheme for the solution of the Euler equations for flows with strong shocks.

5. AN IRREGULAR GRID SCHEME FOR FLOWS WITH STRONG SHOCKS

In this section we propose another irregular grid scheme for the solution of the Euler equations for flows with strong shocks.

Consider again the scalar equation (2.1) and the schemes :
for $a > 0$

$$u^j = u_j - \frac{a\Delta t}{\Delta x_{j+\frac{1}{2}}} (u_j - u_{j-1}) \quad (5.1a)$$

and, for $a < 0$,

$$u^j = u_j - \frac{a\Delta t}{\Delta x_{j-\frac{1}{2}}} (u_{j+1} - u_j) \quad (5.1b)$$

Both of these schemes are conservative in the sense $\sum_{\text{all } j} (u^j - u_j) \Delta x_j = 0$ boundary terms if $\Delta x_j / \Delta x_{j+\frac{1}{2}} = \text{constant}$, equivalently

$\Delta x_{j+\frac{1}{2}} / \Delta x_{j-\frac{1}{2}} = \text{constant}$, r say, i.e. on a grid where the mesh spacings

are in constant ratio. We call this type of grid a 'Geometric grid'.

The two schemes given by equations (3.8) and (3.17) for $a > 0$

can be rewritten as

$$\begin{aligned} u^j &= u_j - \frac{a\Delta t}{\Delta x_{j+\frac{1}{2}}} (u_j - u_{j-1}) \\ &\quad + \frac{1}{2}(r_{j+\frac{1}{2}} - |v_{j-\frac{1}{2}}|)(u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \\ &= \frac{1}{2}(1 - |v_{j+\frac{1}{2}}|)(u_{j+1} - u_j) \frac{a\Delta t}{\Delta x_j} \end{aligned} \quad (5.2)$$

and

$$\begin{aligned}
 u^j &= u_j - \frac{a\Delta t}{\Delta x_{j+\frac{1}{2}}} (u_j - u_{j-1}) \\
 &+ \frac{1}{2}(r_{j-\frac{3}{2}}^+ - |v_{j-\frac{3}{2}}|)(u_{j-1} - u_{j-2}) \frac{a\Delta t}{\Delta x_j} \\
 &= \frac{1}{2}(1 + r_{j-\frac{1}{2}}^+ - r_{j+\frac{1}{2}}^- - |v_{j-\frac{1}{2}}|)(u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \quad (5.3)
 \end{aligned}$$

Similarly, equations (3.9) and (3.20) for $a < 0$ can be rewritten as

$$\begin{aligned}
 u^j &= u_j - \frac{a\Delta t}{\Delta x_{j-\frac{1}{2}}} (u_{j+1} - u_j) \\
 &+ \frac{1}{2}(r_{j-\frac{1}{2}}^+ - |v_{j+\frac{1}{2}}|)(u_{j+1} - u_j) \frac{a\Delta t}{\Delta x_j} \\
 &- \frac{1}{2}(1 - |v_{j-\frac{1}{2}}|)(u_j - u_{j-1}) \frac{a\Delta t}{\Delta x_j} \quad (5.4)
 \end{aligned}$$

and

$$\begin{aligned}
 u^j &= u_j - \frac{a\Delta t}{\Delta x_{j-\frac{1}{2}}} (u_{j+1} - u_j) \\
 &+ \frac{1}{2}(r_{j+\frac{3}{2}}^- - |v_{j+\frac{3}{2}}|)(u_{j+2} - u_{j+1}) \frac{a\Delta t}{\Delta x_j} \\
 &- \frac{1}{2}(1 + r_{j+\frac{1}{2}}^- - r_{j-\frac{1}{2}}^+ - |v_{j+\frac{1}{2}}|)(u_{j+1} - u_j) \frac{a\Delta t}{\Delta x_j} \quad (5.5)
 \end{aligned}$$

We can identify an increment $-\frac{a\Delta t}{\Delta x_{j\pm\frac{1}{2}}} \Delta u_{j\pm\frac{1}{2}}$ in each of equations (5.2-5.5) that is the same as the scheme given by equations (5.1a-b), but it is now not possible to identify a transfer stage in the same way as we did in equations (3.8-3.9), (3.17) and (3.20).

For mildly varying geometric meshes, however, i.e. where $r \sim O(1)$, we can consider approximate transfers of the form $\gamma_{j\pm\frac{1}{2}}, \gamma_{j\pm\frac{3}{2}}$ where $\gamma_{j+\frac{1}{2}} = -\frac{a\Delta t}{2} (1 - |v_{j+\frac{1}{2}}|)(u_{j+1} - u_j)$.

In addition we can limit these transfers as we did in §4 to increase the resolution of the numerical results without introducing spurious oscillations into the solution.

The main reason for considering this different scheme, is that when it is applied to the Euler equations for flows with strong shocks, we obtain significantly better results with this scheme than using the scheme in §4. This could be explained by the fact that for flows with strong shocks, the downwind lengths in the denominator, i.e. $\Delta x_{j+\frac{1}{2}}$, are more appropriate in the following sense. In an upwind scheme the mass entering the cell $j-\frac{1}{2}$ is sent downstream and this mass must be converted to a density by dividing by an appropriate mesh length. It is consistent with the upwind philosophy to regard the mass as smeared over the downwind cell and the value held at j . This subtle difference may be significant for strong shocks.

We shall therefore state here the scalar algorithm that can be used to determine the solution of the Euler equations (4.11a-b) on a mildly varying geometric grid, (typically $5/6 < r < 6/5$), in schematic form

$$\begin{array}{ccc}
 \frac{d_{j-\frac{1}{2}}}{\Delta x_{j-1}} & \xleftarrow{\hspace{2cm}} & \frac{d_{j-\frac{1}{2}}}{\Delta x_j} \\
 j-1 & & j \\
 & & s_{j-\frac{1}{2}} > 0 \\
 & & - \frac{\Delta t}{\Delta x_{j+\frac{1}{2}}} \Delta f_{j-\frac{1}{2}}
 \end{array} \tag{5.6a}$$

$$\begin{array}{ccc}
 \frac{d_{j-\frac{1}{2}}}{\Delta x_{j-1}} & \xleftarrow{\hspace{2cm}} & \frac{d_{j-\frac{1}{2}}}{\Delta x_j} \\
 j-1 & & j \\
 & & s_{j-\frac{1}{2}} < 0 \\
 & & - \frac{\Delta t \Delta f_{j-\frac{1}{2}}}{\Delta x_{j-\frac{3}{2}}}
 \end{array} \tag{5.6b}$$

where

$$v_{j-\frac{1}{2}} = \begin{cases} \frac{\Delta t \Delta f_{j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}}} & u_j \neq u_{j-1} \\ \frac{\Delta t f'(u_j)}{\Delta x_{j-\frac{1}{2}}} & u_j = u_{j-1} \end{cases} \quad (5.6c)$$

$$u_j = u_{j-1} \quad (5.6d)$$

$$s_{j-\frac{1}{2}} = \text{sign}(v_{j-\frac{1}{2}}) \quad (5.6e)$$

$$\Delta f_{j-\frac{1}{2}} = f_j - f_{j-1} = f(u_j) - f(u_{j-1}) \quad (5.6f)$$

$$d_{j-\frac{1}{2}} = -\frac{1}{2}(1 - |v_{j-\frac{1}{2}}|) \Delta t \Delta f_{j-\frac{1}{2}} \psi(M_{j-\frac{1}{2}}) \quad (5.6g)$$

and

$$M_{j-\frac{1}{2}} = \frac{(1 - |v_{j-\frac{1}{2}} - s_{j-\frac{1}{2}}|) \Delta f_{j-\frac{1}{2}} - s_{j-\frac{1}{2}}}{(1 - |v_{j-\frac{1}{2}}|) \Delta f_{j-\frac{1}{2}}} \quad (5.6h)$$

As in §4 we can incorporate a simple device to disperse entropy violating solutions.

In the next section we give a series of problems that can be used to test the schemes given in §3, §4 and §5.

6. TEST PROBLEMS

In this section we look at four test problems used to try out the previously described algorithms. Each problem is concerned with the propagation of a discontinuity (shock) through an irregular grid. Both the linear and non-linear scalar equations (2.1) and (4.9) are considered, together with the Euler equations (4.11a-b).

Two types of grid are considered, a geometric grid and an abutted grid (i.e. one where two regular grids with different mesh spacings are joined). We test all three schemes given in §3, §4 and §5.

Problem 1

The first problem is concerned with the linear equation

$$u_t + u_x = 0$$

i.e. linear advection, with initial data

$$u = 1 \quad x < 0$$

$$u = 0 \quad x > 0$$

The discontinuity moves from left to right with unit speed.

Problem 2

The second problem involves the non-linear equation inviscid Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and the corresponding jump condition

$$\left[\frac{1}{2}u^2\right] = S[u]$$

where S is the shock speed, with initial data

$$u = 1 \quad x < 0$$

$$u = 0 \quad x > 0$$

The discontinuity moves from left to right with speed $\frac{1}{2}$.

Problem 3

The third problem is the well known shock tube problem of Sod for the Euler equations (4.11a-b) with initial data

$$\left. \begin{array}{l} \rho = 1 \\ u = 0 \\ p = 1 \end{array} \right\} \quad x < \frac{1}{2}$$

and

$$\left. \begin{array}{l} \rho = 0.125 \\ u = 0 \\ p = 0.1 \end{array} \right\} \quad x > \frac{1}{2}$$

where $\gamma = 1.4$, (see [6]).

The main features of the exact solution are a shock moving to the right followed by a contact discontinuity also moving to the right but more slowly and an expansion fan moving to the left.

Problem 4

The final problem is concerned with shock reflection using again the Euler equations. We consider a region $0 \leq x \leq 1$ with initial conditions

$$\begin{array}{l} \rho = 1 \\ u = -1 \\ p = p^0 \end{array}$$

i.e. a gas of constant density and pressure moving towards $x = 0$.

The boundary $x = 0$ is a rigid wall and the exact solution represents shock reflection from the wall. The gas is brought to rest at $x = 0$ and, denoting by (0) initial values, by (-) pre-shocked values, and by (+)

post-shocked values, we have $\rho^0 = 1$, $\rho^- = \rho^0$, $\rho^+ = \left(\frac{1}{S} + 1\right)\rho^-$,
 $u^0 = -1$, $u^- = -1$, $u^+ = 0$, $p^0 = p^0$, $p^- = p^0$, $p^+ = \frac{1}{4}\left(\gamma + 1 + \sqrt{(\gamma+1)^2 + 16\gamma p^0}\right)p^0$

and

$$\rho = 1, \quad u = -1, \quad p = p^0 \quad \text{for } x/t \geq S,$$

where

$$S = \frac{1}{4} (\gamma + 3 + \sqrt{(\gamma+1)^2 + 16\gamma p^0}).$$

The shock moves out from the origin with speed S . This includes the special case of an infinite shock when $p^0 = 0$ and $p^+/p^- = \infty$.

Moreover, by suitable choices of p^0 we can vary the shock strength p^+/p^- .

In the next section we display the numerical results for the problems described above.

7. NUMERICAL RESULTS

In this section we show the numerical results obtained for the four test problems described in §6 using the schemes described in §3, §4 and §5. Each of the figures refers to one of Problems 1-4 with either a grid with constant mesh spacings, a geometric grid or an abutted grid. In addition we use either the 'Minmod' limiter i.e.

$$\psi(m) = \max(0, \min(m, 1))$$

or the 'Superbee' limiter i.e.

$$\psi(m) = \max(0, \min(2m, 1), \min(m, 2))$$

(For Problem 1 we also use the non-TVD schemes of §3.) With a geometric grid or an abutted grid we call the scheme in §4 'Scheme 1' and the scheme in §5 'Scheme 2'. In Problem 4 we apply a reflection condition[†] at $x = 0$.

Problem 1

Figure 1	Constant Grid	Warming/Beam	
Figure 2	Constant Grid	Superbee	
Figure 3	Abutted Grid	Warming/Beam	Scheme 1
Figure 4	Abutted Grid	Superbee	Scheme 1
Figure 5	Geometric Grid	Superbee	Scheme 1
Figure 6	Geometric Grid	Minmod	Scheme 2

Problem 2

Figure 7	Constant Grid	Superbee	
Figure 8	Abutted Grid	Superbee	Scheme 1
Figure 9	Geometric Grid	Superbee	Scheme 1
Figure 10	Geometric Grid	Superbee	Scheme 2

(†) A reflected boundary condition can be implemented by considering an 'image' cell at the boundary and imposing equal density and pressure, and equal and opposite velocity at either end of the cell. This results in no net movement in the cell.

Problem 3

Figures 11-16 refer to Problem 3 and only that part of the density featuring the shock and contact discontinuities is shown. Figure 16a shows the complete solution for the density with increasing and decreasing geometric grids throughout.

Figure 11	Constant Grid	Minmod and Superbee
Figure 12	Constant Grid	Minmod and Superbee
Figure 13	Abutted Grid	Schemes 1 and 2
Figure 14	Abutted Grid	Schemes 1 and 2
Figure 15	Geometric Grid	Schemes 1 and 2
Figure 16	Geometric Grid	Schemes 1 and 2
Figure 16a	Geometric Grid	Schemes 1 and 2

Problem 4

Figures 17-34 refer to Problem 4 using the Minmod' limiter. We choose three geometric grid ratios, r , two values for the ratio of specific heat capacities, γ , and different shock strengths P^+/P^- . (N.B. only the density is given, as generally no difficulty is found in computing the velocity and pressure with either scheme 1 or 2.)

Figure 17	$r = 1.05$	$\gamma = 5/3$	$P^+/P^- = \infty$
Figure 18	$r = 1.05$	$\gamma = 5/3$	$P^+/P^- = 10$
Figure 19	$r = 1.05$	$\gamma = 5/3$	$P^+/P^- = 2$
Figure 20	$r = 1.05$	$\gamma = 1.4$	$P^+/P^- = \infty$
Figure 21	$r = 1.05$	$\gamma = 1.4$	$P^+/P^- = 10$
Figure 22	$r = 1.05$	$\gamma = 1.4$	$P^+/P^- = 2$
Figure 23	$r = 1.1$	$\gamma = 5/3$	$P^+/P^- = \infty$
Figure 24	$r = 1.1$	$\gamma = 5/3$	$P^+/P^- = 10$
Figure 25	$r = 1.1$	$\gamma = 5/3$	$P^+/P^- = 2$
Figure 26	$r = 1.1$	$\gamma = 1.4$	$P^+/P^- = \infty$
Figure 27	$r = 1.1$	$\gamma = 1.4$	$P^+/P^- = 10$
Figure 28	$r = 1.1$	$\gamma = 1.4$	$P^+/P^- = 2$
Figure 29	$r = 1.15$	$\gamma = 5/3$	$P^+/P^- = \infty$
Figure 30	$r = 1.15$	$\gamma = 5/3$	$P^+/P^- = 10$
Figure 31	$r = 1.15$	$\gamma = 5/3$	$P^+/P^- = 2$
Figure 32	$r = 1.15$	$\gamma = 1.4$	$P^+/P^- = \infty$
Figure 33	$r = 1.15$	$\gamma = 1.4$	$P^+/P^- = 10$
Figure 34	$r = 1.15$	$\gamma = 1.4$	$P^+/P^- = 2$

For Problems 1, 2 and 3 we see very little difference in the results using schemes 1 and 2. The discontinuity (shock) in each problem passes through the variable mesh producing little spurious oscillation. (The upwind scheme of §3, however, yields spurious oscillations in Problem 1 as expected.) For Problem 4 we notice that the scheme of §5 performs better than the scheme of §4 when the shock strength is large. This is true for both choices of the ratio of specific heat capacities, γ , and each value of the mesh ratio, r . For mild shock strengths the results from using scheme 1 and 2 are very similar.

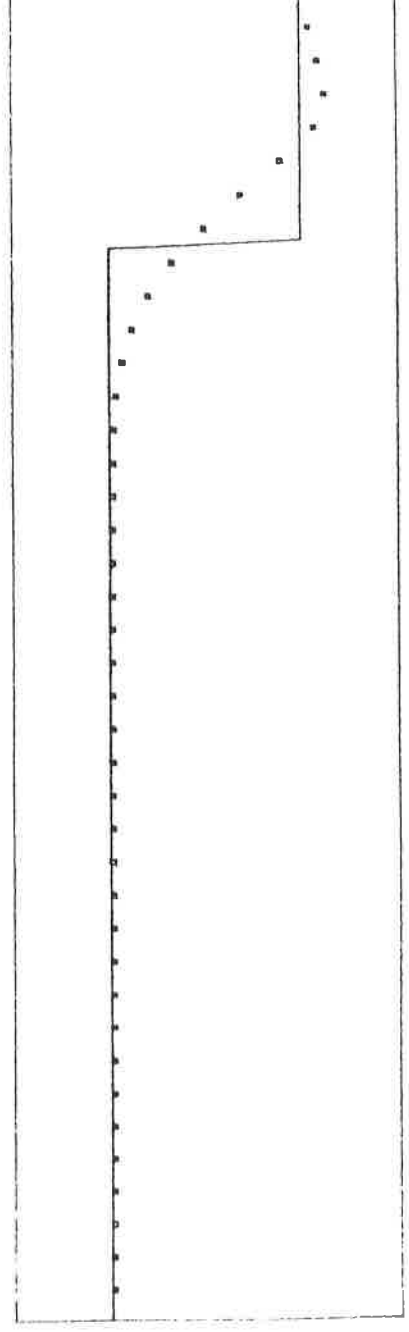
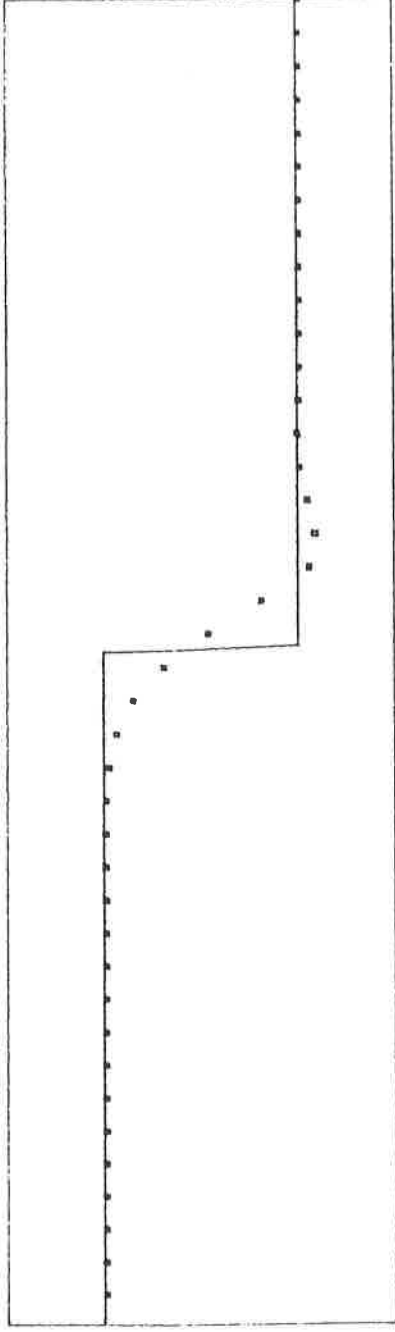
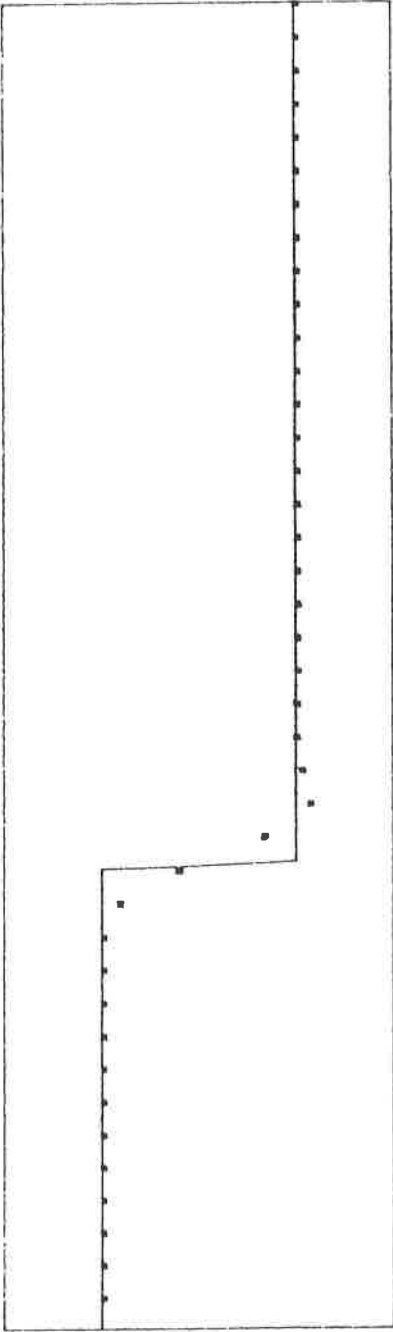
Linear Advection

