ROBUSTNESS IN PARTIAL POLE PLACEMENT

N.K. Nichols

Department of Mathematics
University of Reading
P.O. Box 220, Reading RG6 2AX, U.K.

ABSTRACT

The robustness of state feedback solutions to the problem of partial pole placement obtained by a new projection procedure is examined. The projection procedure gives a reduced order pole assignment problem. It is shown that the sensitivities of the assigned poles in the complete closed loop system are bounded in terms of the sensitivities of the assigned reduced-order poles, and the sensitivities of the unaltered poles are bounded in terms of the sensitivities of the corresponding open loop poles. If the assigned poles are well-separated from the unaltered poles, these bounds are expected to be tight. The projection procedure is described in [3], and techniques for finding robust (or insensitive) solutions to the reduced order problem are given in [1] [2].

1. Introduction

A projection procedure for solving the problem of partial pole placement has recently been proposed by Saad [3]. The problem is stated as follows:

Given system pair (A,B), $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, where A has eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_n\}$, find matrix $F \in \mathbb{R}^{m \times n}$ such that $A+BF$ has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_{k+1}, \ldots, \mu_n\}$. In other words, find a feedback which reassigns the eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$ to be $\lambda_1, \lambda_2, \ldots, \lambda_k$, while leaving the rest of the spectrum of A unchanged. We remark that the sets $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ and $\{\mu_{k+1}, \ldots, \mu_n\}$ must be closed with respect to complex conjugation and are assumed to be disjoint. It is also assumed that if a multiple eigenvalue belongs to the set $\{\mu_1, \mu_2, \ldots, \mu_k\}$, then it is represented several times in the set according to its algebraic multiplicity.

The procedure of [3] determines the left invariant subspace of the eigenvalues which are to be reassigned and projects the initial problem into that subspace in order to obtain a reduced order pole assignment problem. The partial pole placement problem could, of course, be solved using any full eigenvalue assignment technique which allows the open loop poles to be re-assigned to the closed loop system. For very large (sparse) systems, however, where there are only

a few poles to be stabilized, the advantage of the projection procedure is that the computational work required to solve the problem is very significantly reduced.

A projection procedure similar to that of [3] is proposed in [6]. The method of [6] uses the matrix of eigenvectors associated with the poles which are to be replaced in order to carry out the projection. This procedure is unstable numerically and <u>can</u> give rise to very large computational errors. The method of [3], on the other hand, uses an orthogonal basis for the left invariant subspaces and the computations are numerically stable. The accuracy of the solution may still be affected, however, by the conditioning, or sensitivity, of the open loop poles which are replaced. (See discussion in [3]).

The reduced problem obtained by the projection procedures can be solved by any standard pole assignment technique. For a <u>robust</u> solution to the over-all problem, it is necessary that the assigned poles be insensitive to perturbations in the system matrices (A,B) and the gain matrix F. For the single-input case the solution to the pole assignment problem is unique and the sensitivity cannot be controlled. For the multi-input case, however, there are a number of degrees of freedom, and it is important to select these so as to guarantee the <u>robustness</u> of the solution. Procedures which determine robust solutions to the standard pole assignment problem by assigning the eigenvectors associated with the specified poles are derived and discussed in [1] [2]. These techniques can also be applied to the reduced pole assignment problem obtained by the partial projection procedure.

In this note we show that robustness of the over-all problem can be achieved by the partial procedure - more specifically that a bound on the sensitivity of the assigned poles can be minimized and that the sensitivity of the unaltered poles remains reasonably bounded under certain conditions. In the next section we describe briefly the projection procedure and introduce notation. In the following section we examine the robustness properties and in the final section we present a numerical example.

2. The Procedure

We let Q_1 represent an orthonormal basis for the left invariant subspace of A associated with the eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$, and assume that

$$Q_1^H A = R_1 Q_1^H, \tag{1}$$

where Q_1 is nxk such that $Q_1^H Q_1 = I$, and R_1 is a kxk lower triangular matrix with diagonal entries equal to the eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_k\}$. (We remark that in practice the _real_ form of the partial Schur decomposition (1) is used, where Q_1 is _real_ with orthonormal columns, and R_1 is a _real_ quasi-triangular matrix with 1x1 or 2x2 block diagonal entries equal to the real eigenvalues, or the _real_ representation of complex conjugate pairs.) A solution to the pole assignment problem in the form

