Nonlinear Optimal Control Problems

in Tidal Power Calculations.

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1. INTRODUCTION

A recent study [11] has shown that the use of tidal energy to generate electrical power is economically viable. In evaluating a tidal power scheme it is important to know that the plant is operating at maximum efficiency and several studies have been made [1], [7], [12], [16] using a variety of mathematical models and techniques in order to find the best operating strategies. One technique which has proved to be particularly useful is that of applying the mathematical theory of optimal control. In previous studies [2], [3], [4], [5], [6] optimal control methods are applied to the problem of maximising the average power functional subject to the satisfaction of a system of linear differential equations which model the fluid dynamics in an estuary. It has been found that this approach is attractive, since it is both computationally feasible and easy to generalise to take into account such things as ebb or two way generation, non-linear head-flow relationships at the tidal barrage, and variable estuarine geometries. The use of linear dynamical equations has, however, restricted investigation to estuary channels of approximately rectangular cross-section, whereas, in practice, channels are far from rectangular and are subject to effects such as the drying out of sand bars. Modelling of such non-linear effects requires the use of non-linear dynamical equations.

In this report we present the application of optimal control techniques to the problem of maximising the output from a tidal power plant where the estuarine dynamics are described by a non-linear ordinary differential equation which more realistically models the estuarine flow. As a further generalisation to previous studies we introduce a tariff function to weight the power integral and also allow the turbines to be of variable efficiency, depending on the head difference.

In the next section the mathematical model of the tidal power scheme is described and the corresponding optimal control problem is formulated. The model is analysed and necessary conditions for the solution of the optimal control problem are given in the following sections. In Section 3 a numerical method for determining the optimal control strategy is developed and a computational algorithm is given. Results are presented in Section 4 with data approximating that for a scheme using the Severn Estuary. Conclusions are given in Section 5.

2. THE MATHEMATICAL MODEL

2.1 The Equation of Flow

We assume that the water surface upstream of the tidal barrier remains horizontal and that the tidal elevation on the seaward side of the barrier is a known function of time, independent of flows through the barrier. The fluid dynamics inside the tidal basin are then governed by the conservation of fluid law given by

$$S(\eta)\dot{\eta} = \alpha_1(t)K_1P(\Delta H) + \alpha_2(t)K_2R(\Delta H), \tag{2.1}$$

where $\eta(t)$ is the water surface elevation above the datum, $S(\eta)$ is the horizontal surface area of the water, K_1, K_2 are the total number of turbines and sluices respectively, P, R are the head-flow functions for each turbine and sluice respectively. The head difference $\Delta H = f(t) - \eta(t)$, where $f(t)$ is the tidal elevation above the datum and α_1, α_2 give the instantaneous proportion of turbines and sluices respectively in operation. Thus α_1, α_2 satisfy the inequalities

$$0 \leqq \alpha_1(t),\ \alpha_2(t) \leqq 1 \quad . \tag{2.2}$$

Over short intervals of time the tides are approximately periodic, and so we also assume that $f(t + T) = f(t)$, where T is the tidal period, and require that η is also periodic satisfying

$$\eta(0) = \eta(T) \quad . \tag{2.3}$$

The maximum power p available to each turbine is given by

$$p(t) = \rho g P(\Delta H)\Delta H \quad ,$$

and the electrical energy J derived is given by

$$J = p(t)e(\Delta H) \quad ,$$

where e is an efficiency function satisfying

$$0 \leq e(\Delta H) \leq 1 \quad .$$

If we introduce a tariff function $C(t) \geq 0$ then the average profit $\overline{P}$ over a tidal cycle is given by

$$\overline{P} = \frac{\rho g K_1}{T} \int_0^T C(t)\alpha_1(t)P(\Delta H)\Delta H e(\Delta H)\, dt, \tag{2.4}$$

where ρ is the fluid density and g is the acceleration due to gravity.

2.2 The Optimal Control Problem

The optimisation problem is then to determine the control functions $\alpha_1(t)$, $\alpha_2(t)$ in order to maximise $\overline{P}$ given by (2.4) subject to equations (2.1), (2.3) and constraints (2.2). Admissible controls are assumed to be measurable on $[0,T]$, [15].

2.3 Analysis of the model

For the optimal control problem to be well-posed it is necessary that for any admissible non-trivial control function $\underline{\alpha} = (\alpha_1,\alpha_2)^T$, the equation of motion (2.1) together with periodic condition (2.3), has a unique solution. In order to show this we impose some conditions on the data f, P, R, S.

We make the following assumptions:

(i) $f(t)$, $P(y)$, $R(y)$ are continuous everywhere

(ii) P', R' exist everywhere and satisfy $0 < \underline{K} \leq P'(y)$, $R'(y) \leq \overline{K} < \infty$ where $\overline{K}$, $\underline{K}$ are positive real numbers

(iii) $S(y)$ is continuous everywhere and there exists $\sigma > 0$, $\sum < \infty$ such that $\sigma \leq S(y) \leq \sum$

Assumptions (i)-(iii) then imply that for each control $\underline{\alpha}$ such that $\int_0^T (\alpha_1 + \alpha_2)dt > 0$, there exists a unique continuous solution $\eta(t)$ satisfying (2.1), (2.3). Conditions (i)-(iii) also imply continuity of the solution η with respect to the control α in the sense that for each pair of non-trivial controls $\underline{\alpha}$, $\underline{\beta}$ with corresponding responses η_α, η_β, there exists a constant $M_\alpha < \infty$ such that

$$|\eta_\beta(t) - \eta_\alpha(t)| \leq M_\alpha \|\underline{\beta} - \underline{\alpha}\|_1 \quad , \tag{2.5}$$

where $\|\underline{\alpha}\|_1 = \int_0^T (\alpha_1(t) + \alpha_2(t))dt.$ The proofs of existence of periodic solutions and continuity property (2.5) are given in appendix 1. Both of these results are required in the next section where necessary conditions are derived for the solution of the optimal control problem.