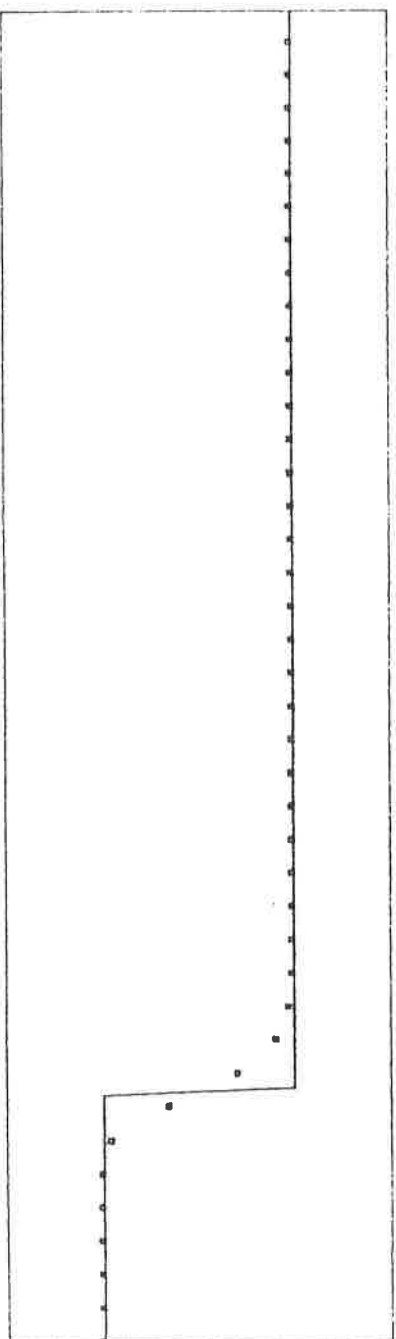
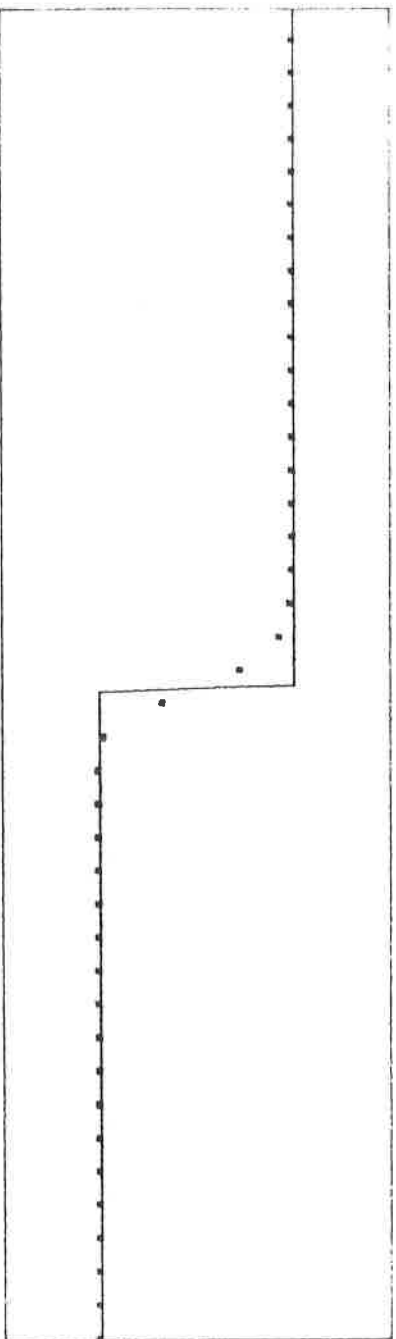
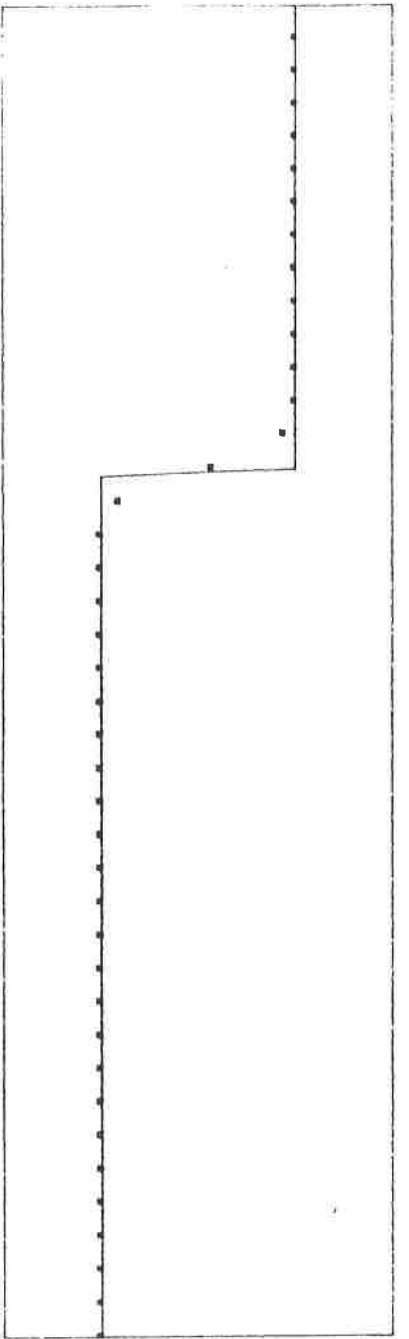
Harming/Beam scheme

— Exact solution

..... Approximate solution

Constant Grid





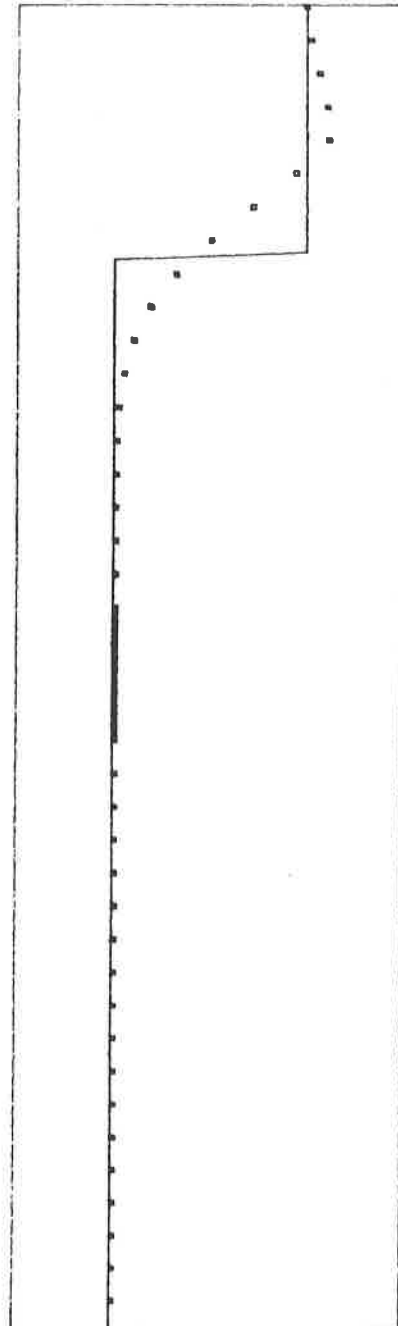
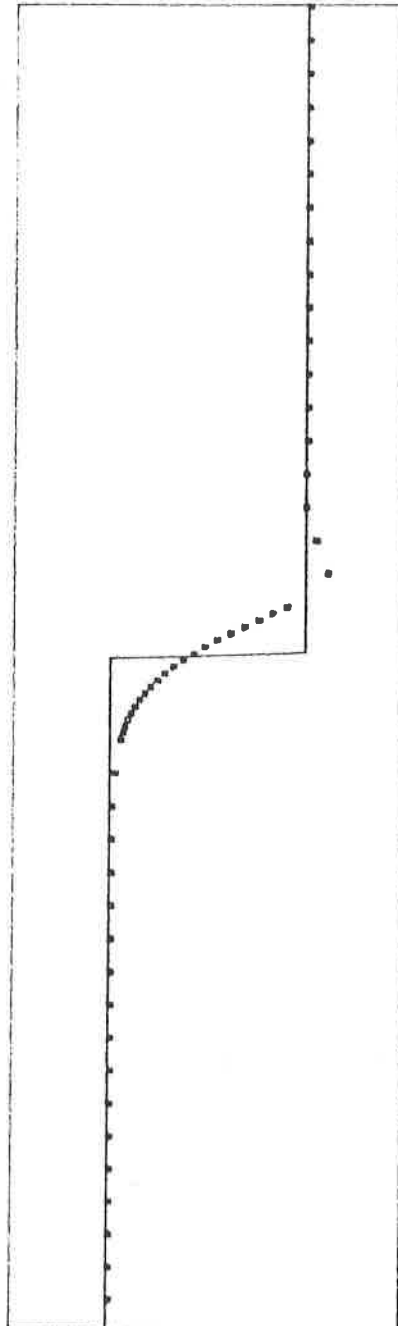
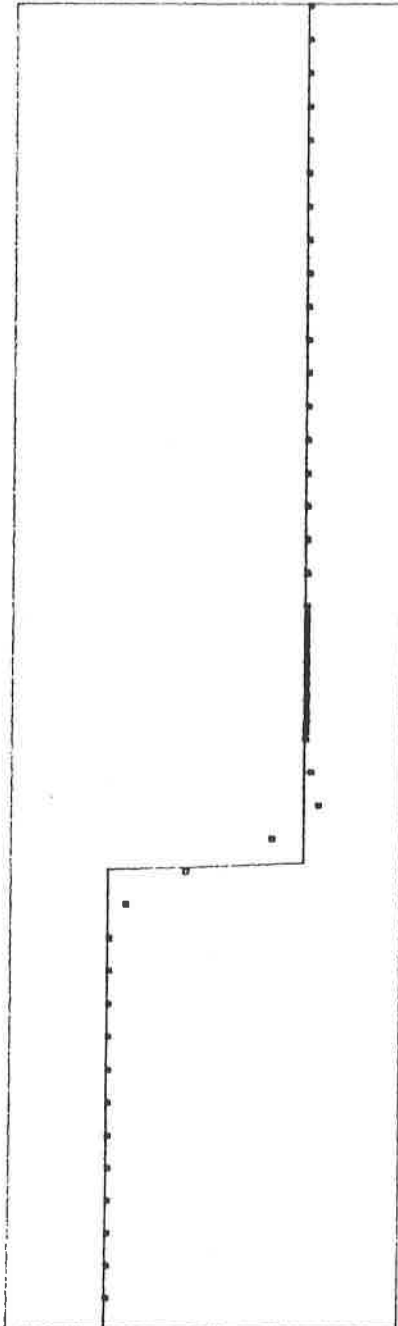
Linear Advection

"Superbee" limiter

— Exact solution

..... Approximate solution

Constant Grid



Linear Advection

Warming/Beam scheme

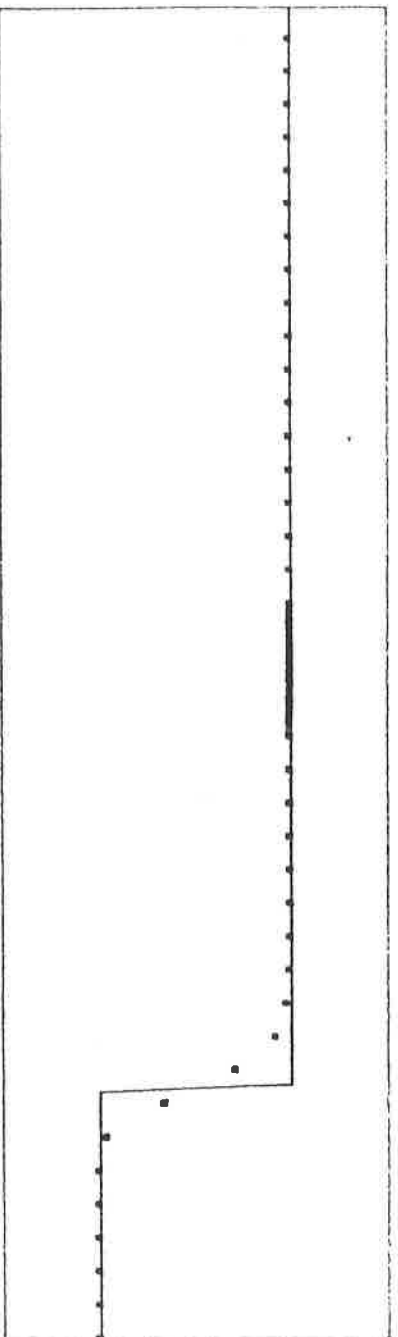
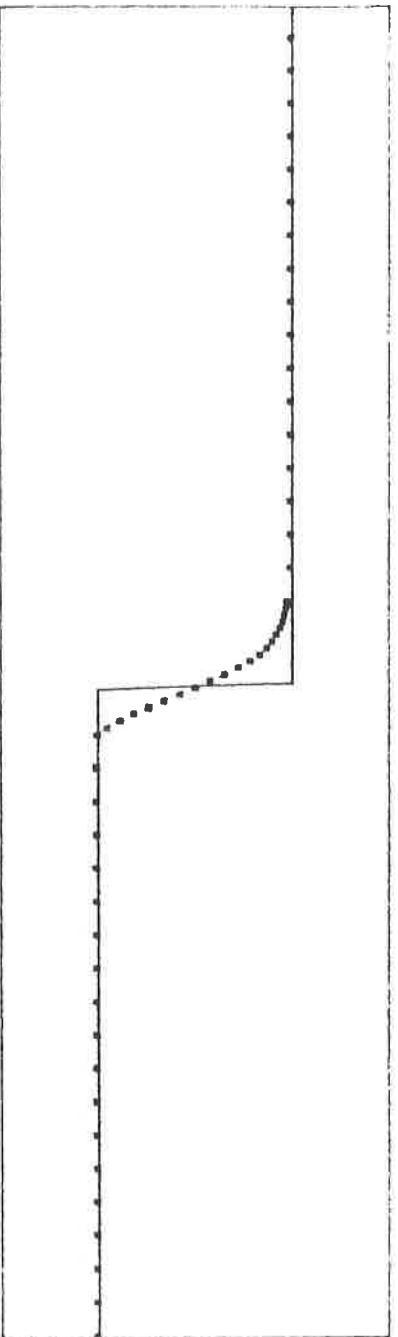
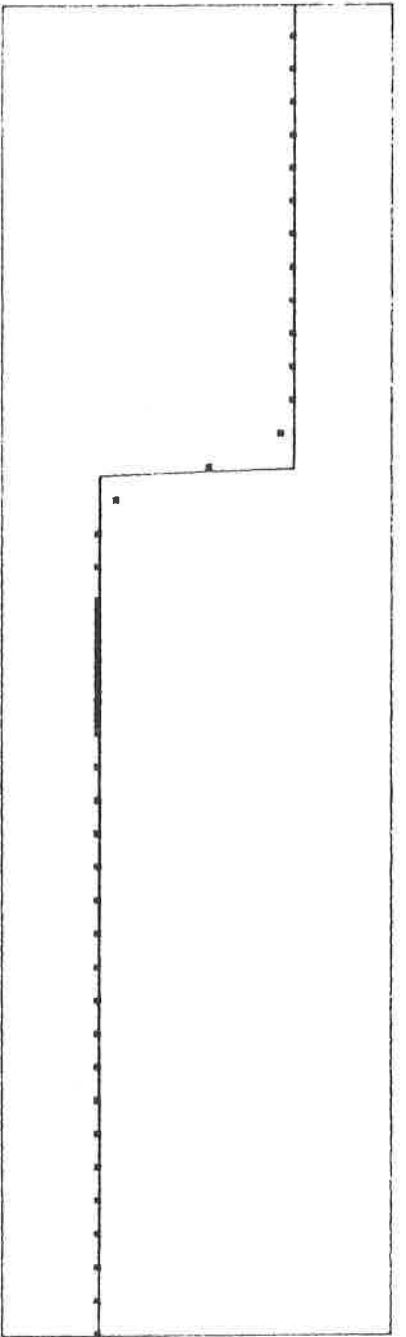
— Exact solution

..... Approximate solution

Abutted Grid

Mesh Jump = 5.0

Scheme 1



Linear Advection

Superbee' limiter

— Exact solution

..... Approximate solution

Abutted Grid

Mesh Jump = 5.0

Scheme 1

Linear Advection

"Superbee" limiter

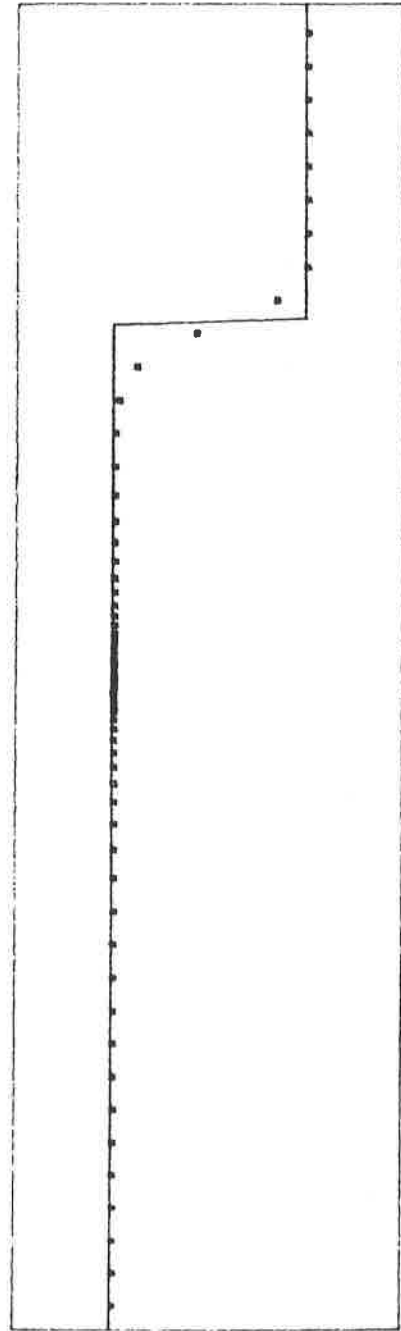
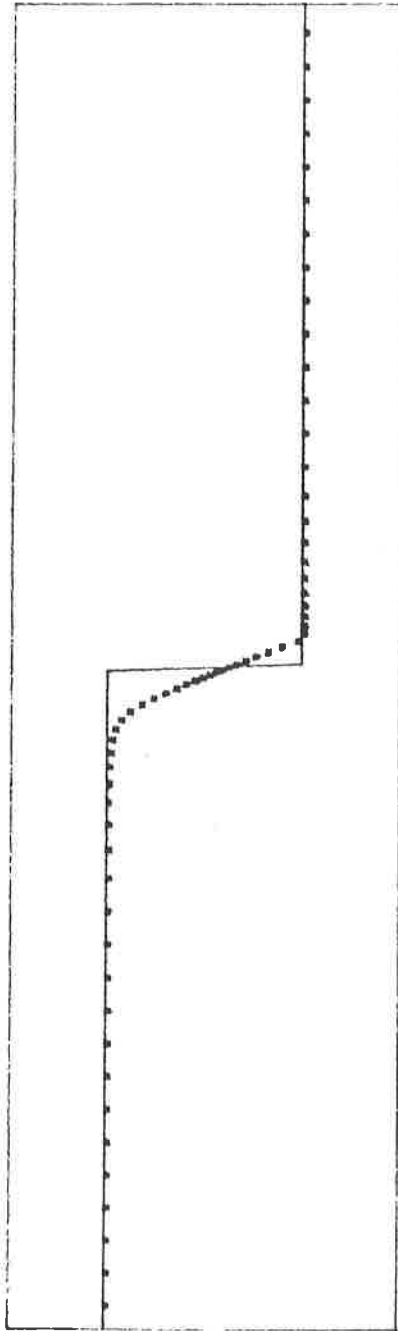
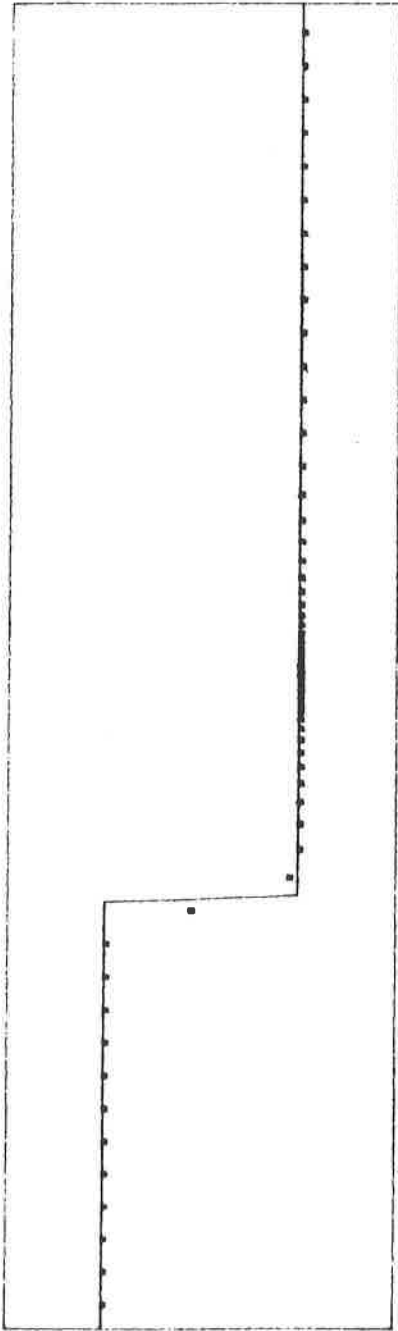
— Exact solution