$$F = \tilde{F}_1 Q_1^H \tag{2}$$

is sought. Applying the transformation from the left to the full closed loop system matrix and denoting $\tilde{B}_1 = Q_1^H B$, we obtain

$$Q_1^H (A + BF) \equiv (R_1 + \tilde{B}_1 \tilde{F}_1) Q_1^H. \tag{3}$$

If now $\tilde{F}_1$ is chosen to assign the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ to the reduced closed loop system matrix $R_1 + \tilde{B}_1 \tilde{F}_1$, then F given by (2) solves the partial pole assignment problem, as shown in [3]. Techniques for finding the partial Schur decomposition (1) are also shown in [3] and in [4].

We remark that if the original system pair is completely controllable, then the reduced system pair $(R_1, \tilde{B}_1)$ is also controllable; more precisely, if the poles $\mu_1, \ldots, \mu_k$ of A are controllable, then the pair $(R_1, \tilde{B}_1)$ is a completely controllable pair. This can be seen as follows: The pair $(R_1, \tilde{B}_1)$ is completely controllable provided,

$$\{v^T R_1 = v^T \mu \quad \text{and} \quad v^T \tilde{B}_1 = 0 \implies v^T = 0\}. \tag{4}$$

Now $v^T R_1 = v^T \mu \implies y^T A = \mu y^T$, where $y^T = v^T Q_1^H$, and $v^T \tilde{B}_1 = 0 \implies y^T B = 0$; therefore, if μ is a controllable pole of A, then, by definition, $y^T \equiv v^T Q_1^H = 0$. Since Q_1^H is of full row rank, we have also $v^T = 0$, and hence controllability of $(R_1, \tilde{B}_1)$.

3. <u>Robustness Properties</u>

The sensitivity of an eigenvalue λ_j of a (non-defective) matrix M to arbitrary perturbations in M is well-known [5] to be proportional to the condition number

$$c_j = \|\underline{y}_j\| \, \|\underline{x}_j\| / |\underline{y}_j^T \underline{x}_j|, \tag{5}$$

where $\underline{x}_j$, $\underline{y}_j$ are the associated right- and left-eigenvectors of M satisfying

$$M\underline{x}_j = \lambda_j \underline{x}_j, \qquad \underline{y}_j^T M = \lambda_j \underline{y}_j^T . \tag{6}$$

and $\|\cdot\|$ denotes the ℓ_2-norm. For a robust solution to the over-all pole assignment problem, then, we choose the feedback F such that the condition numbers of the assigned poles of the closed loop system matrix $M \equiv A + BF$ are small. (See [1] [2]).

For the reduced 'partial' feedback problem we may choose $\tilde{F}_1$ such that the assigned poles $\{\lambda_1, \lambda_2, \ldots \lambda_k\}$ of the closed loop matrix $\tilde{R}_1 + \tilde{B}_1\tilde{F}_1$ have small condition numbers. We show now that the assigned poles of the full feedback problem are also then insensitive, and furthermore, that the sensitivity of the unchanged poles is not greatly altered.

We let Q_2 denote an orthonormal basis for the left invariant subspace of A corresponding to eigenvalues $\{\mu_{k+1}, \ldots \mu_n\}$, and let Q denote the unitary matrix $Q = [Q_1, Q_2]$. We assume that

$$Q^H A = \begin{bmatrix} R_1 & O \\ R_2 & R_3 \end{bmatrix} Q^H, \tag{7}$$

where R_3 is $(n-k) \times (n-k)$ lower triangular, and denote $Q_2^H B = \tilde{B}_2$. It follows that

$$Q^H M \equiv Q^H(A + BF) = \begin{bmatrix} R_1 + \tilde{B}_1\tilde{F}_1 & O \\ R_2 + \tilde{B}_2\tilde{F}_1 & R_3 \end{bmatrix} Q^H \equiv \tilde{M} Q^H, \tag{8}$$

where $F = \tilde{F}_1 Q_1^H = [\tilde{F}_1, \ O]Q^H$.