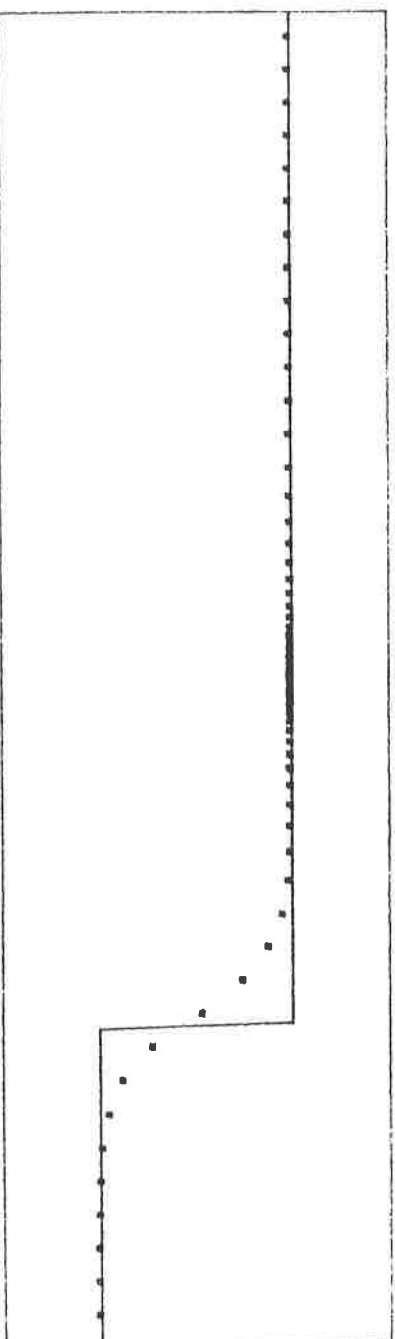
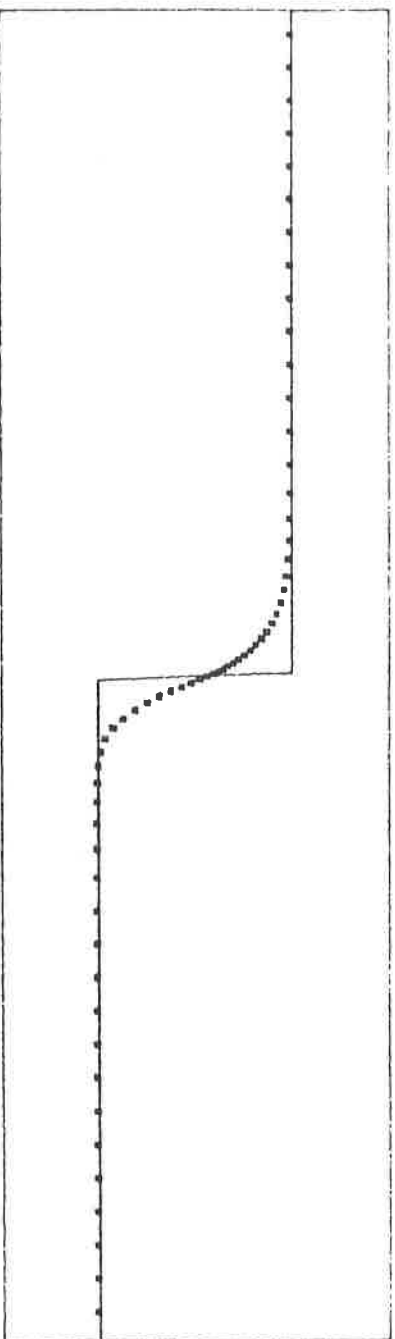
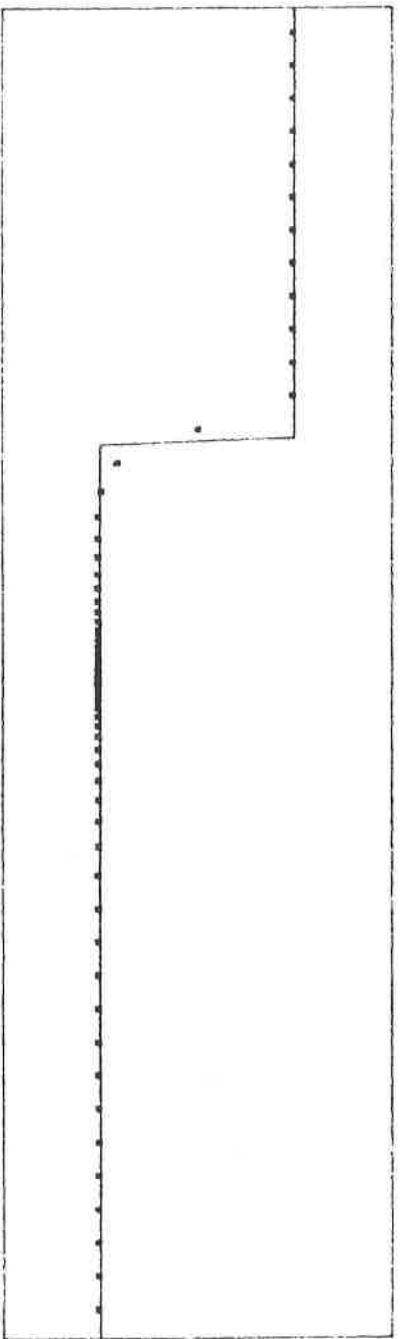
..... Approximate solution

Geometric Grid

Mesh Ratio = 1.15

Scheme 1





Linear Advection

— Exact solution

..... Approximate solution

Geometric Grid

Mesh Ratio = 1.15

Scheme 2

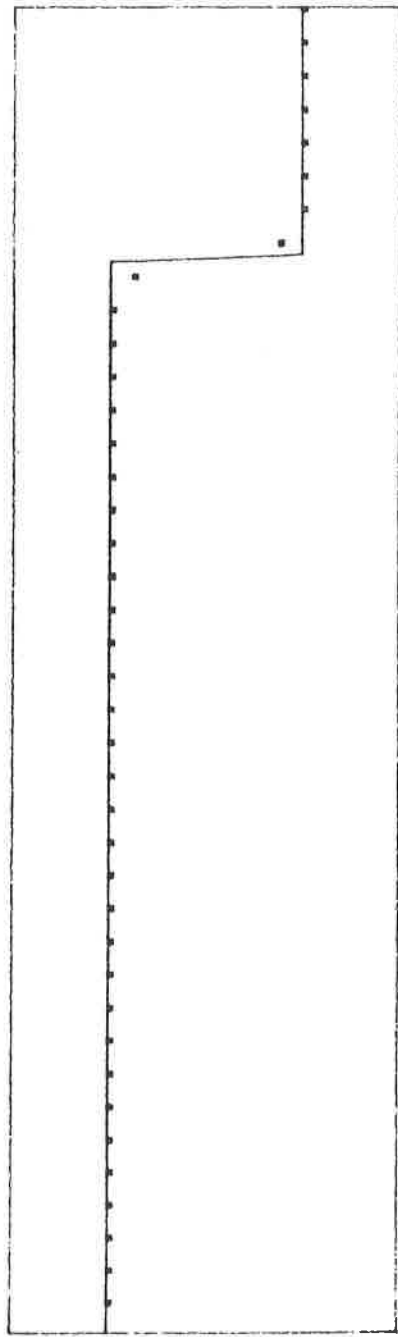
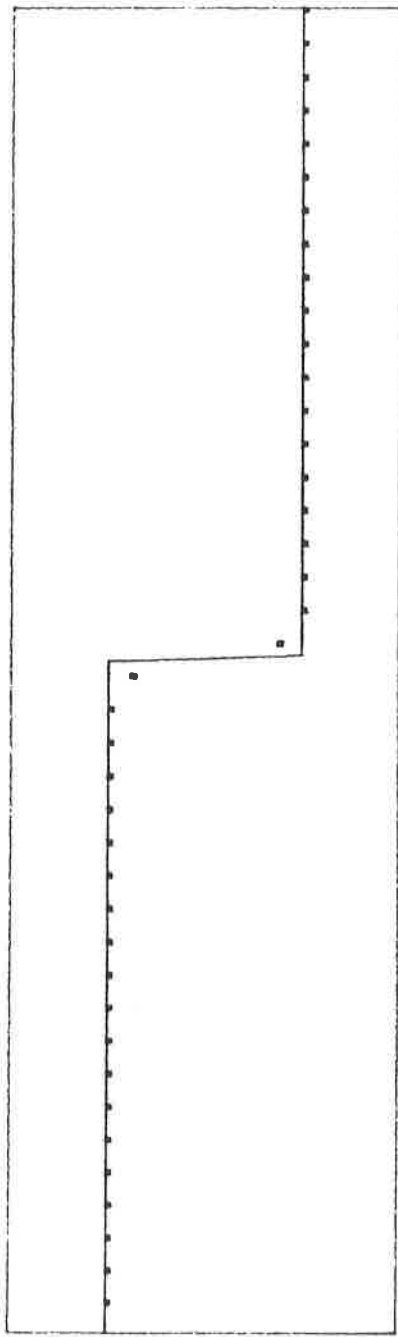
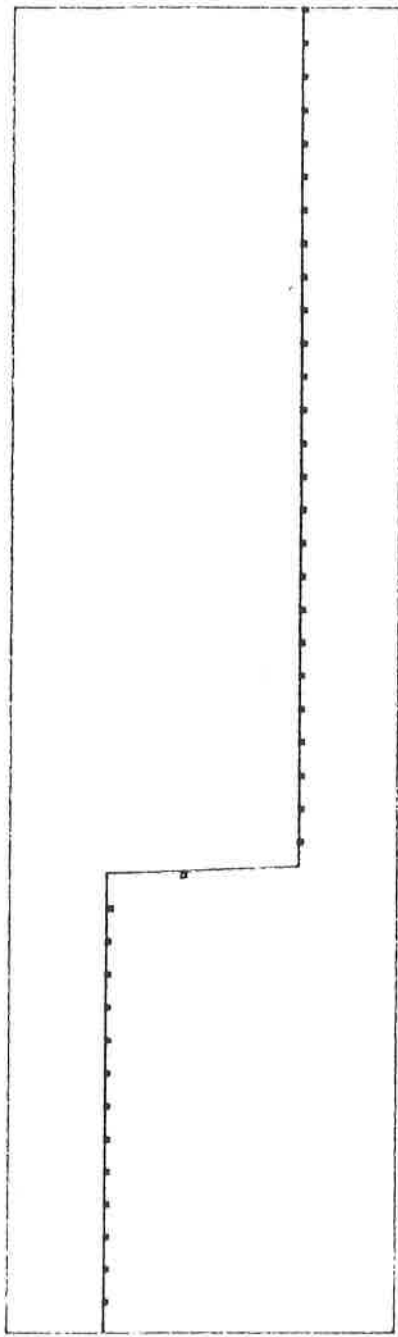
Burgers' Equation

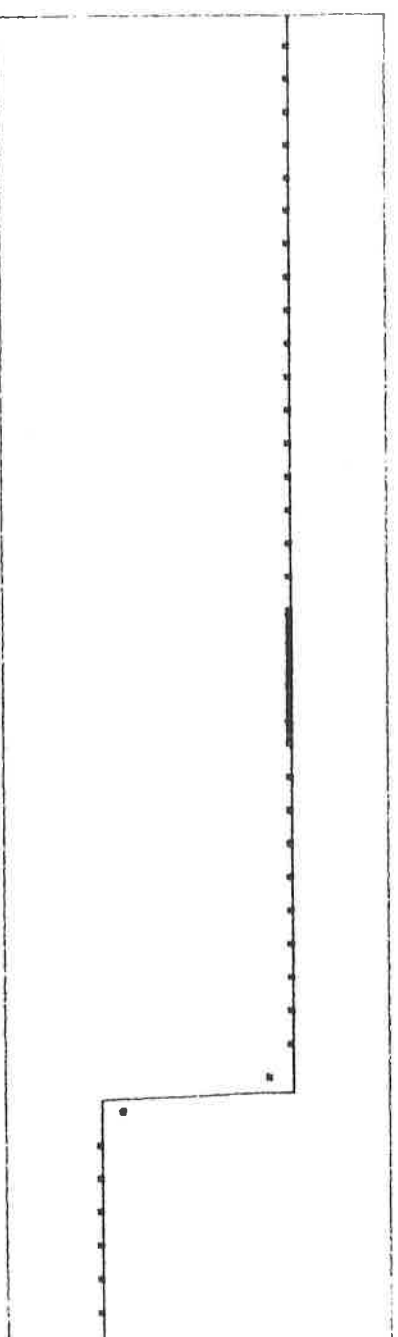
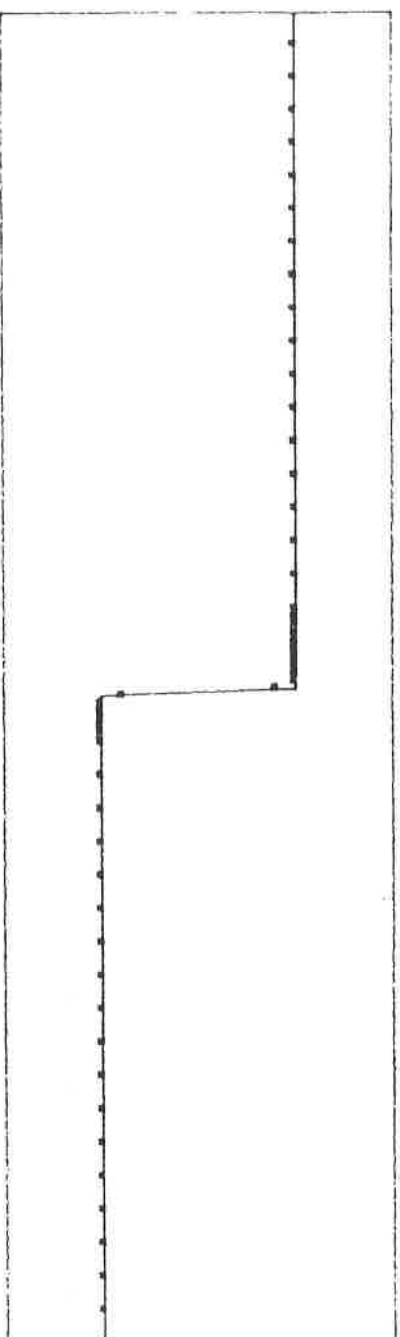
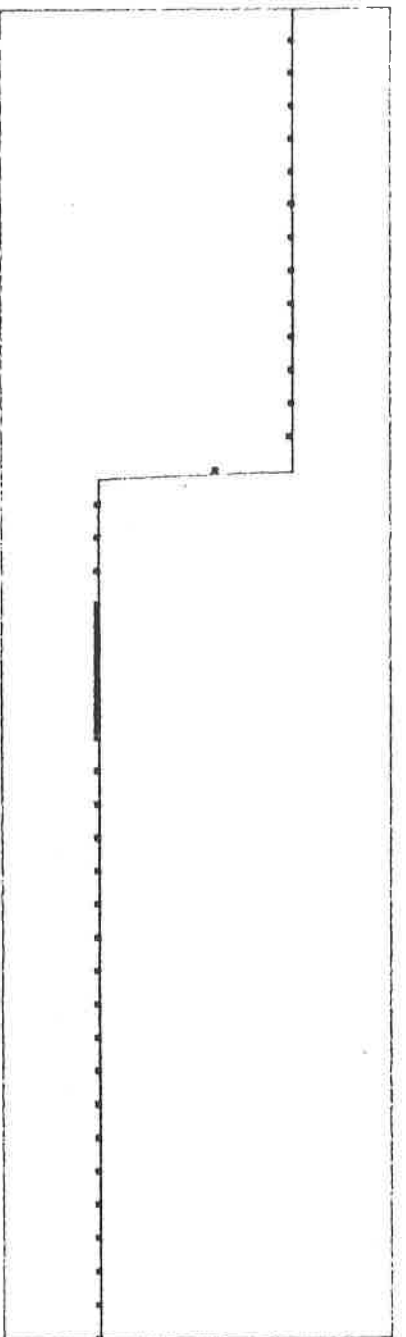
'Superbee' limiter

--- Exact solution

..... Approximate solution

Constant Grid





Burgers' Equation

Superbee' limiter

— Exact solution

..... Approximate solution

Abutted Grid

Mesh Jump = 5.0

Scheme 1

Burgers' Equation

"Superbee" limiter

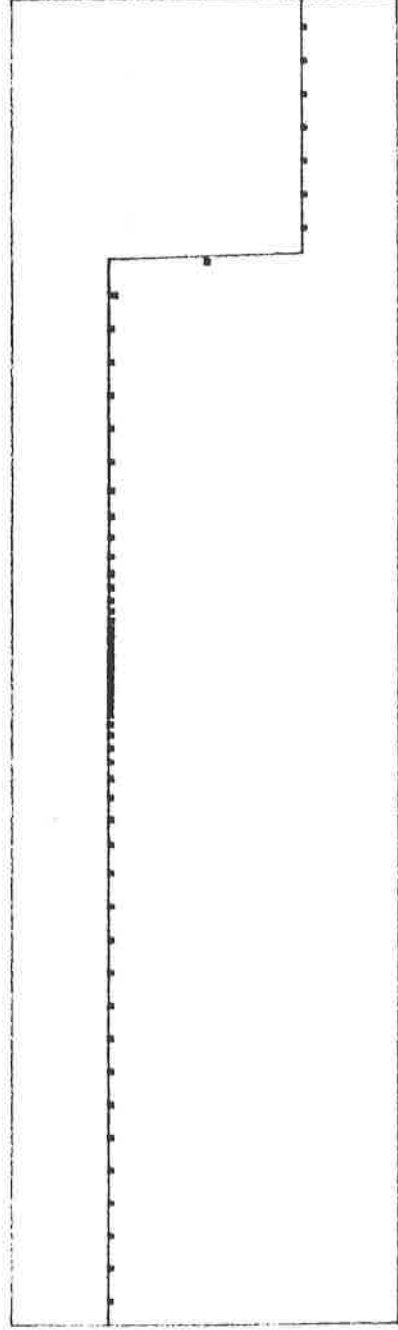
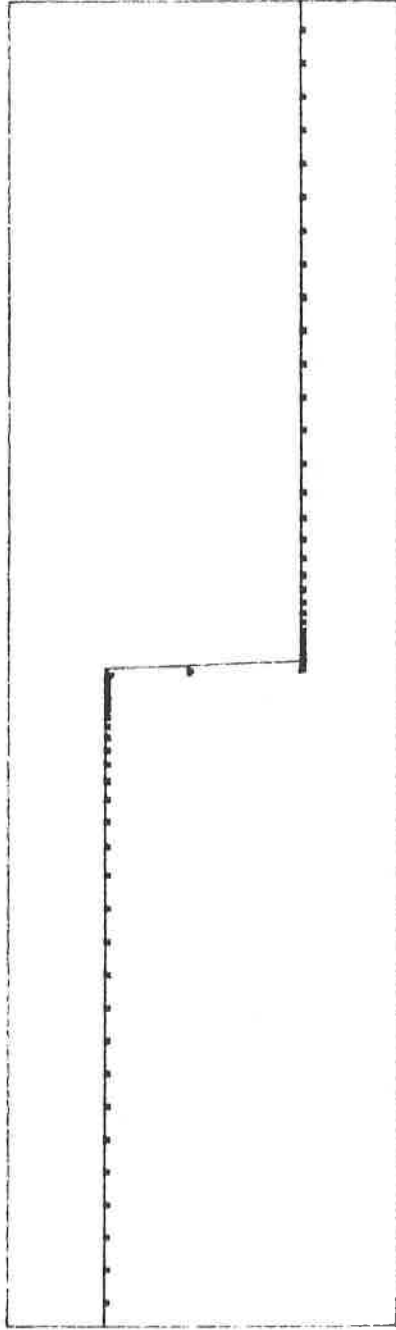
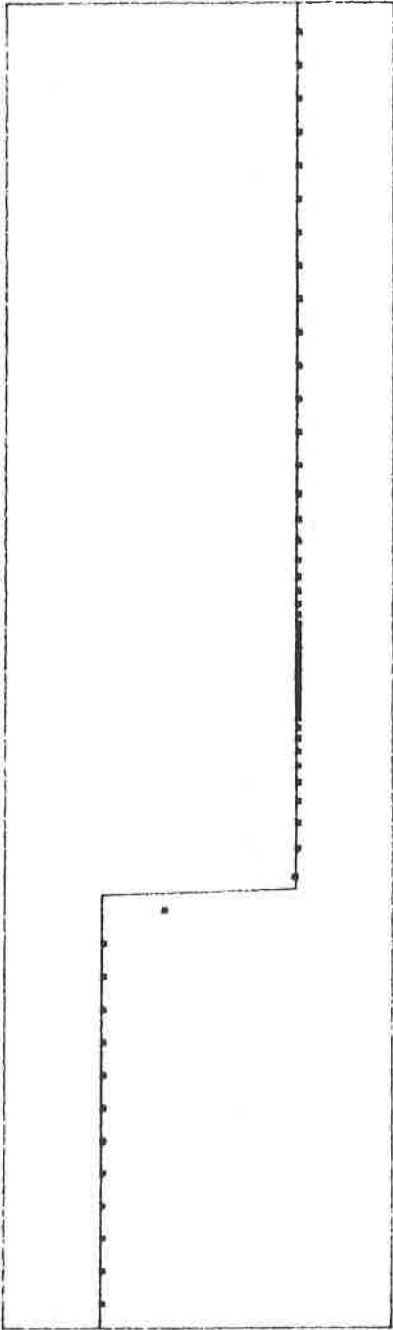
— Exact solution

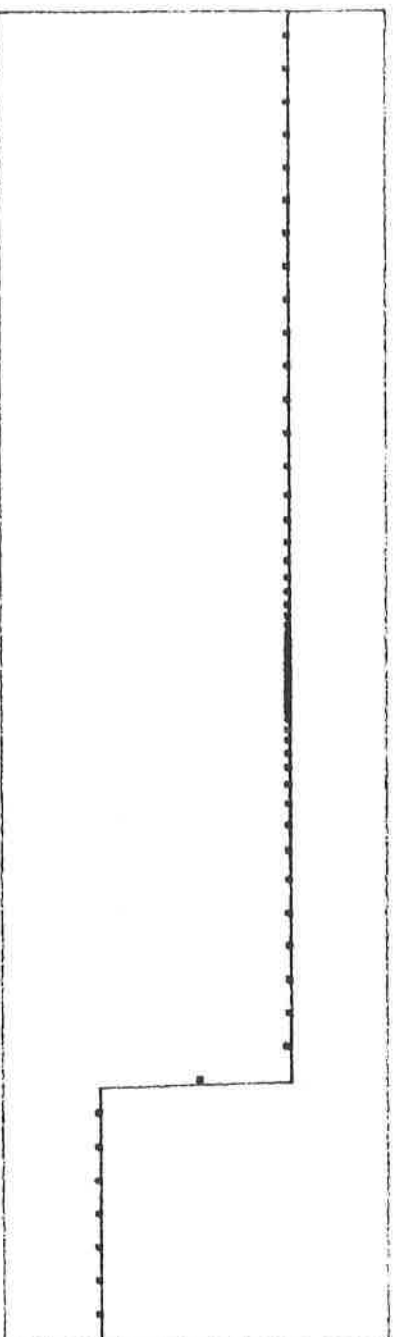
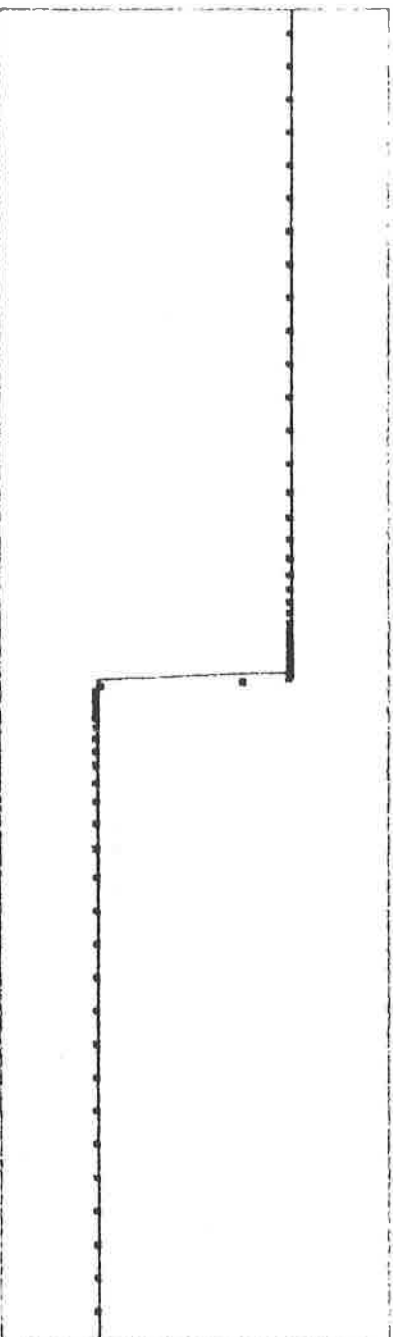
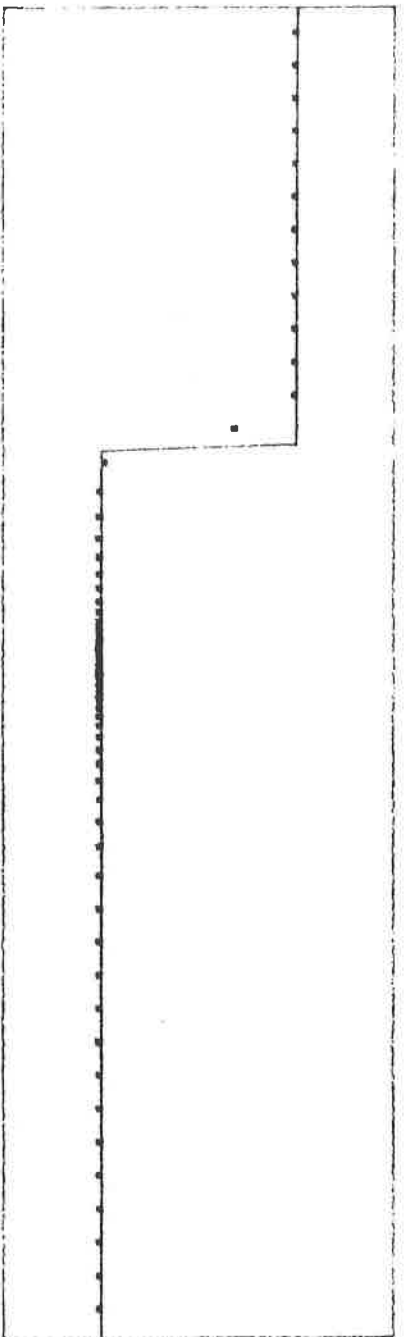
..... Approximate solution

Geometric Grid

Mesh Ratio = 1.15

Scheme 1





Burgers' Equation

— Superbee' limiter

— Exact solution

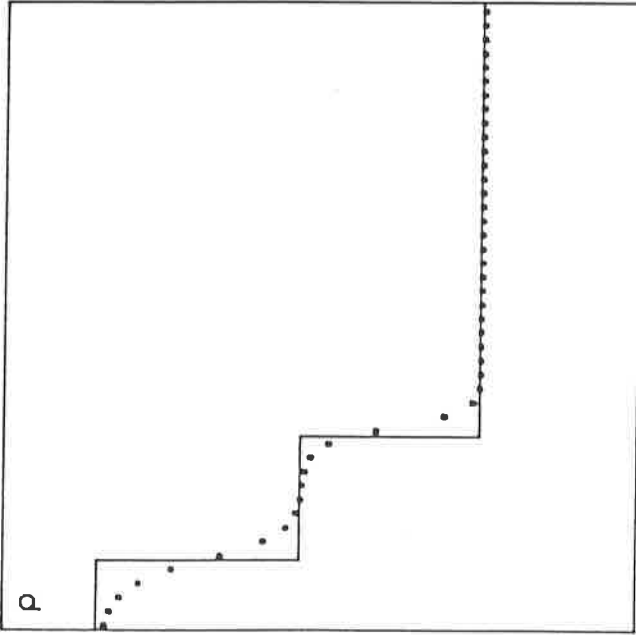
..... Approximate solution

Geometric Grid

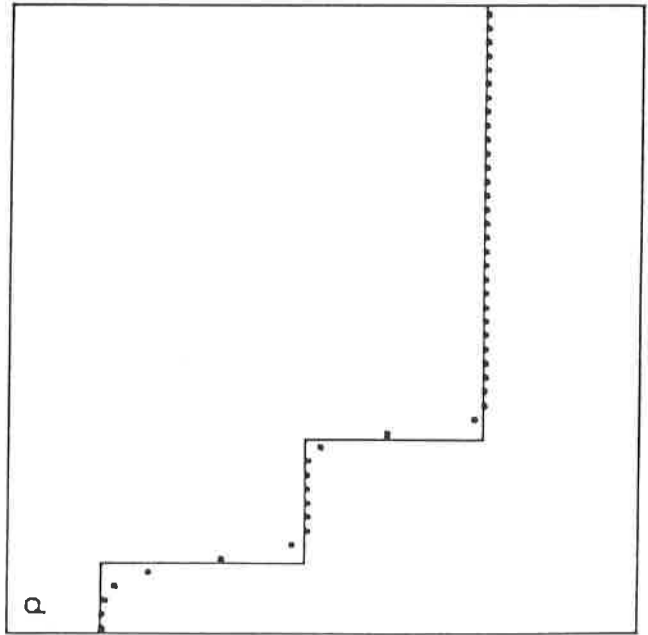
Mesh Ratio = 1.15

Scheme 2

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Sod's Shock Tube Problem



'Minmod' limiter



'Superbee' limiter

KEY

ρ - Density

— Exact Solution
 at time $t = 0.108 \text{ s}$
 Approximate solution
 at time $t = 0.108 \text{ s}$

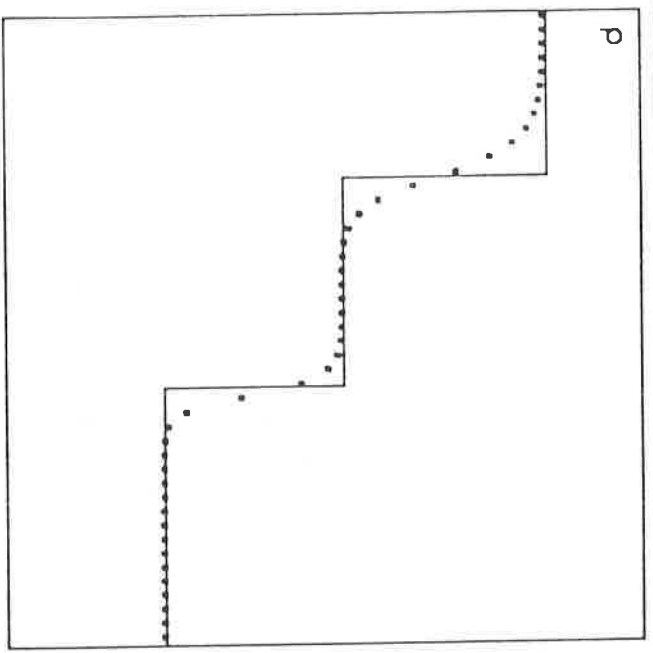
PARAMETERS

$\gamma = 1.4$
 Constant Grid

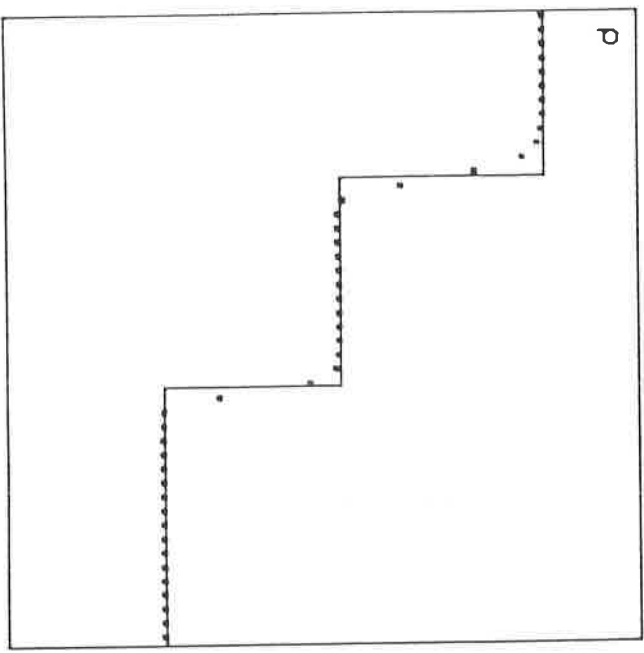
INITIAL CONDITIONS

$\rho = 1$	$\rho = 0.125$
$u = 0$	$u = 0$
$p = 1$	$p = 0.1$
0.5	
0	1

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Sod's Shock Tube Problem



'Minmod' Limiter



'Superbee' Limiter

KEY

p - Density

— Exact Solution

at time $t = 0.18$ s

..... Approximate solution

at time $t = 0.18$ s

PARAMETERS

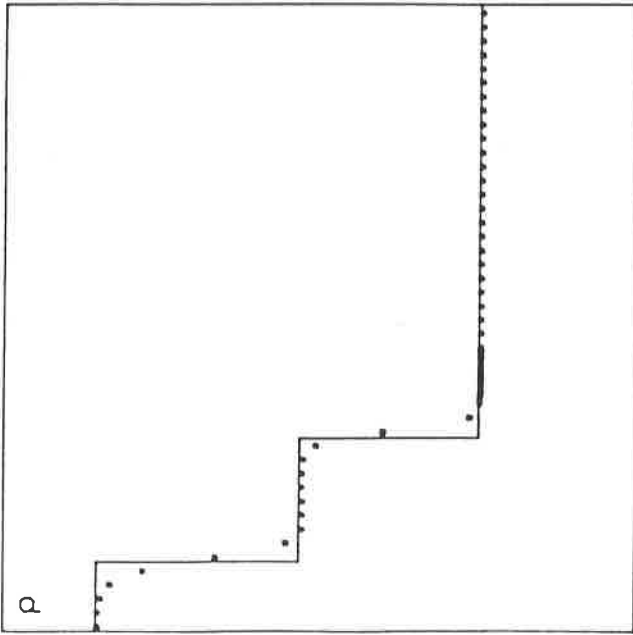
$\gamma = 1.4$

Constant Grid

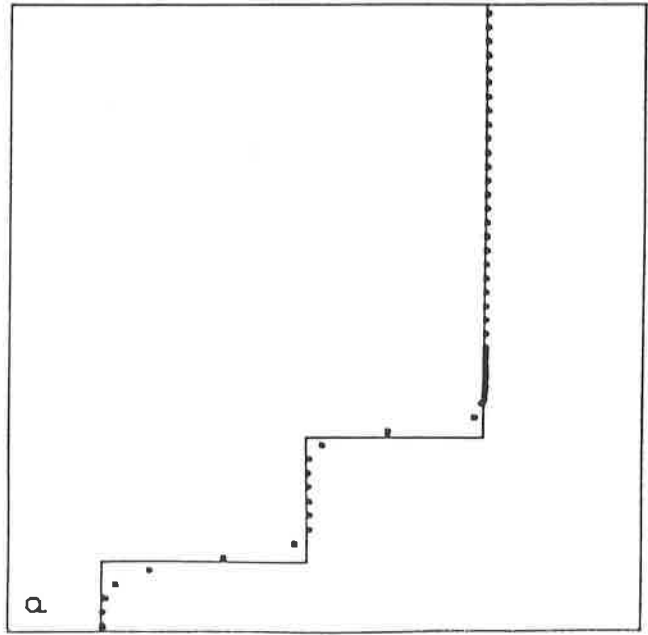
INITIAL CONDITIONS

$p = 1$	$p = 0.125$
$u = 0$	$u = 0$
$p = 1$	$p = 0.1$
0	1
0.5	

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Sod's Shock Tube Problem



Scheme 1



Scheme 2

KEY

ρ - Density

— Exact Solution
 at time $t = 0.108$ s
 Approximate solution
 at time $t = 0.108$ s