We now denote the right- and left-eigenvectors of the transformed closed loop system matrix $\tilde{M}$, respectively, by

$$W = \begin{bmatrix} W_1 & 0 \\ W_2 & W_3 \end{bmatrix}, \qquad V^T = \begin{bmatrix} V_1^T & 0 \\ V_2^T & V_3^T \end{bmatrix}. \tag{9}$$

Then

$$\tilde{M}W = W\Lambda, \qquad V^T\tilde{M} = \Lambda V^T. \tag{10}$$

implies that

$$(\tilde{R}_1 + \tilde{B}_1\tilde{F}_1)W_1 = W_1\Lambda_1, \qquad V_1^T(\tilde{R}_1 + \tilde{B}_1\tilde{F}_1) = \Lambda_1 V_1^T, \tag{11}$$

and

$$R_3W_3 = W_3\Lambda_2, \qquad V_3^T R_3 = \Lambda_2 V_3^T, \tag{12}$$

where

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \qquad \Lambda_1 = \text{diag}\{\lambda_1,\ldots,\lambda_k\}, \quad \Lambda_2 = \text{diag}\{\mu_{k+1},\ldots,\mu_n\}. \tag{13}$$

Hence, W_1 and V_1^T are the assigned right- and left-eigenvectors associated with the assigned poles $\{\lambda_1,\lambda_2,\ldots\lambda_k\}$ of the reduced pole placement problem, and W_3, V_3^T are the right- and left-eigenvectors of the transformed open loop system matrix R_3, which is unchanged by the feedback.

If we also denote the right- and left-eigenvectors of the complete closed loop system matrix $M \equiv A + BF$, respectively, by

$$X = [X_1, X_2], \quad Y^T = \begin{bmatrix} Y_1^T \\ Y_2^T \end{bmatrix},$$

where

$$MX = X\Lambda, \qquad Y^TM = \Lambda Y^T, \tag{14}$$

and Λ is defined by (13), then we may write, from (10) and (14)

$$\tilde{M}Q^H QW \equiv Q^H MQW = W\Lambda,$$

$$\tag{15}$$

$$V^T\tilde{M}Q^H \equiv V^TQ^HM = \Lambda V^TQ^H.$$

Premultiplying the first of (15) by Q, we deduce that

$$X \equiv [X_1, X_2] = QW \equiv [Q_1W_1 + Q_2W_2, Q_2W_3],$$

(16)

$$Y^T \equiv \begin{bmatrix} Y_1^T \\ Y_2^T \end{bmatrix} = V^T Q^H \equiv \begin{bmatrix} V_1^T Q_1^H \\ V_2^T Q_1^H + V_3^T Q_2^H \end{bmatrix}.$$

It follows that the condition numbers of the assigned poles $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ are given by

$$c_j = \|\underline{e}_j^T Y_1^T\| \ \|X_1 \underline{e}_j^T\| / |\underline{e}_j^T Y_1^T X_1 \underline{e}_j|$$

(17)

$$= \|\underline{e}_j^T V_1^T\| \ \| \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \underline{e}_j \| / |\underline{e}_j^T V_1^T W_1 \underline{e}_j|, \qquad j = 1, 2, \ldots k.$$

Here $\underline{e}_j$ denotes the jth unit vector. The condition number is invariant under scaling of the eigenvectors, and we may assume that the right- and left-eigenvectors are scaled to unit length, such that $\| \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \underline{e}_j \| = 1, \qquad \|\underline{e}_j^T V_1^T\| = 1.$ The condition numbers are then given precisely by

$$c_j = 1 / |\underline{e}_j^T V_1^T W_1 \underline{e}_j|, \qquad j = 1, 2, \ldots, k$$

(18)

Similarly, the condition numbers of the unaltered eigenvalues $\{\mu_{k+1}, \ldots, \mu_n\}$ are given by

$$c_j = \|\underline{e}_s^T Y_2^T\| \ \|X_2 \underline{e}_s\| / |\underline{e}_s^T Y_2^T X_2 \underline{e}_s|$$

(19)

$$= \|\underline{e}_s^T [V_2^T, V_3^T]\| \ \|W_3 \underline{e}_s\| / |\underline{e}_s^T V_3^T W_3 \underline{e}_s|, \quad s = j-k, \ j = k+1, \ldots n .$$

Assuming again that the eigenvectors are scaled to unit length, such that $\|W_3 \underline{e}_s\| = 1$ and $\|\underline{e}_s^T [V_2^T, V_3^T]\| = 1$, then

$$c_j = 1 / |\underline{e}_s^T V_3^T W_3 \underline{e}_s|, \ s = j-k, \quad j = k+1, \ldots, n.$$