PARAMETERS

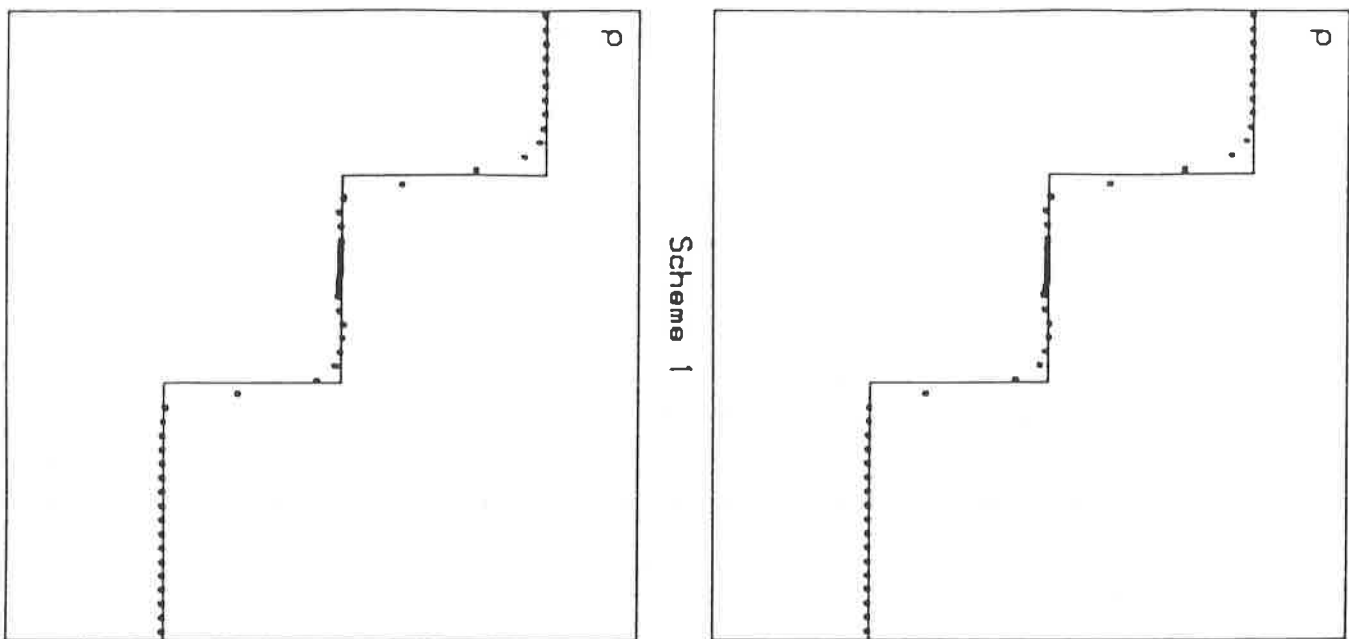
$\gamma = 1.4$
 Abutted Grid
 Mesh Jump = 5.0
 "Superbee" limiter used

INITIAL CONDITIONS

$\rho = 1$	$\rho = 0.125$
$u = 0$	$u = 0$
$p = 1$	$p = 0.1$
0.5	
0	1

Figure 14

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Sod's Shock Tube Problem



KEY

p - Density

— Exact Solution

at time $t = 0.18$ s

..... Approximate solution

at time $t = 0.18$ s

PARAMETERS

$\gamma = 1.4$

Abutted Grid

Mesh Jump = 5.0

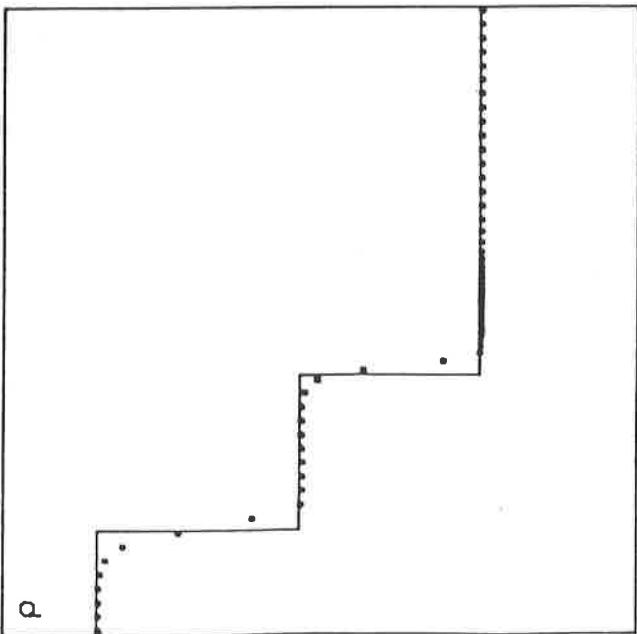
'Superbee' limiter used

INITIAL CONDITIONS

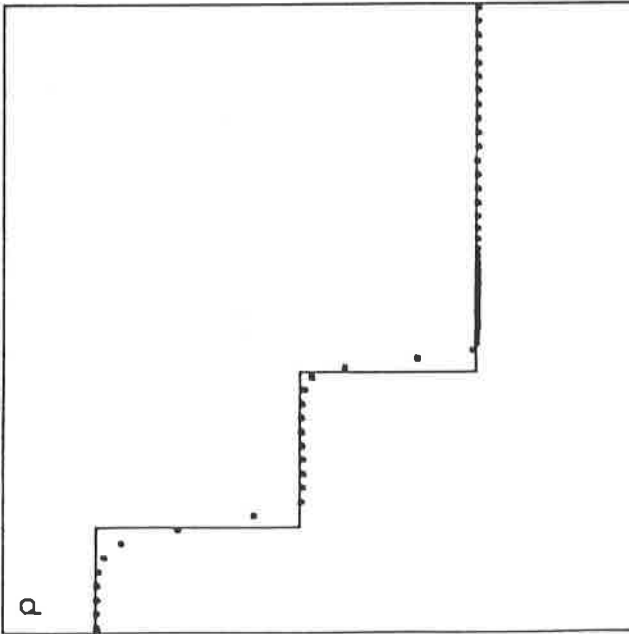
$p = 1$	$p = 0.125$
$u = 0$	$u = 0$
$p = 1$	$p = 0.1$
0	1
0.5	

Scheme 2

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Sod's Shock Tube Problem



Scheme 1



Scheme 2

KEY

ρ - Density

- Exact Solution
- at time $t = 0.135$ s
- Approximate solution
- at time $t = 0.135$ s

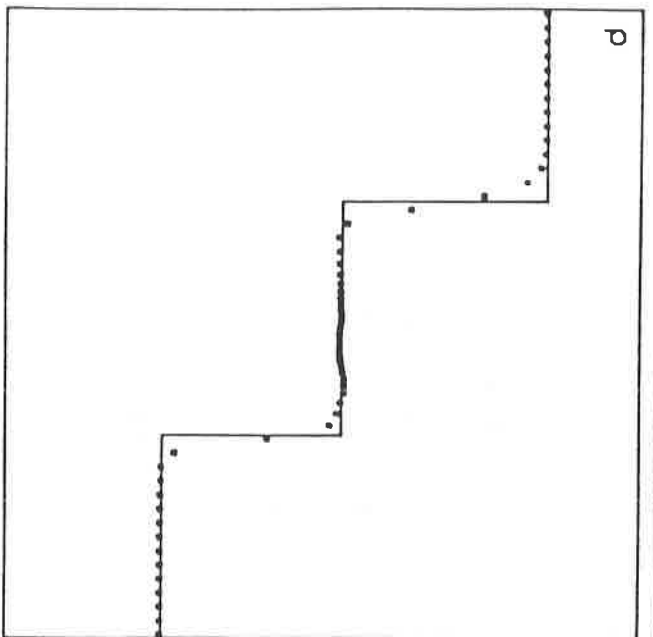
PARAMETERS

$\gamma = 1.4$
 Geometric Grid
 Mesh Ratio = 1.15
 'Superbee' limiter used

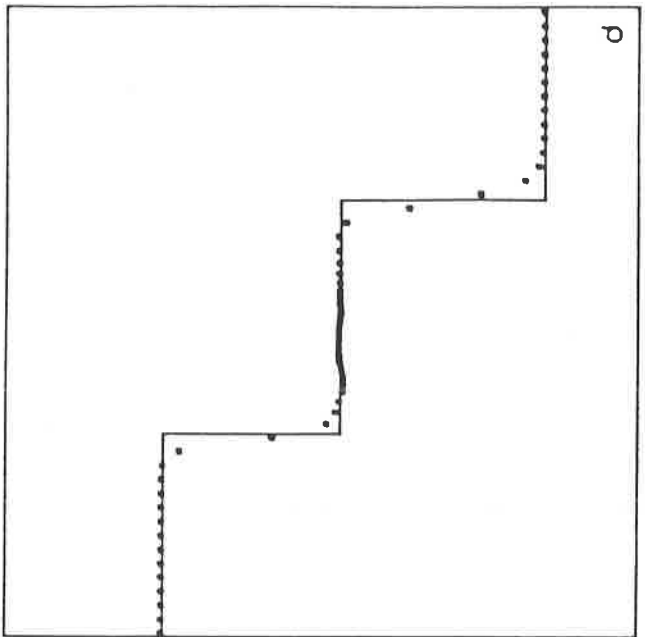
INITIAL CONDITIONS

$\rho = 1$	$\rho = 0.125$
$u = 0$	$u = 0$
$p = 1$	$p = 0.1$
	0.5
	1

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Sod's Shock Tube Problem



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution

at time $t = 0.203$ s

..... Approximate solution

at time $t = 0.203$ s

PARAMETERS

$\gamma = 1.4$

Geometric Grid

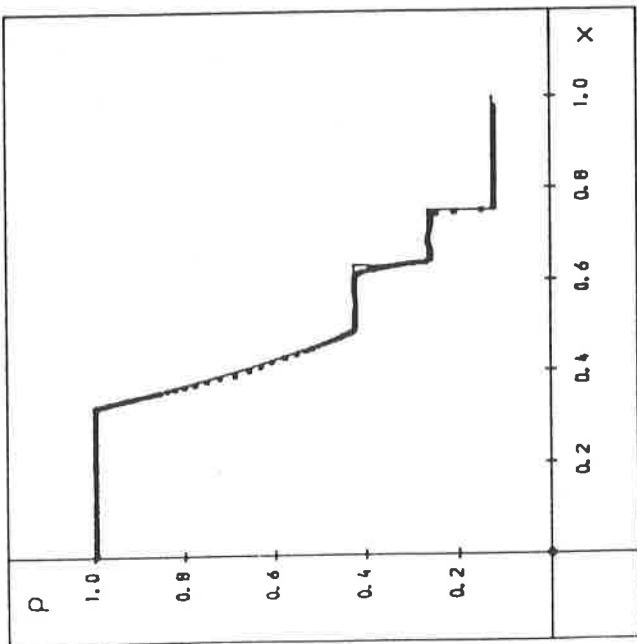
Mesh Ratio = 1.15

'Superbee' limiter used

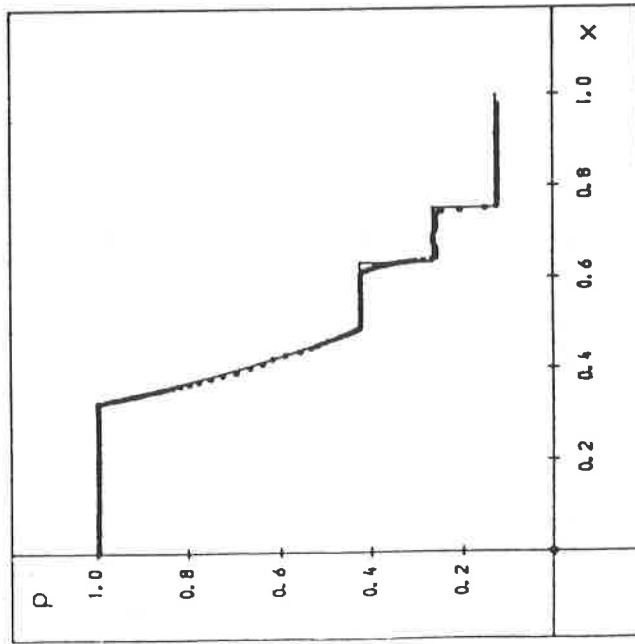
INITIAL CONDITIONS

$p = 1$	$p = 0.125$
$u = 0$	$u = 0$
$p = 1$	$p = 0.1$
0	0.5
	1

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Sod's Shock Tube Problem



Scheme 1



Scheme 2

KEY

ρ - Density

— Exact Solution
 at time $t = 0.144$ s
 Approximate solution
 at time $t = 0.144$ s

PARAMETERS

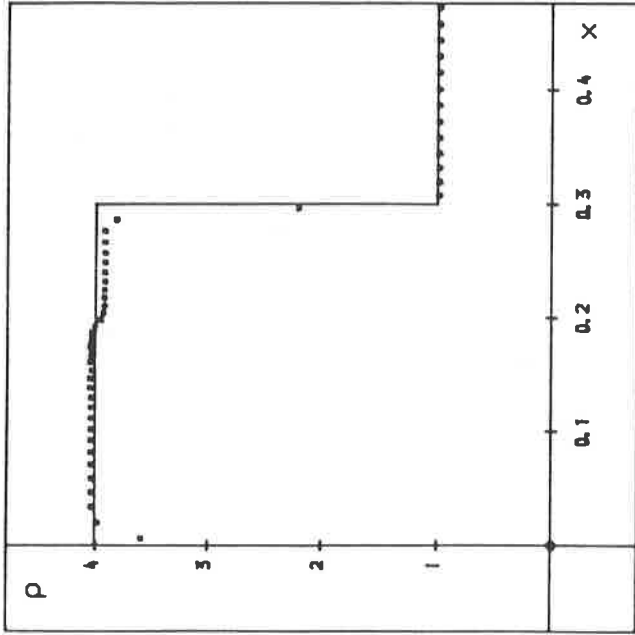
$\gamma = 1.4$
 Geometric Grid
 Mesh Ratio = 1.15
 Superbee limiter used

INITIAL CONDITIONS

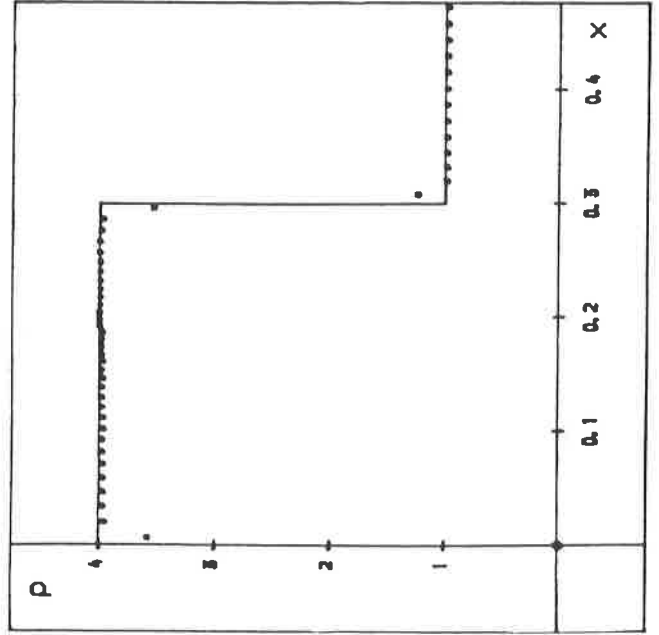
$\rho = 1$	$\rho = 0.125$
$u = 0$	$u = 0$
$p = 1$	$p = 0.1$
0	0.5
	1

Figure 17

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

ρ - Density

- Exact Solution at time $t = 0.9$ s
- Approximate solution at time $t = 0.9$ s

PARAMETERS

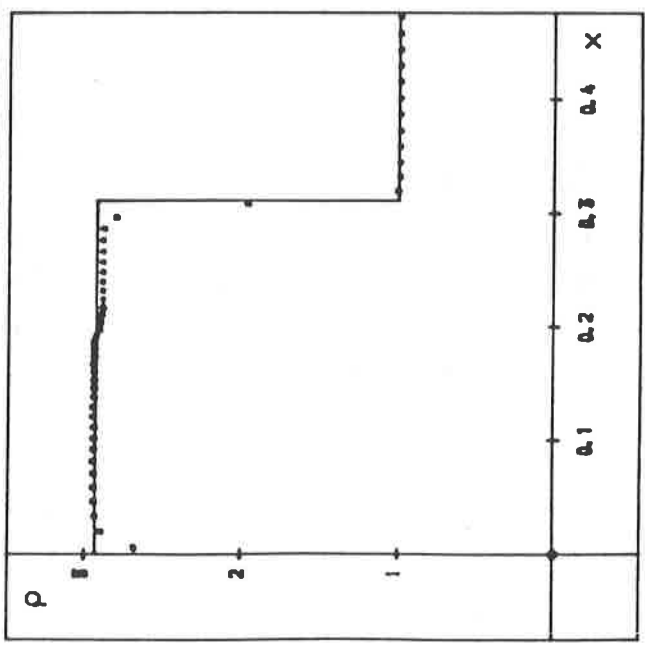
$\gamma = 5/3$
 Pressure Ratio = ∞
 Mesh Ratio = 1.05

INITIAL CONDITIONS

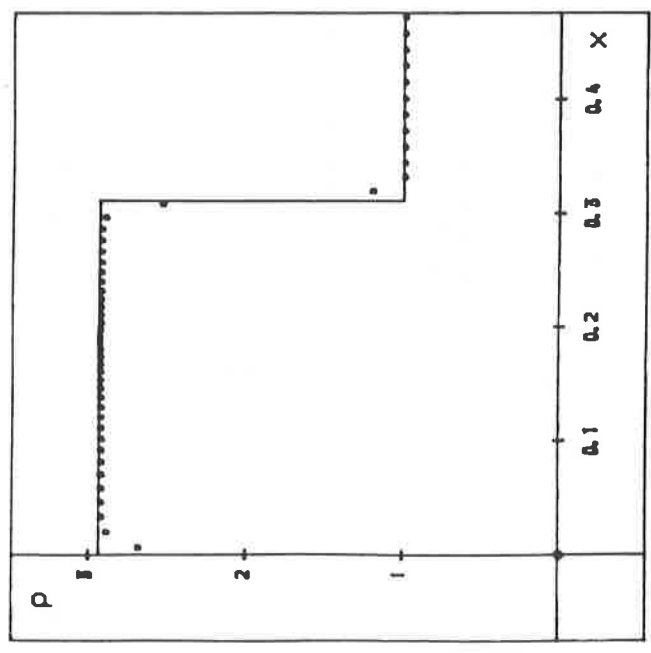
$\rho = 1$
 $u = -1$
 $p = 0.000$

Reflected Boundary Conditions
 at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

- ρ - Density
- Exact Solution
at time $t = 0.6$ s
- Approximate solution
at time $t = 0.6$ s

PARAMETERS

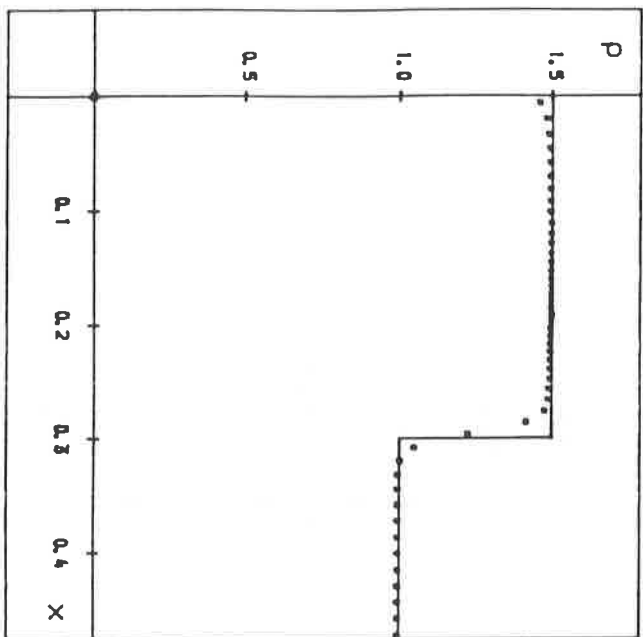
$\gamma = 5/3$
 Pressure Ratio = 10
 Mesh Ratio = 1.05