(20)

We remark that with a full feedback of the form $F = [\tilde{F}_1, \tilde{F}_2] Q^H$, it is possible to assign all the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \mu_{k+1}, \ldots, \mu_n\}$ such that an over-all measure of all the condition numbers c_j, $j = 1, 2, \ldots, n$ is minimized. This requires the solution of the __full__ feedback problem, however, which for large systems may not be practicable. Instead, we wish to solve the 'partial' pole placement problem in such a way that the condition numbers (18) of the assigned

poles λ_j, $j = 1,2,\ldots,k$, are as close to unity as possible and the condition

numbers (20) of the remaining poles μ_j, $j = k+1,\ldots,n$ are not much worse than in

the open loop system.

In practice, to minimize the condition numbers c_j, $j = 1,2,\ldots,k$, given by

(18) we require information about the matrix W_2, and, in particular, about the

matrix R_2, which is not available from the partial Schur decomposition (3). We

therefore aim, instead, to find a feedback solution which minimizes the

sensitivities of the assigned poles of the reduced problem. We now examine the

effect of such a feedback choice on the robustness of the complete closed loop

system.

For robust pole placement in the reduced 'partial' problem we select $\tilde{F}_1$ to

assign eigenvalues Λ_1 and right- and left-eigenvectors W_1 and V_1^T satisfying (11),

such that the condition numbers

$$\tilde{c}_j = \|\underline{e}_j^T V_1^T\| \; \|W_1 \underline{e}_j\| / |\underline{e}_j^T V_1^T W_1 \underline{e}_j|, \quad j = 1,2,\ldots,k. \tag{21}$$

are small. (In essence we select the eigenvectors to be as close to an

orthonormal set as possible.) We emphasize here that the condition number $\tilde{c}_j$, given

by (21), gives the sensitivity of the eigenvalue λ_j with respect to the reduced problem,

and the condition number c_j, given by (18), gives the sensitivity of the same eigenvalue

with respect to the entire system. From (18) and (21) we find that

$$\tilde{c}_j = \|W_1 \underline{e}_j\| \; c_j \leq c_j \tag{22}$$

Furthermore, from (8)-(10) we find that

$$W_2 \Lambda_1 - R_3 W_2 = (R_2 + \tilde{B}_2 \tilde{F}_1) W_1, \tag{23}$$

and by the scaling assumption we have

$$1 = \|W_1 \underline{e}_j\|^2 + \|(\lambda_j I - R_3)^{-1}(R_2 + \tilde{B}_2 \tilde{F}_1) W_1 \underline{e}_j\|^2$$

$$\leq \|W_1 \underline{e}_j\|^2 (1 + \omega_j^2), \tag{24}$$

where $\omega_j = \|(\lambda_j I - R_3)^{-1}(R_2 + \tilde{B}_2 \tilde{F}_1)\|$. We find then that

$$\tilde{c}_j \geq \frac{1}{\sqrt{1 + \omega_j^2}} \; c_j. \tag{25}$$

From (22) and (25) it can be seen that the condition measure $\tilde{c}_j$ is, in a mathematical sense, equivalent to the condition number c_j; that is, the measures c_j and $\tilde{c}_j$ are bounded in terms of each other.

In practice, an <u>over-all</u> robustness measure of the system, such as

$$\tilde{\nu} \equiv (\sum_{j=1}^{k} \tilde{d}_j^2 \tilde{c}_j^2)^{1/2}, \tag{26}$$

where $\tilde{d}_j$ are given weights, is optimized [2]. From the inequalities (22) and (25) we may deduce that for each $j = 1, 2, \ldots, k$, there exists a constant $\tilde{\omega}_j$ such that $\tilde{\omega}_j c_j = \tilde{c}_j$, where $(1 + \omega_j^2)^{-1/2} \leq \tilde{\omega}_j \leq 1$. Hence we have

$$\tilde{\nu} = (\sum_{j=1}^{k} d_j^2 c_j^2)^{1/2} \tag{27}$$

where

$$(1 + \omega_j^2)^{-1/2} \tilde{d}_j \leq d_j \equiv \tilde{\omega}_j \tilde{d}_j \leq \tilde{d}_j, \tag{28}$$

and optimizing the robustness of the 'partial' pole placement solution thus corresponds to minimizing exactly a <u>particular</u> weighted sum of the squares of the condition numbers of the assigned poles of the complete closed loop system. Moreover, since

$$\tilde{\nu} \geq (\sum_{j=1}^{k} \tilde{d}_j^2 c_j^2)^{1/2} / (1 + \omega_j^2)^{1/2}. \tag{29}$$

it follows that minimizing $\tilde{\nu}$, also minimizes an upper bound on the weighted sum of the squares of the assigned pole sensitivities for the weights $\tilde{d}_j / (1 + \omega_j^2)^{1/2}$.

We now show that the sensitivities of the remaining, unaltered poles of the closed loop system can be bounded in terms of their condition numbers in the original open loop system.

If we denote by $[U_2^T, V_3^T]$ the left-eigenvectors corresponding to the poles μ_j, $j = k+1, \ldots, n$, of the transformed open loop system matrix $Q^H A Q$, given by (7), then, using similar arguments to those for the closed loop system, we find that the open loop

condition numbers of the poles $\{\mu_{k+1}, \mu_{k+2}, \ldots, \mu_n\}$ are given by

$$c_j^{\circ} = \|\underline{e}_s^T[U_2^T, V_3^T]\| \; \|W_3\underline{e}_s\|/|\underline{e}_s^T V_3^T W_3 \underline{e}_s|,$$

$$= \frac{\|\underline{e}_{-s}^T[U_2^T, V_3^T]\|}{\|\underline{e}_{-s}^T[V_2^T, V_3^T]\|} \, c_j, \qquad s = j-k, \; j = k+1, \ldots, n. \tag{30}$$

From (10) we have that

$$\Lambda_2 V_2^T - V_2^T(R_1 + \tilde{B}_1 \tilde{F}_1) = V_3^T(R_2 + \tilde{B}_2 \tilde{F}_1) \tag{31}$$

and it can be shown, similarly, that

$$\Lambda_2 U_2^T - U_2^T R_1 = V_3^T R_2 \tag{32}$$

We have, therefore, that

$$\|\underline{e}_s^T V_2^T\| \leq \|\underline{e}_s^T V_3^T\| \gamma_j, \qquad \|\underline{e}_s^T U_2^T\| \leq \|\underline{e}_s^T V_3^T\| \delta_j, \qquad s = j-k, \tag{33}$$

where

$$\gamma_j = \|(R_2 + \tilde{B}_2 \tilde{F}_1)(\mu_j I - (R_1 + \tilde{B}_1 \tilde{F}_1))^{-1}\|,$$

$$\tag{34}$$

$$\delta_j = \|R_2(\mu_j I - R_1)^{-1}\|, \qquad j = k+1, \ldots, n.$$

Now, using the assumption that the closed loop eigenvectors are normalized to unit length, such that $\|\underline{e}_s^T[V_2^T, V_3^T]\| = 1$, we also have

$$1 = \|\underline{e}_s^T V_3^T\|^2 + \|\underline{e}_s^T V_2^T\|^2 \leq \|\underline{e}_s^T V_3^T\|^2 (1 + \gamma_j^2). \tag{35}$$

It follows, then, from (33) and (35) that

$$c_j^{\circ} \leq (\|\underline{e}_s^T U_2^T\|^2 + \|\underline{e}_s^T V_3^T\|^2)^{1/2} c_j \leq (1 + \delta_j^2)^{1/2} c_j \tag{36}$$

and

$$c_j^{\circ} \geq \|\underline{e}_s^T V_3^T\| \, c_j \geq \frac{1}{(1 + \gamma_j^2)^{1/2}} \, c_j. \tag{37}$$

Hence the closed loop condition numbers c_j of the unaltered poles $\{\mu_{k+1}, \ldots \mu_n\}$ are bounded in terms of the open loop condition numbers.