INITIAL CONDITIONS

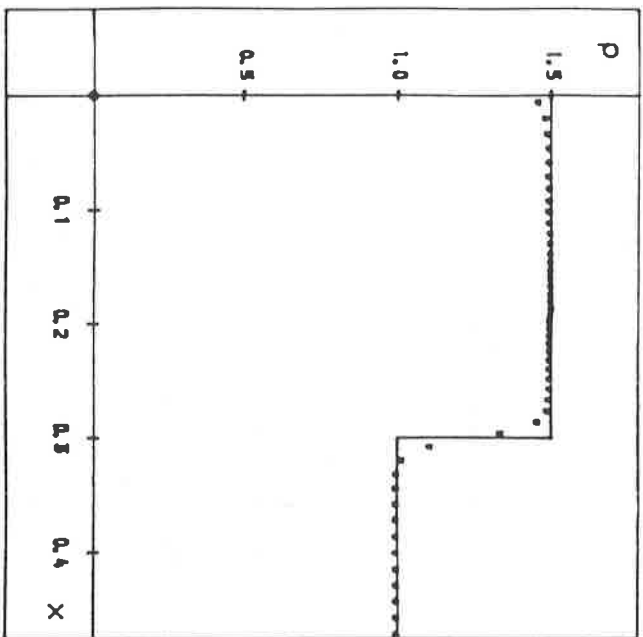
$\rho = 1$
 $u = -1$
 $p = 0.169$

Reflected Boundary Conditions
 at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution

at time $t = 0.15$

..... Approximate solution

at time $t = 0.15$

PARAMETERS

$\gamma = 5/3$

Pressure Ratio = 2

Mesh Ratio = 1.05

INITIAL CONDITIONS

$\rho = 1$

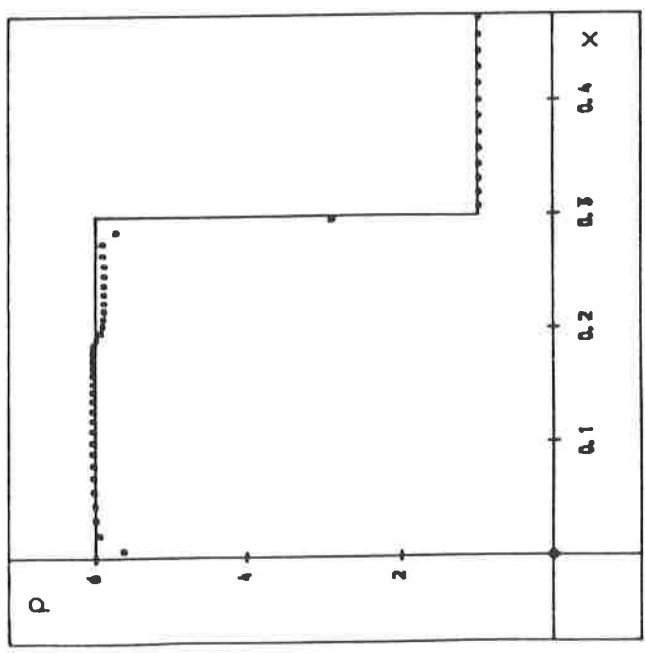
$u = -1$

$p = 3.000$

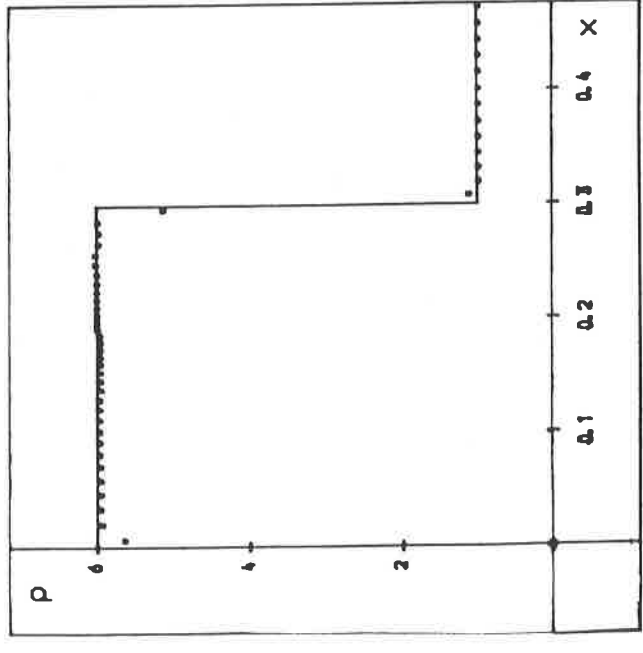
Reflected Boundary Conditions

at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution

at time $t = 1.5 \text{ s}$

..... Approximate solution

at time $t = 1.5 \text{ s}$

PARAMETERS

$\gamma = 1.4$

Pressure Ratio = ∞

Mesh Ratio = 1.05

INITIAL CONDITIONS

$p = 1$

$u = -1$

$p = 0.000$

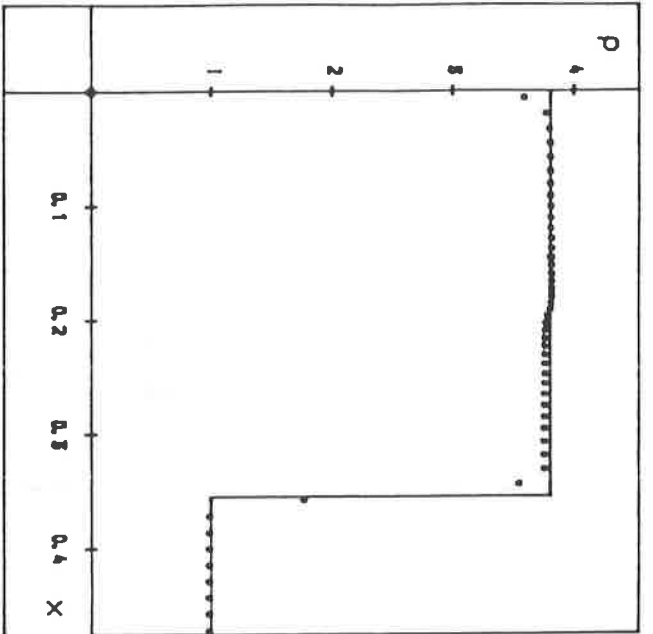
0

Reflected Boundary Conditions

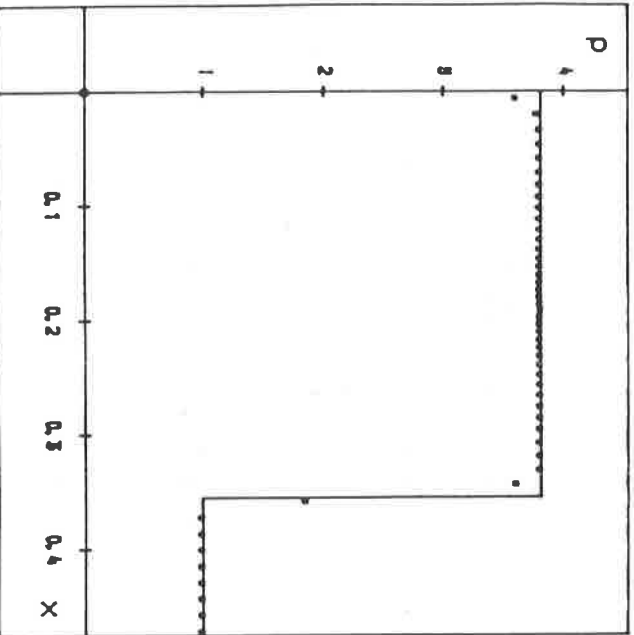
at $x = 0$

1

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution

at time $t = 1.0$ ●

..... Approximate solution

at time $t = 1.0$ ●

PARAMETERS

$\gamma = 1.4$

Pressure Ratio = 10

Mesh Ratio = 1.05

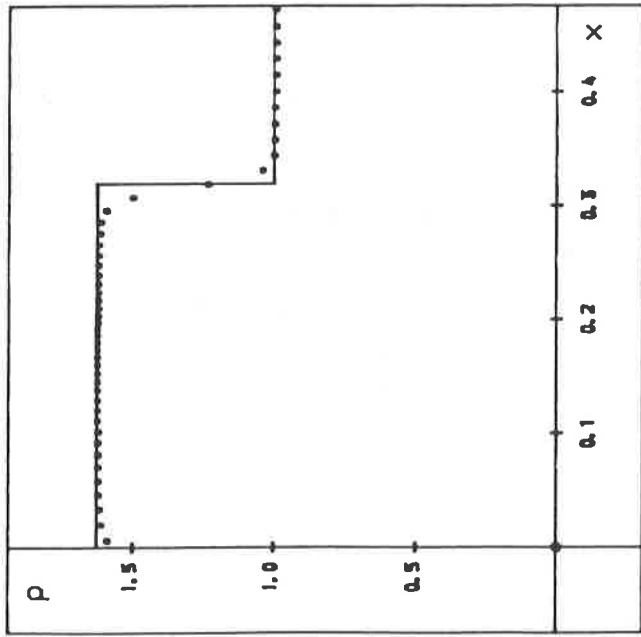
INITIAL CONDITIONS

$p = 1$
 $u = -1$
 $p = 0.151$

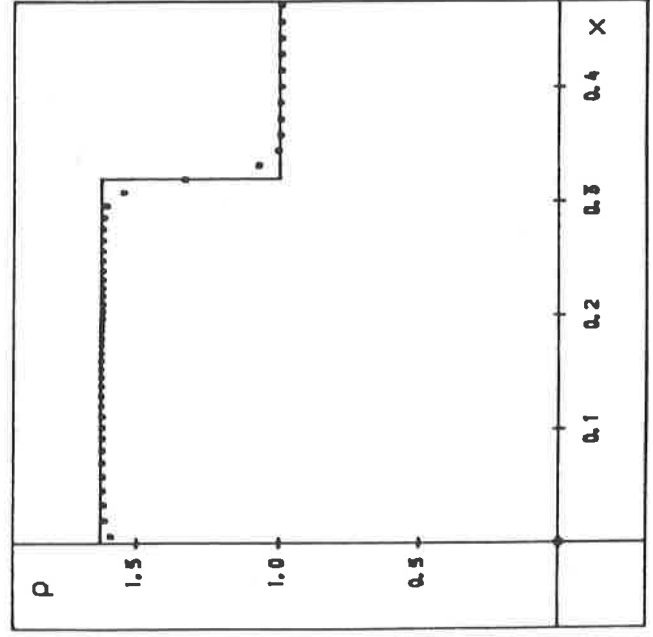
Reflected Boundary Conditions

at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

ρ - Density

— Exact Solution

at time $t = 0.2$ s

..... Approximate solution

at time $t = 0.2$ s

PARAMETERS

$\gamma = 1.4$

Pressure Ratio = 2

Mesh Ratio = 1.05

INITIAL CONDITIONS

$p = 1$

$u = -1$

$\rho = 2.600$

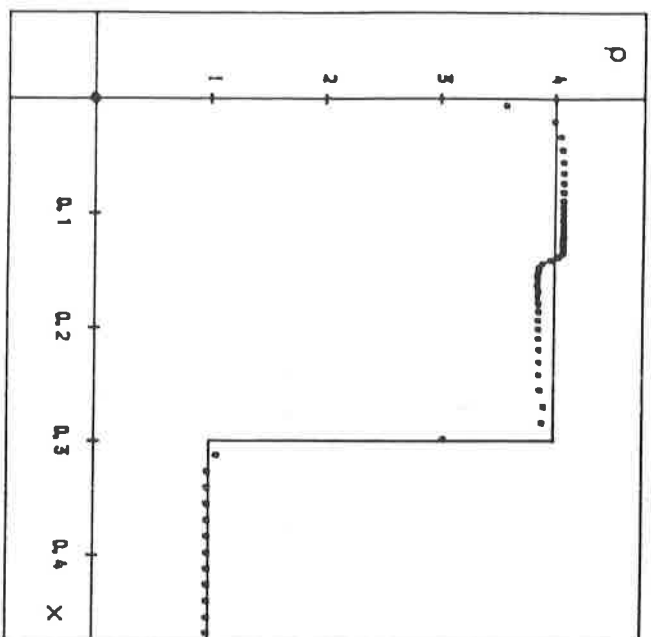
0

Reflected Boundary Conditions

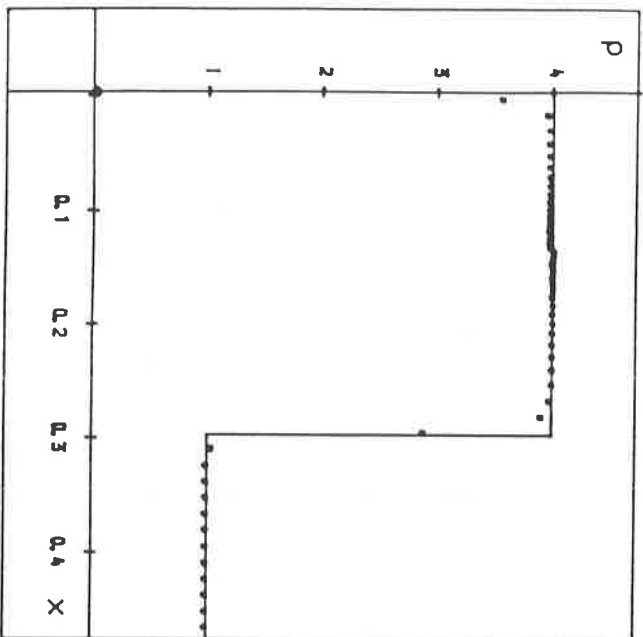
at $x = 0$

1

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution

at time $t = 0.9$ s

..... Approximate solution

at time $t = 0.9$ s

PARAMETERS

$\gamma = 5/3$

Pressure Ratio = ∞

Mesh Ratio = 1.1

Minmod Limiter used

INITIAL CONDITIONS

$p = 1$

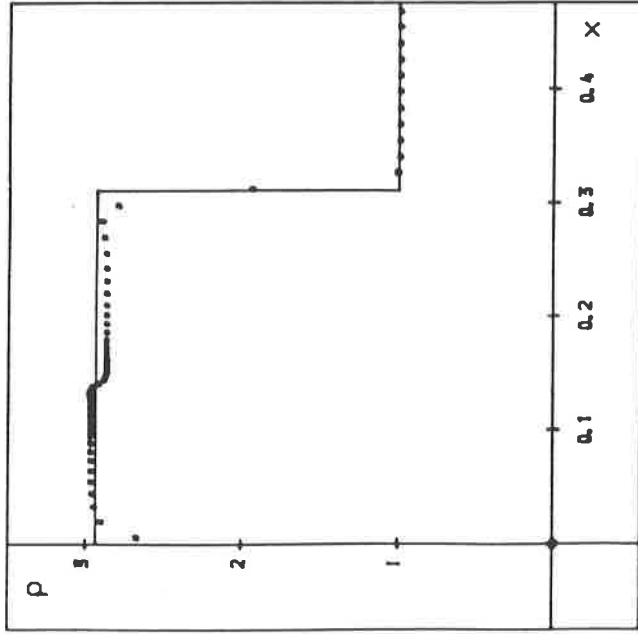
$u = -1$

$p = 0.000$

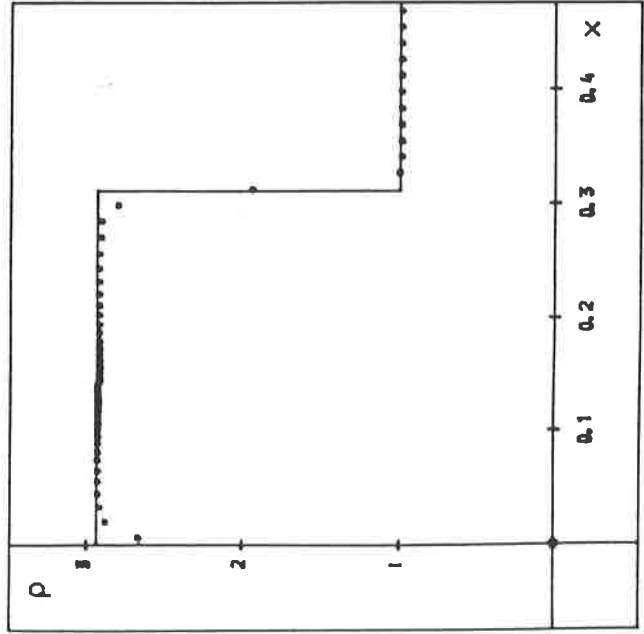
Reflected Boundary Conditions

at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY -- Shock Reflection



Scheme 1



Scheme 2

KEY

ρ - Density

- Exact Solution at time $t = 0.6$ s
- Approximate solution at time $t = 0.6$ s

PARAMETERS

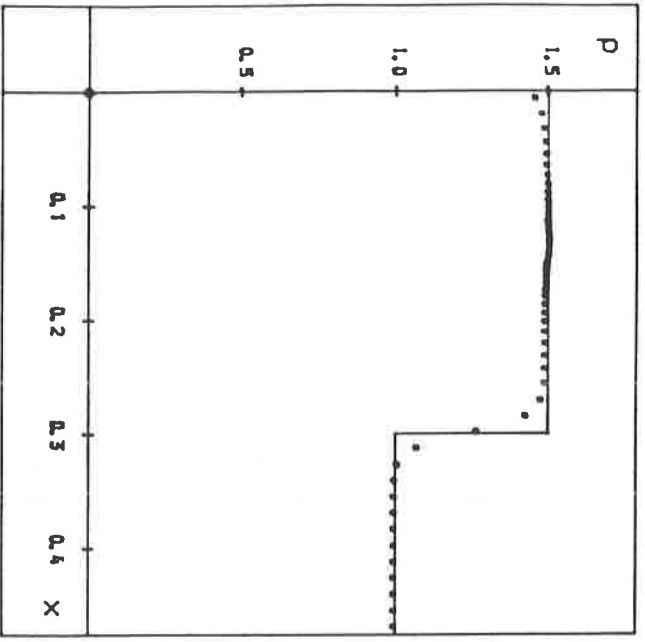
$\gamma = 5/3$
 Pressure Ratio = 10
 Mesh Ratio = 1.1
 "Minmod" limiter used

INITIAL CONDITIONS

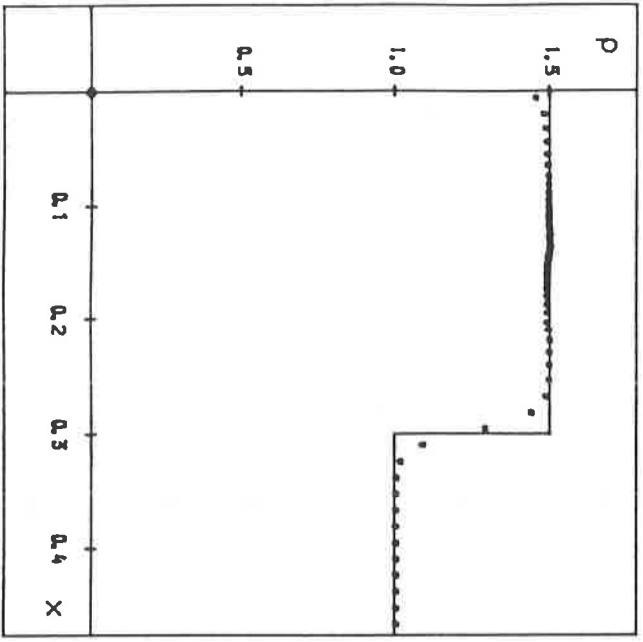
$$\begin{aligned} p &= 1 \\ u &= -1 \\ p &= 0.169 \end{aligned}$$

Reflected Boundary Conditions
 at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution
at time $t = 0.15$ s

..... Approximate solution
at time $t = 0.15$ s

PARAMETERS

$\gamma = 5/3$

Pressure Ratio = 2

Mesh Ratio = 1.1

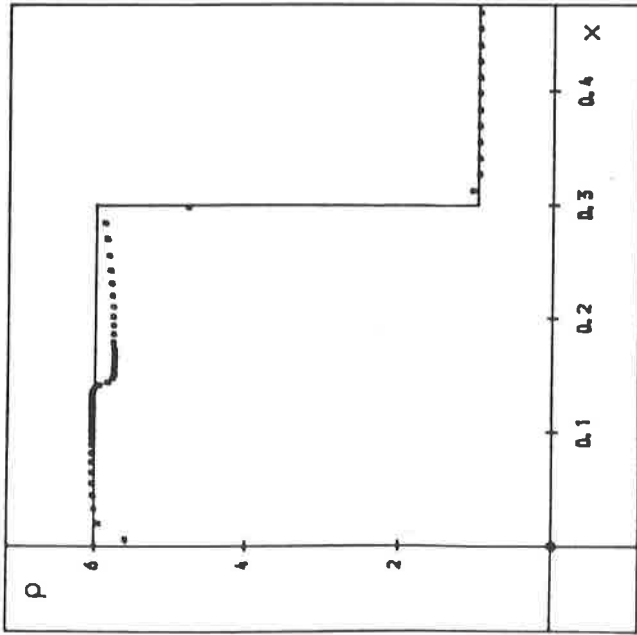
'Minmod' limiter used

INITIAL CONDITIONS

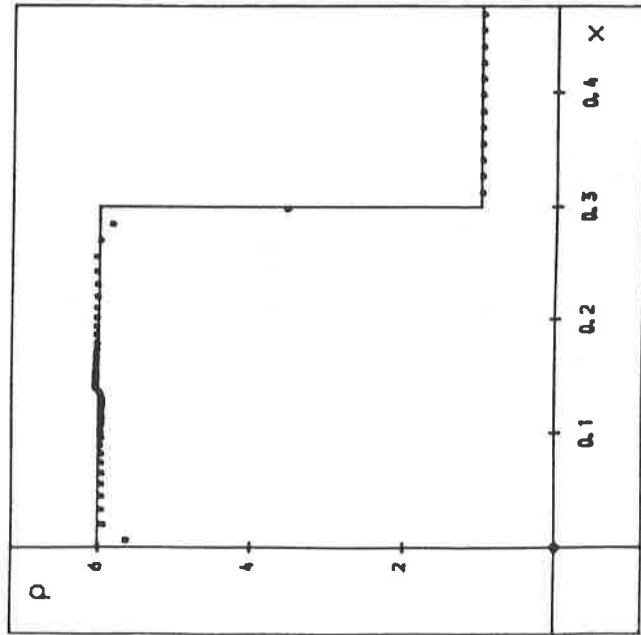
$p = 1$	0	1
$u = -1$	0	1
$p = 3.000$	0	1