We summarize these results in the following:

THEOREM The condition-numbers, or sensitivities, c_j of the poles $\{\lambda_1,\ldots,\lambda_k,\mu_{k+1},\ldots,\mu_n\}$ of the closed loop system matrix $A + BF$ obtained by the partial pole placement projection algorithm are bounded by

$$\tilde{c}_j \leq c_j \leq (1 + \omega_j^2)^{1/2}\,\tilde{c}_j, \qquad\qquad j = 1,2,\ldots,k \qquad\qquad (38)$$

and

$$\frac{1}{(1 + \delta_j^2)^{1/2}}\,\overset{\circ}{c}_j \leq c_j \leq (1 + \gamma_j^2)^{1/2}\,\overset{\circ}{c}_j, \qquad j = k+1,\ldots,n, \qquad\qquad (39)$$

where $\tilde{c}_j$ are the condition numbers of the assigned poles $\{\lambda_1,\ldots,\lambda_k\}$ of the reduced closed loop system matrix $R_1 + \tilde{B}_1\tilde{F}_1$, and $\overset{\circ}{c}_j$ are the condition numbers of the poles $\{\mu_{k+1},\ldots,\mu_n\}$ of the open loop system matrix A, and

$$\omega_j = \|(\lambda_j I - R_3)^{-1}(R_2 + \tilde{B}_2\tilde{F}_1)\|,$$

$$\gamma_j = \|(R_2 + \tilde{B}_2\tilde{F}_1)(\mu_j I - (R_1 + \tilde{B}_1\tilde{F}_1))^{-1}\|,$$

$$\delta_j = \|R_2(\mu_j I - R_1)^{-1}\|. \qquad\qquad\qquad (40)$$

□

We remark that the parameters ω_j, γ_j, δ_j exist provided the matrices $(\lambda_j I - R_3)$, $j = 1,2,\ldots,k$, and $(\mu_j I - (R_1 + \tilde{B}_1\tilde{F}_1))$ and $(\mu_j I - R_1)$, $j = k+1,\ldots,n$, are invertible, and the order of magnitude of the parameters depends essentially on the conditioning of these matrices. The constants γ_j exist, therefore, by virtue of the assumption that the assigned eigenvalues $\{\lambda_1,\ldots,\lambda_k\}$ of $R_1 + \tilde{B}_1\tilde{F}_1$ and the fixed eigenvalues $\{\mu_{k+1},\ldots,\mu_n\}$ of Λ_2 are disjoint. Similarly, the constants γ_j exist, since the eigenvalues of Λ_1 and those of R_3 are disjoint by the same assumption. The constants δ_j exist provided the sets $\{\mu_1,\ldots,\mu_k\}$ and $\{\mu_{k+1},\ldots,\mu_n\}$ are also disjoint, which holds by the initial assumption that repeated eigenvalues are not contained in different sets.

From these remarks it follows that the order of magnitude of the parameters ω_j and γ_j depends on the separation measure

$$\left(\begin{array}{c} \min \\ 1 \leq i \leq k \\ k+1 \leq j \leq n \end{array} |\lambda_i - \mu_j|\right)^{-1}. \qquad\qquad (41)$$

If this value is $0(1)$, or smaller, then the condition numbers $\tilde{c}_j$ of the reduced problem may be expected to give a tight bound on the condition numbers c_j, $j = 1,2,\ldots,k$, of the assigned poles in the complete closed loop system; furthermore, the condition numbrs c_j, $j = k+1,\ldots,n$ of the unaltered poles may be expected to be very little

worse than their open loop condition numbers. In selecting the dimension of the reduced problem, it is therefore advisable to ensure that the unassigned eigenvalues are well separated from the poles to be assigned, so that the separation measure (41) is small.

The magnitude of the gain matrix $\tilde{F}_1$ is expected to depend on the distance between the sets $\{\mu_1, \ldots, \mu_k\}$ and $\{\lambda_1, \ldots, \lambda_k\}$ and on the sensitivity of the assigned closed loop poles. In fact the feedback $\tilde{F}_1$ is given explicitly by

$$\tilde{F}_1 = \tilde{B}_1^+ V_1^{-T}(\Lambda_1 V_1^T - V_1^T R_1), \tag{42}$$

where + denotes the Moore-Penrose pseudo-inverse and V_1^T is defined as previously. (We remark that V_1^T is taken to be of full rank n, since otherwise the closed loop matrix of the reduced system would be <u>defective</u> and the conditioning of some of the poles would be infinite.) From (42) we obtain

$$\|\tilde{F}_1\| \leq \sigma_{min}^{-1}\{\tilde{B}_1\} \| V_1^{-T} \|_F \| \Lambda V_1^T - V_1^T R_1 \|_F$$

$$= \sigma_{min}^{-1}\{\tilde{B}_1\} (\sum_{i=0}^{k} \tilde{c}_i^2)^{\frac{1}{2}} (\sum_{i=0}^{k} \| \underline{e}_i^T V_1^T (\lambda_i I - R_1) \|^2)^{\frac{1}{2}},$$

where $\| \cdot \|_F$ denotes the Frobenius norm and $\sigma_{min}\{\cdot\}$ denotes minimum singular value. The parameters ω_j, γ_j then depend implicitly on the conditioning numbers $\tilde{c}_i$, and on the separation between the assigned poles λ_i, i = 1, 2, ..., k and the eigenvalues μ_j, j = 1, 2, ..., k, of R_1, as well as upon the separation measure (41). Small constants ω_j, γ_j and a small separation measure thus imply a good separation between $\{\mu_1, \ldots, \mu_k\}$ and $\{\mu_{k+1}, \ldots, \mu_n\}$. The reduced system should, therefore, also be selected such that its poles are well-separated from the unassigned poles in the complete system. To ensure good separation between the set $\{\mu_{k+1}, \ldots, \mu_n\}$ and both sets $\{\lambda_1, \ldots, \lambda_k\}$ <u>and</u> $\{\mu_1, \ldots, \mu_k\}$ it may be advisable to include in the set to be reassigned any open loop poles near the sites of the specified closed loop poles. By this technique, the conditioning of all such poles could also be controlled.

We observe that it is not necessary to assume the disjointness of the sets $\{\lambda_1, \ldots, \lambda_k\}$ and $\{\mu_{k+1}, \ldots, \mu_n\}$ in order to <u>solve</u> the partial pole placement problem. If this assumption is <u>not</u> made, however, it is possible for the complete closed loop system matrix A + BF obtained to be defective, in which case the sensitivity of some

eigenvalue of the system must be an order of magnitude worse - that is, perturbations of $0(\varepsilon)$ in $A + BF$ must cause perturbations larger by at least an order of magnitude in some eigenvalue. In practice, therefore, we make the disjointness assumption to ensure robustness.

The principal advantage of the projection procedure arises for very large sparse systems. In these cases the necessary information to determine the constants ω_j, γ_j, δ_j, is not available from the partial Schur decomposition (1), but it is expected that good estimates are attainable and further research on this problem is in progress. In the next section we examine a small numerical example to illustrate the conclusions of the theorem, and compare the results of the robust partial pole placement with robust pole placement for the complete problem.

4. Numerical Example

We consider a model of an unstable chemical reactor with $n = 4$ states and $m = 2$ controls and system matrices

$$A = \begin{bmatrix} 1.380 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.290 & 0 & 0.6750 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.0480 & 4.273 & 1.343 & -2.104 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & 5.679 & 1.136 & 1.136 \\ 0 & 0 & -3.146 & 0 \end{bmatrix}$$

The open loop eigenvalues $\{\mu_j\}$ are given by the set $\{1.991, 0.06351, -5.057, -8.666\}$ with condition numbers $\{c_j^\circ\}$ given by $\{1.631, 1.304, 1.506, 1.451\}$.

The partial pole assignment problem is to move the unstable poles μ_1, μ_2 into stable positions given by $\{\lambda_1, \lambda_2\} = \{-0.2, -0.5\}$. Applying the partial Schur decomposition we obtain (to four figures accuracy) the reduced system pair

$$R_1 = \begin{bmatrix} 1.991 & 0 \\ -0.5446 & 0.06351 \end{bmatrix}$$

$$\tilde{B}_1 = \begin{bmatrix} -1.560 & 1.771 \\ -5.117 & 0.2882 \end{bmatrix}$$

Since the matrix $\tilde{B}_1$ is square and invertible, the poles $\{\lambda_1, \lambda_2\}$ can be assigned to the reduced problem with perfect conditioning $\tilde{c}_1 = \tilde{c}_2 = 1$. The feedback is obtained from $\tilde{F}_1 = \tilde{B}_1^{-1}(\Lambda_1 - R_1)$ and is given (to four figures) by

$$\tilde{F}_1 = \begin{bmatrix} -0.1853 & 0.1159 \\ -1.401 & 0.1021 \end{bmatrix} .$$

The feedback for the complete system is constructed from (2) as

$$F = \begin{bmatrix} 0.1522 & -0.04904 & 0.09370 & -0.1159 \\ 1.0856 & 0.2124 & 0.7790 & -0.3767 \end{bmatrix} .$$