Reflected Boundary Conditions
at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution

at time $t = 1.5$ s

..... Approximate solution

at time $t = 1.5$ s

PARAMETERS

$\gamma = 1.4$

Pressure Ratio = ∞

Mesh Ratio = 1.1

'Minmod' limiter used

INITIAL CONDITIONS

$p = 1$

$u = -1$

$p = 0.000$

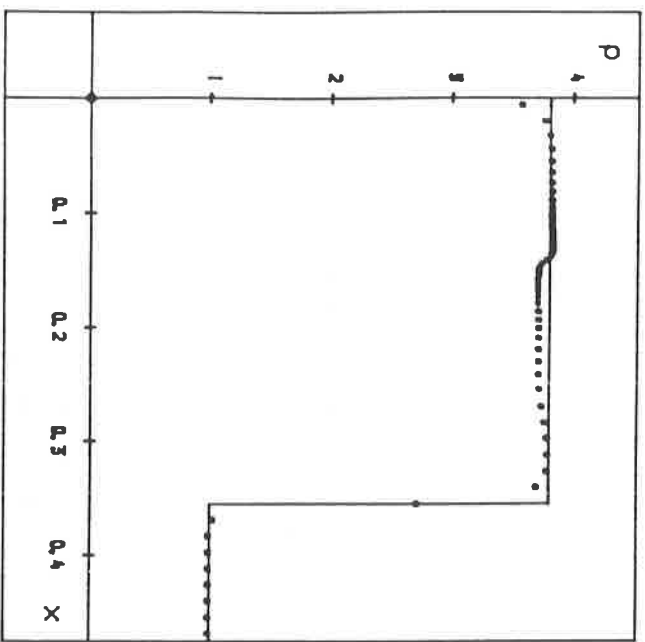
0

1

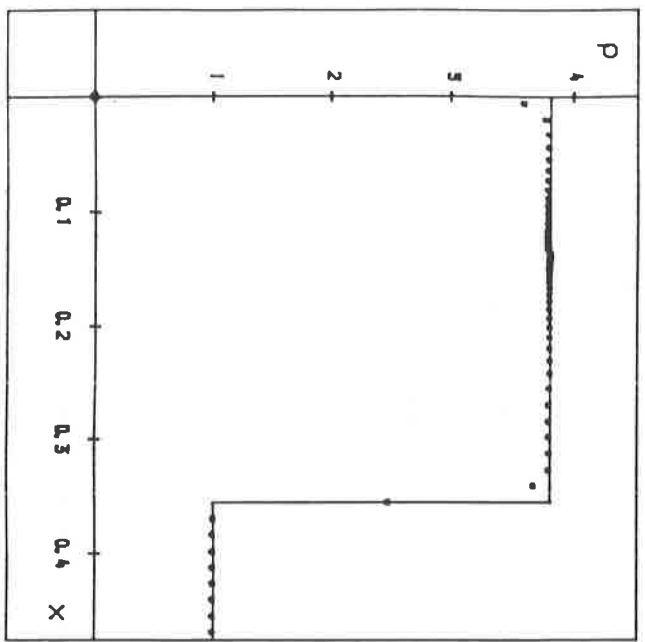
Reflected Boundary Conditions

at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution
at time $t = 1.0$ s

..... Approximate solution
at time $t = 1.0$ s

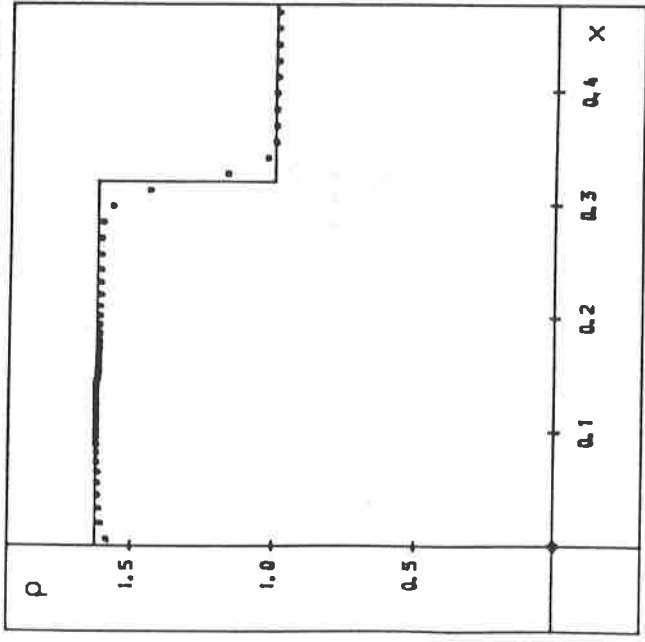
PARAMETERS

- $\gamma = 1.4$
- Pressure Ratio = 10
- Mach Ratio = 1.1
- 'Minmod' limiter used

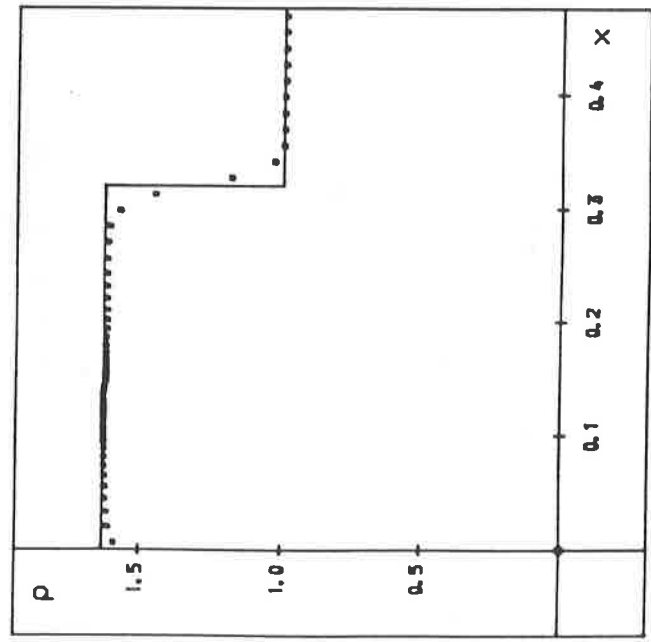
INITIAL CONDITIONS

$p = 1$ $u = -1$ $p = 0.151$	Reflected Boundary Conditions at $x = 0$
------------------------------------	---

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

ρ - Density

- Exact Solution at time $t = 0.2$ s
- Approximate solution at time $t = 0.2$ s

PARAMETERS

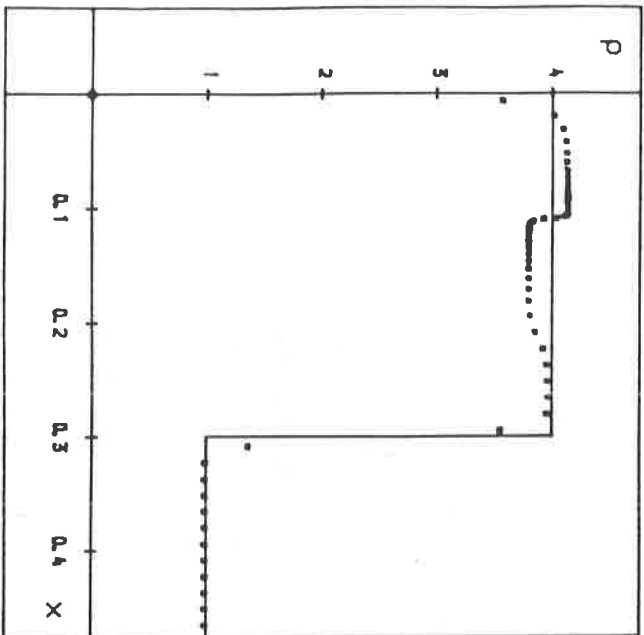
- $\gamma = 1.4$
- Pressure Ratio = 2
- Mesh Ratio = 1.1
- 'Minmod' Limiter used

INITIAL CONDITIONS

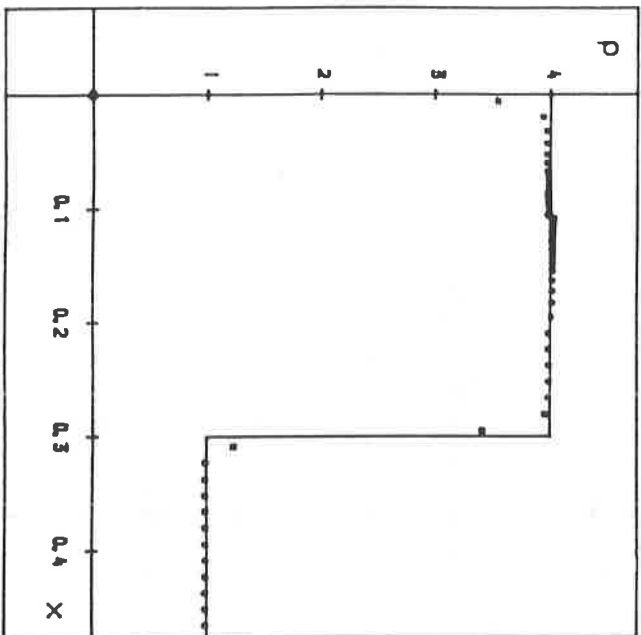
$$\begin{aligned} \rho &= 1 \\ u &= -1 \\ p &= 2.600 \end{aligned}$$

Reflected Boundary Conditions at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution

at time $t = 0.9$ s

..... Approximate solution

at time $t = 0.9$ s

PARAMETERS

$\gamma = 5/3$

Pressure Ratio = ∞

Mesh Ratio = 1.15

'Minmod' limiter used

INITIAL CONDITIONS

$p = 1$

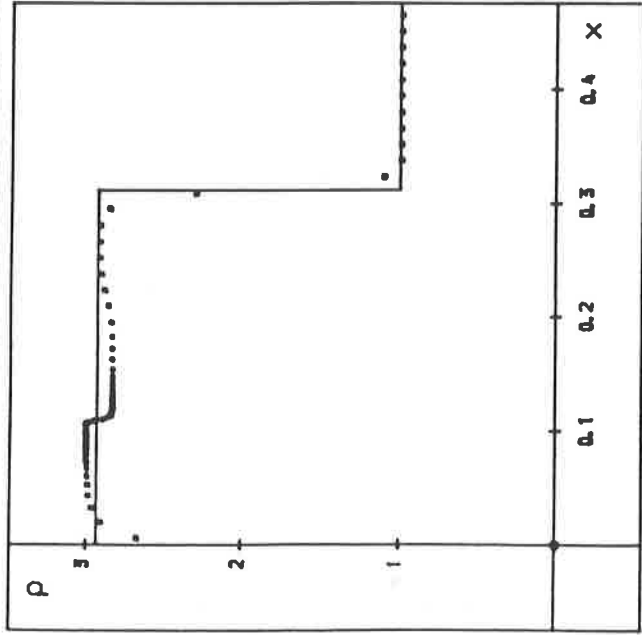
$u = -1$

$p = 0.000$

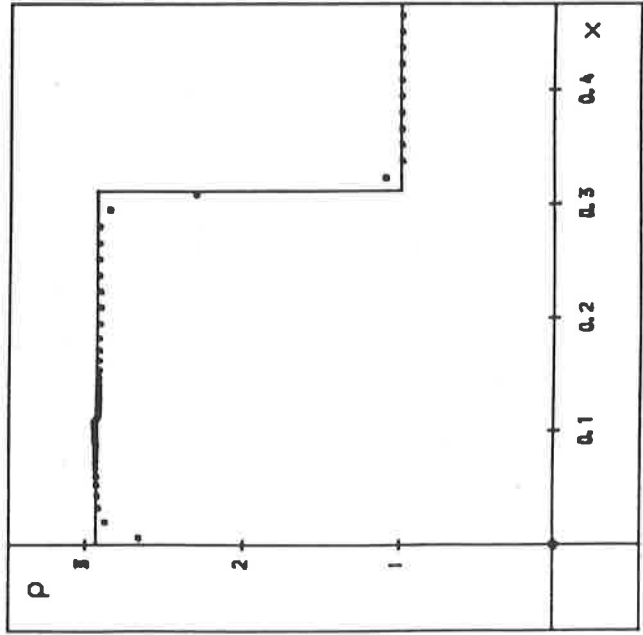
Reflected Boundary Conditions

at $x = 0$

SOLUTION OF THE FUJER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

ρ - Density

— Exact Solution

at time $t = 0.6$ s

..... Approximate solution

at time $t = 0.6$ s

PARAMETERS

$\gamma = 5/3$

Pressure Ratio = 10

Mesh Ratio = 1.15

'Minmod' limiter used

INITIAL CONDITIONS

$$p = 1$$

$$u = -1$$

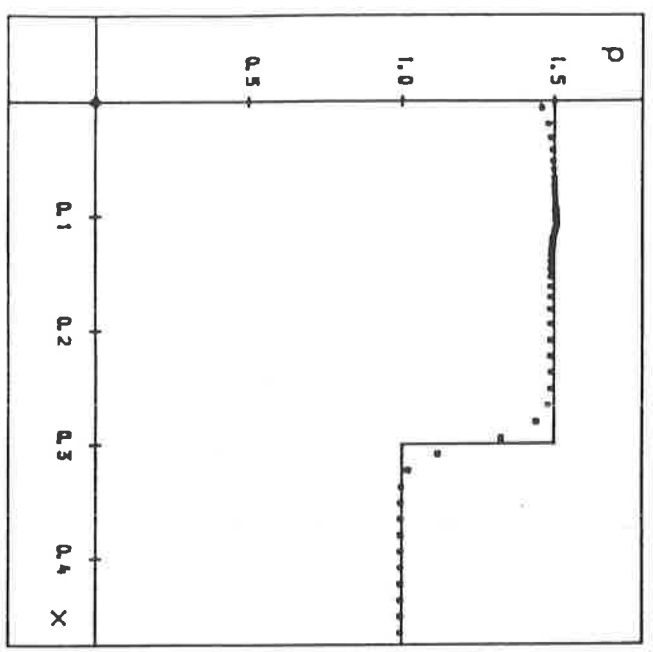
$$p = 0.169$$

0 1

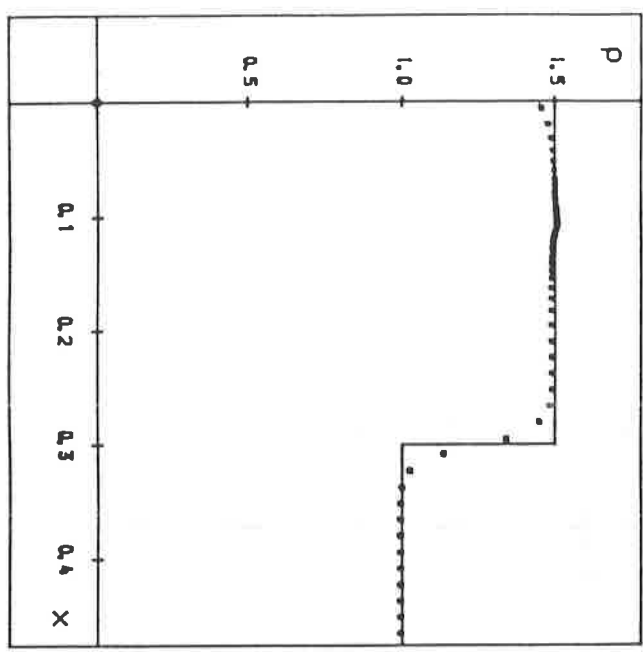
Reflected Boundary Conditions

at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution

at time $t = 0.15$ s

..... Approximate solution

at time $t = 0.15$ s

PARAMETERS

$\gamma = 5/3$

Pressure Ratio = 2

Mach Ratio = 1.15

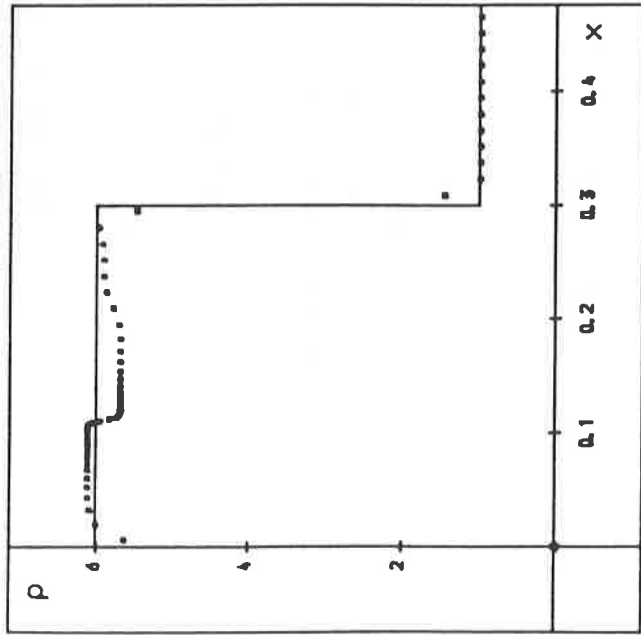
'Minmod' limiter used

INITIAL CONDITIONS

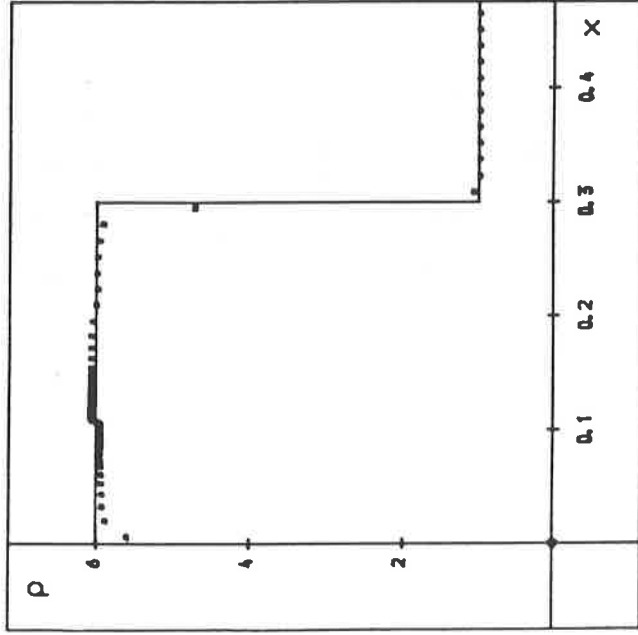
$p = 1$	$u = -1$
$p = 3.000$	$u = 0$

Reflected Boundary Conditions
at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