The condition numbers of the assigned poles λ_1, λ_2 in the complete closed loop system $A + BF$ are now given by $c_1 = 1.515$ and $c_2 = 1.735$. The condition numbers of the unaltered poles in the closed loop system are given by $c_3 = 1.5698$ and $c_4 = 1.795$, which compare well with the original open loop condition numbers. For this example we compute the constants ω_j, γ_j, and δ_j to be

$$\omega_1 = 1.558, \quad \gamma_3 = 2.580, \quad \delta_3 = 1.638 ,$$
$$\omega_2 = 1.627, \quad \gamma_4 = 1.463, \quad \delta_4 = 0.9945,$$

and it can be seen that the inequalities of the Theorem are satisfied by the condition numbers c_j, $j = 1, 2, \ldots, 4$.

We may also compare the partial pole placement results with those of a full robust pole assignment for the problem, where the poles $\lambda_1 = -0.2$, $\lambda_2 = -0.5$, $\lambda_3 = \mu_3$ and $\lambda_4 = \mu_4$ are assigned so as to make all the poles of the system matrix $A + BF$ as robust as possible. Results for this problem are given in [2], and are reproduced in Table 1, together with the results of the partial pole placement algorithm.

TABLE 1   Condition Numbers

	Partial Pole Placement	Complete Pole Placement (Method 1)
c_1	1.52	1.48
c_2	1.74	1.76
c_3	1.57	1.45
c_4	1.80	1.81
$\|\underline{c}\|_2$	3.32	3.23
$\kappa_2(X)$	3.59	3.32
$\|F\|$	1.42	1.40

We observe that the condition numbers of the assigned poles are very close to each other. The over-all measures of conditioning given by $\|\underline{c}\|_2 = (\sum_4^4 c_j^2)^{1/2}$ and $\kappa_2(X)$ are also close together. We remark that the minimal condition number $\kappa_2(X)$ which is achievable for this problem is bounded below by

$$\kappa_2(X) \geq 1.88,$$

and the solutions obtained by both methods are close to optimal (See [2]). Clearly, from this result and from the values of the constants $\omega_j, \gamma_j, \delta_j$ we see that this a very well-conditioned system and the results of the partial pole placement may be expected to be satisfactory. We remark that the magnitudes of the gains in both feedback solutions are nearly the same, but the computed feedback matrices are themselves rather different.

5. Conclusions

In this paper we have examined the robustness of state feedback solutions to the partial pole placement problem obtained by the projection method of [3]. We have shown that the sensitivity of the assigned poles to perturbations in the closed loop system matrices can be bounded in terms of the sensitivities, or condition numbers, of the assigned poles of the reduced order pole placement problem. Methods for finding robust solutions to the reduced pole placement problem are given in [1][2]. We have also shown that the closed loop sensitivities of the unaltered poles can be bounded in terms of their open loop sensitivity. If robust solutions to the problem exist, that is, if the over-all problem is well-conditioned, then the bounds may be expected to be tight, provided there is good separation between the assigned poles and those which are unaffectecd by the feedback. Further numerical experiments and techniques for estimating the parameters in the bounds will be discussed elsewhere.

References

[1]. Kautsky, J. and Nichols, N.K., Robust multiple eigenvalue assignment by state feedback in linear systems, in Linear Algebra and its Role in Systems Theory.AMS Contemp. Math. Series, 47, 253-264, 1985.

[2] Kautsky, J., Nichols, N.K., and Van Dooren P., Robust pole assignment in linear state feedback, Int. J. Contr. 41, 1129-1155, 1985.

3] Saad, Y., A Projection Method for Partial Pole Assignment in Linear State Feedback. Yale University Technical Rpt. YALEU/DCS/RR-449, 1986.

4] Saad, Y., Partial Eigensolutions of Large Non-Symmetric Matrices, Yale University Technical Rpt. YALEU/DCS/RR-397, 1985.

5] Wilkinson, J.H. The Algebraic Eigenvalue Problem. Oxford University Press, 1965.

] Mielke, R.R. et al, Design of Multivariable Feedback Control Systems via Spectral Assignment Using Reduced Order Models and Reduced Order Observers, NASA Contractor Rpt. 3889, April, 1985.