ρ - Density

— Exact Solution

at time $t = 1.5$ s

..... Approximate solution

at time $t = 1.5$ s

PARAMETERS

$\gamma = 1.4$

Pressure Ratio = ∞

Mesh Ratio = 1.15

'Minmod' Limiter used

INITIAL CONDITIONS

$$\rho = 1$$

$$u = -1$$

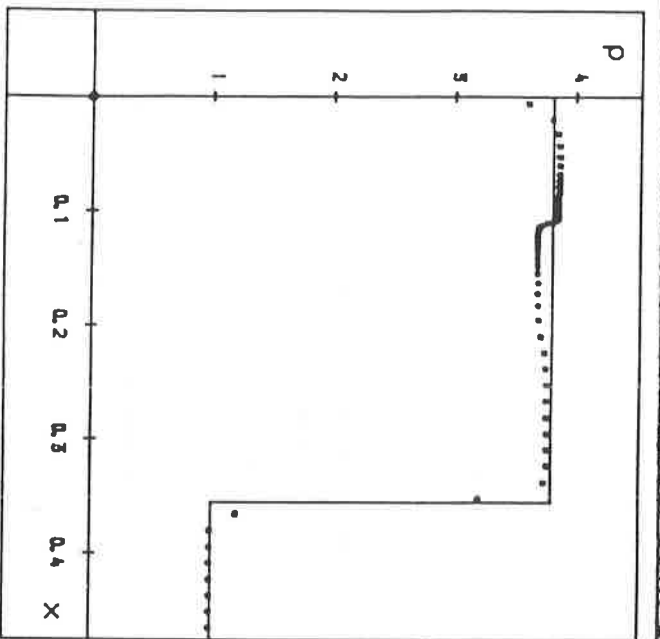
$$p = 0.000$$

0 1

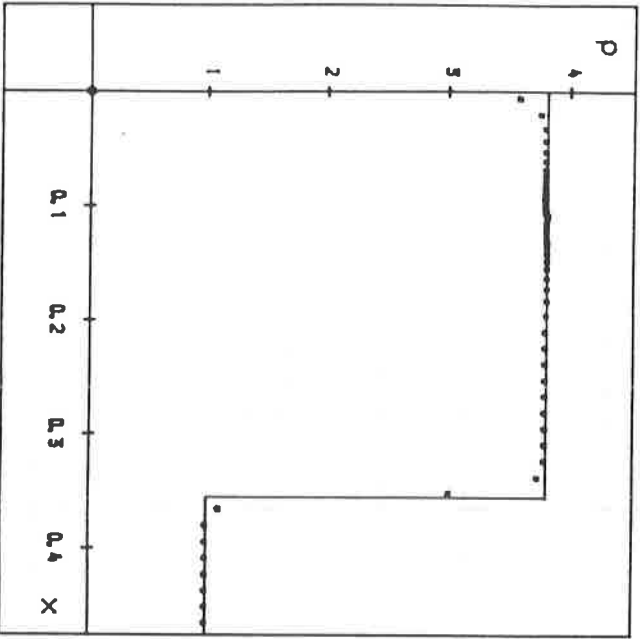
Reflected Boundary Conditions

at $x = 0$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

ρ - Density

— Exact Solution

at time $t = 1.0$ s

..... Approximate solution

at time $t = 1.0$ s

PARAMETERS

$$\gamma = 1.4$$

$$\text{Pressure Ratio} = 10$$

$$\text{Mesh Ratio} = 1.15$$

'Minmod' limiter used

INITIAL CONDITIONS

$$p = 1$$

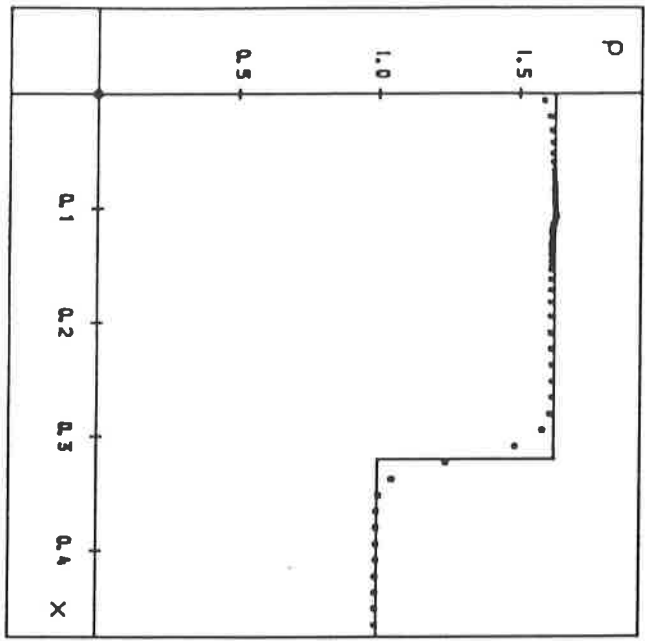
$$u = -1$$

$$p = 0.151$$

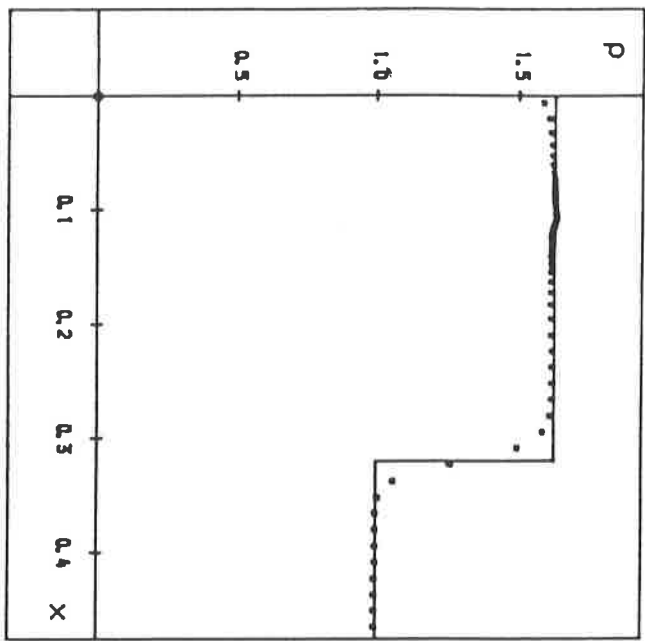
Reflected Boundary Conditions

$$\text{at } x = 0$$

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



Scheme 1



Scheme 2

KEY

p - Density

— Exact Solution
 at time $t = 0.2$ s

..... Approximate solution
 at time $t = 0.2$ s

PARAMETERS

$\gamma = 1.4$
 Pressure Ratio = 2
 Mesh Ratio = 1.15
 "Minmod" limiter used

INITIAL CONDITIONS

$p = 1$	
$u = -1$	
$p = 2.600$	

Reflected Boundary Conditions
 at $x = 0$

8. CONCLUSION

We have devised two new irregular grid schemes, based on Roe's (linearised) Riemann solver, for the solution of the Euler equations of gas dynamics. Both schemes give similar results for flows with weak shocks. The scheme of §5, however, gives better results for a strongly shocked flow.

ACKNOWLEDGEMENTS

I would like to express my thanks to Dr. M. J. Baines for many useful discussions and to Dr. P. K. Sweby for the use of his exact Riemann solver for the shock tube problem of §6.

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4. SWEBY, P. K. (1984) High Resolution Schemes using Flux Limiters for Hyperbolic Conservation Laws. SIAM J. Numer. Anal. 21, 995.
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APPENDIX

We show in this Appendix that the scheme given by equations (4.10a-h) together with the same restrictions on the limiter ψ as in §4, i.e.

$$2 \geq \psi(m) \geq 0, \quad 2 \geq \psi(m)/m \geq 0$$

and

$$\psi(m) = 0 \quad \text{when} \quad m < 0,$$

is TVD if

$$-\min(r_{j-\frac{1}{2}}^-, \frac{1}{2}) \leq v_{j-\frac{1}{2}} \leq \min(r_{j-\frac{1}{2}}^+, \frac{1}{2}) \quad (\text{A1})$$

for all j .

The scheme given by equations (4.10a-h) can be written in the form

$$u^j = u_j - \frac{1}{2}(s_{j-\frac{1}{2}} + 1) \frac{\Delta t}{\Delta x_j} \Delta f_{j-\frac{1}{2}} - \frac{1}{2}(1 - s_{j+\frac{1}{2}}) \frac{\Delta t}{\Delta x_j} \Delta f_{j+\frac{1}{2}} - s_{j-\frac{1}{2}} \frac{b_{j-\frac{1}{2}}}{\Delta x_j} + s_{j+\frac{1}{2}} \frac{b_{j+\frac{1}{2}}}{\Delta x_j} \quad (\text{A2})$$

where

$$v_{j-\frac{1}{2}} = \begin{cases} \frac{\Delta t \Delta f_{j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}}} & u_j \neq u_{j-1} \\ \frac{\Delta t}{\Delta x_{j-\frac{1}{2}}} f'(u_j) & u_j = u_{j-1} \end{cases} \quad (\text{A3})$$

$$b_{j-\frac{1}{2}} = -\frac{1}{2}(1 - |v_{j-\frac{1}{2}}|) \frac{\Delta t}{\Delta x_j} \Delta f_{j-\frac{1}{2}} \psi(M_{j-\frac{1}{2}}) \quad (\text{A4})$$

and

$$M_{j-\frac{1}{2}} = \frac{\left(\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}} - s_{j-\frac{1}{2}}} - |v_{j-\frac{1}{2}} - s_{j-\frac{1}{2}}| \right) \Delta f_{j-\frac{1}{2}} - s_{j-\frac{1}{2}}}{(1 - |v_{j-\frac{1}{2}}|) \Delta f_{j-\frac{1}{2}}} \quad (\text{A5})$$

Firstly, we define

$$v_{j-\frac{1}{2}}^+ = \begin{cases} v_{j-\frac{1}{2}} & v_{j-\frac{1}{2}} > 0 \\ 0 & v_{j-\frac{1}{2}} \leq 0 \end{cases} \quad (A6)$$

$$v_{j-\frac{1}{2}}^- = \begin{cases} 0 & v_{j-\frac{1}{2}} \geq 0 \\ v_{j-\frac{1}{2}} & v_{j-\frac{1}{2}} < 0 \end{cases} \quad (A7)$$

$$\delta_{j-\frac{1}{2}}^\pm = \frac{1}{2}(1 \mp v_{j-\frac{1}{2}}^\pm) \quad (A8)$$

$$\tau_{j-\frac{1}{2}}^\pm = \frac{1}{2}(r_{j-\frac{1}{2}}^\pm \mp v_{j-\frac{1}{2}}^\pm) \quad (A9)$$

$$M_{j-\frac{1}{2}}^+ = \begin{cases} M_{j-\frac{1}{2}} = \frac{\tau_{j-\frac{3}{2}}^+ v_{j-\frac{3}{2}}^+ \Delta u_{j-\frac{3}{2}} \Delta x_{j-\frac{3}{2}}}{\delta_{j-\frac{1}{2}}^+ v_{j-\frac{1}{2}}^+ \Delta u_{j-\frac{1}{2}} \Delta x_{j-\frac{1}{2}}} & v_{j-\frac{1}{2}} > 0 \\ 0 & v_{j-\frac{1}{2}} \leq 0 \end{cases} \quad (A10)$$

and

$$M_{j-\frac{1}{2}}^- = \begin{cases} 0 & v_{j-\frac{1}{2}} \geq 0 \\ M_{j-\frac{1}{2}} = \frac{\tau_{j+\frac{1}{2}}^- v_{j+\frac{1}{2}}^- \Delta u_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}{\delta_{j-\frac{1}{2}}^- v_{j-\frac{1}{2}}^- \Delta u_{j-\frac{1}{2}} \Delta x_{j-\frac{1}{2}}} & v_{j-\frac{1}{2}} < 0 \end{cases} \quad (A11)$$

(N.B. $r_{j-\frac{1}{2}}^\pm = \frac{\Delta x_{j-\frac{1}{2} \pm 1}}{\Delta x_{j-\frac{1}{2}}}$) .

Thus

$$v_{j-\frac{1}{2}}^+ \geq 0$$

$$v_{j-\frac{1}{2}}^- \leq 0$$

and as a result of the restrictions given by equation (A1) we find that

$$\delta_{j-\frac{1}{2}}^{\pm} \geq 0$$

and

$$\tau_{j-\frac{1}{2}}^{\pm} \geq 0 .$$

Using the definitions given by equations (A3)-(A11) we can rewrite equation (A2) as

$$\begin{aligned} u^j = & u_j - \frac{v_{j-\frac{1}{2}}^+ \Delta x_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}}}{\Delta x_j} - \frac{v_{j+\frac{1}{2}}^- \Delta x_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}}}{\Delta x_j} \\ & - \frac{\Delta_-}{\Delta x_j} \left(\delta_{j+\frac{1}{2}}^+ v_{j+\frac{1}{2}}^+ \Delta u_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}} \psi(M_{j+\frac{1}{2}}^+) \right. \\ & \left. - \delta_{j+\frac{1}{2}}^- v_{j+\frac{1}{2}}^- \Delta u_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}} \psi(M_{j+\frac{1}{2}}^-) \right) \end{aligned} \quad (A12)$$

If we now use equations (A10) and (A11) to replace the terms

$$\delta_{j\pm\frac{1}{2}}^{\pm} v_{j\pm\frac{1}{2}}^{\pm} \Delta u_{j\pm\frac{1}{2}} \Delta x_{j\pm\frac{1}{2}} \quad \text{in equation (A12) by } \tau_{j+\frac{1}{2}}^{\pm} v_{j+\frac{1}{2}}^{\pm} \Delta u_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}} / M_{j\pm\frac{1}{2}}^{\pm}$$

we obtain

$$u^j = u_j - C_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}} + D_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} \quad (A13)$$

where

$$\begin{aligned} C_{j-\frac{1}{2}} = & \frac{\Delta x_{j-\frac{1}{2}} v_{j-\frac{1}{2}}^+}{\Delta x_j} \left(1 + \frac{\psi(M_{j+\frac{1}{2}}^+)}{M_{j+\frac{1}{2}}^+} \tau_{j-\frac{1}{2}}^+ \right. \\ & \left. - \delta_{j-\frac{1}{2}}^+ \psi(M_{j-\frac{1}{2}}^+) \right) \end{aligned}$$

and

$$\begin{aligned} D_{j+\frac{1}{2}} = & - \frac{\Delta x_{j+\frac{1}{2}} v_{j+\frac{1}{2}}^-}{\Delta x_j} \left(1 + \frac{\psi(M_{j-\frac{1}{2}}^-)}{M_{j-\frac{1}{2}}^-} \tau_{j+\frac{1}{2}}^- \right. \\ & \left. - \delta_{j+\frac{1}{2}}^- \psi(M_{j+\frac{1}{2}}^-) \right) \end{aligned}$$

Now

$$\begin{aligned}
 C_{j-\frac{1}{2}} &\geq \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} v_{j-\frac{1}{2}}^+ \left(1 - \frac{1}{2}(1 - v_{j-\frac{1}{2}}^+) \psi(M_{j-\frac{1}{2}}^+) \right) \\
 &\geq \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} v_{j-\frac{1}{2}}^+ \left(1 - \frac{1}{2} \psi(M_{j-\frac{1}{2}}^+) \right) \\
 &\geq \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} v_{j-\frac{1}{2}}^+ (1 - \frac{1}{2} \cdot 2) \geq 0
 \end{aligned} \tag{A14}$$

since $v_{j-\frac{1}{2}}^+ \geq 0$, $\psi(M_{j+\frac{1}{2}}^+)/M_{j+\frac{1}{2}}^+ \geq 0$, $\tau_{j-\frac{1}{2}}^+ \geq 0$ and $2 \geq \psi(M_{j-\frac{1}{2}}^+) \geq 0$.

Similarly

$$\begin{aligned}
 D_{j+\frac{1}{2}} &\geq - \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} v_{j+\frac{1}{2}}^- \left(1 - \frac{1}{2}(1 + v_{j+\frac{1}{2}}^-) \psi(M_{j+\frac{1}{2}}^-) \right) \\
 &\geq - \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} v_{j+\frac{1}{2}}^- \left(1 - \frac{1}{2} \psi(M_{j+\frac{1}{2}}^-) \right) \\
 &= - \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} v_{j+\frac{1}{2}}^- (1 - \frac{1}{2} \cdot 2) \geq 0
 \end{aligned} \tag{A15}$$

since $v_{j+\frac{1}{2}}^- \leq 0$, $\psi(M_{j-\frac{1}{2}}^-)/M_{j-\frac{1}{2}}^- \geq 0$, $\tau_{j+\frac{1}{2}}^- \geq 0$ and $2 \geq \psi(M_{j+\frac{1}{2}}^-) \geq 0$.

Also

$$\begin{aligned}
 C_{j+\frac{1}{2}} &\leq \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}} v_{j+\frac{1}{2}}^+ \left(1 + \frac{\psi(M_{j+\frac{3}{2}}^+)}{M_{j+\frac{3}{2}}^+} \cdot \frac{1}{2}(r_{j+\frac{1}{2}}^+ - v_{j+\frac{1}{2}}^+) \right) \\
 &\leq \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}} v_{j+\frac{1}{2}}^+ (1 + 2 \cdot \frac{1}{2} r_{j+\frac{1}{2}}^+) \\
 &= \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}} v_{j+\frac{1}{2}}^+ \left(\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} + \frac{\Delta x_{j+\frac{3}{2}}}{\Delta x_{j+\frac{1}{2}}} \right) \\
 &= \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j+1}} v_{j+\frac{1}{2}}^+ \cdot \frac{2\Delta x_{j+1}}{\Delta x_{j+\frac{1}{2}}} = 2v_{j+\frac{1}{2}}^+
 \end{aligned}$$

since $v_{j+\frac{1}{2}}^+ \geq 0$, $\psi(M_{j+\frac{1}{2}}^+) \geq 0$, $\delta_{j+\frac{1}{2}}^+ \geq 0$ and $2 \geq \psi(M_{j+\frac{3}{2}}^+)/M_{j+\frac{3}{2}}^+ \geq 0$.

Similarly

$$\begin{aligned}
 D_{j+\frac{1}{2}} &\leq -\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} v_{j+\frac{1}{2}}^- \left(1 + \frac{\psi(M_{j-\frac{1}{2}}^-)}{M_{j-\frac{1}{2}}^-} \cdot \frac{1}{2} (r_{j+\frac{1}{2}}^- + v_{j+\frac{1}{2}}^-) \right) \\
 &\leq -\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} v_{j+\frac{1}{2}}^- (1 + 2 \cdot \frac{1}{2} r_{j+\frac{1}{2}}^-) \\
 &= -\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} v_{j+\frac{1}{2}}^- \left(\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} + \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} \right) \\
 &= -\frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} v_{j+\frac{1}{2}}^- \cdot \frac{2\Delta x_j}{\Delta x_{j+\frac{1}{2}}} = -2v_{j+\frac{1}{2}}^-
 \end{aligned}$$

since $v_{j+\frac{1}{2}}^- \leq 0$, $\psi(M_{j+\frac{1}{2}}^-) \geq 0$, $\delta_{j+\frac{1}{2}}^- \geq 0$ and $\psi(M_{j+\frac{1}{2}}^-) \geq 0$.

Thus

$$0 \leq C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 2(v_{j+\frac{1}{2}}^+ - v_{j+\frac{1}{2}}^-)$$

Now, either $v_{j+\frac{1}{2}} \geq 0$ so that

$$v_{j+\frac{1}{2}}^+ = v_{j+\frac{1}{2}} \leq \frac{1}{2} \text{ and } v_{j+\frac{1}{2}}^- = 0$$

giving

$$0 \leq C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1 \tag{A16}$$

or $v_{j+\frac{1}{2}} \leq 0$ so that

$$v_{j+\frac{1}{2}}^- = v_{j+\frac{1}{2}} \geq -\frac{1}{2} \text{ and } v_{j+\frac{1}{2}}^+ = 0$$

giving

$$0 \leq C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1 \tag{A17}$$

It follows from equations (2.6)-(2.7) (A13)-(A17) that the scheme given by equations (4.10a-h) for the solution of equation (4.9) is TVD if

$$-\min(r_{j-\frac{1}{2}}^-, \frac{1}{2}) \leq v_{j-\frac{1}{2}} \leq \min(r_{j-\frac{1}{2}}^+, \frac{1}{2})$$

for all j .