2.4 Necessary Conditions for the Optimal

Necessary conditions for the solution of the optimal control problem are derived using the Lagrangian formulation of the problem [9]. This approach also provides a basis for the numerical procedure described in the next section.

The Lagrangian functional $L(\underline{\alpha})$ associated with the optimal control problem is defined by

$$L(\underline{\alpha}) = \int_0^T C\alpha_1 P(\Delta H)\Delta He(\Delta H) + \lambda[-S(\eta)\dot{\eta} + \alpha_1 K_1 P(\Delta H) + \alpha_2 K_2 R(\Delta H)]dt \ , \tag{2.6}$$

where $\lambda(t)$ is a Lagrange multiplier known as the adjoint state. For $\underline{\alpha}$ to be optimal it is necessary that the first variation $\delta L(\underline{\alpha},\delta\underline{\alpha})$ of the functional L is non-positive where δL is linear in $\delta\underline{\alpha} = \underline{\beta} - \underline{\alpha}$ and such that

$$L(\beta) - L(\alpha) = \delta L(\alpha,\delta\alpha) + o\|\beta - \alpha\|_1 \quad ,$$

for all admissible non-trivial controls $\underline{\beta}$.

If we assume that the efficiency function $e(y)$ is also differentiable, then the first variation of the Lagrangian (2.7) can be written in the form

$$\begin{aligned}\delta L(\underline{\alpha},\delta\underline{\alpha}) = &\int_0^T \delta\alpha_1[CP(\Delta H)\Delta He(\Delta H) + \lambda K_1 P(\Delta H)]\ dt \\ &+ \int_0^T \delta\alpha_2 \lambda K_2 R(\Delta H)\ dt \quad ,\end{aligned} \tag{2.7}$$

where integration by parts has been used and $\lambda(t)$ has been taken to satisfy the adjoint problem

$$S(\eta)\dot{\lambda} = \lambda(\alpha_1 K_1 P'(\Delta H) + \alpha_2 K_2 R'(\Delta H)) + C\alpha_1 \frac{\partial}{\partial \Delta H}(P(\Delta H)\Delta He(\Delta H))$$

$$\lambda(0) = \lambda(T) \quad . \tag{2.8}$$

Under conditions (i)-(iii) of §2.3, it can be shown that for every non-trivial admissible control $\underline{\alpha}$, problem (2.9) also posesses a unique continuous solution $\lambda(t)$. If we regard $\overline{P}$ in (2.4) as a function solely of α_1, α_2 we therefore have $\frac{\rho g K_1}{T} \delta L(\underline{\alpha}, \delta\underline{\alpha})$ equal to the first variation of $\overline{P}$ with respect to $\underline{\alpha}$, and we may write

$$\langle \underline{\nabla \overline{P}}(\underline{\alpha}), \underline{\delta\alpha} \rangle = \frac{\rho g K_1}{T} \delta L(\underline{\alpha}, \underline{\delta\alpha})$$

where $\delta\alpha$ is an admissible variation, $\langle \cdot , \cdot \rangle$ is the inner product defined by

$$\langle \underline{p}, \underline{q} \rangle = \int_0^T \underline{p}^T(t)\underline{q}(t)\, dt$$

and the function space gradient $\underline{\nabla \overline{P}}(\underline{\alpha})(t)$

$$\underline{\nabla \overline{P}}(\underline{\alpha})(t) = \frac{\rho g K_1}{T} \begin{pmatrix} C(t)P(\Delta H)\Delta He(\Delta H) + \lambda K_1 P(\Delta H) \\ \lambda K_2 R(\Delta H) \end{pmatrix} , \tag{2.9}$$

where $\Delta H = f - \eta$, η satisfies (2.1),(2.3) and λ satisfies (2.8). For the control $\underline{\alpha}$ to be optimal it is necessary, then that

$$\langle \nabla \overline{P}(\alpha), \underline{\beta} - \underline{\alpha} \rangle \leq 0 \tag{2.10}$$

is satisfied for all admissible controls $\underline{\beta}$. For a given control, the gradient vector can be computed from (2.9) and since the values of the controls α_1, α_2 belong to a closed interval for any t, the inequality (2.10) is easily tested. Gradient methods can therefore be applied to determine numerical approximations to the optimal control problem. The numerical procedure described in the next section uses a gradient technique to generate a sequence of approximations to the solution of the optimal control problem.

3. A NUMERICAL METHOD

The computational method which we use to solve the optimal control problem consists of a constrained optimisation technique for determining the control, together with a finite difference method for solving the state and adjoint problems.

3.1 The Conditional Gradient Method

Many optimisation techniques are described in the literature [8], but previous investigation [3] indicates that the Conditional Gradient method is suitable for application to the tidal power problem when the dynamics are approximated by an ordinary differential equation model. This method generates a sequence of piecewise continuous controls $\underline{\alpha}^K(t)$, $K = 1,2,...$ approximating the optimal control $\underline{\alpha}(t)$. Since the set of admissible controls U is closed and convex, there exists a maximal displacement $\underline{\delta\alpha}^K$ in the direction of the gradient $\underline{\nabla}\overline{P}(\underline{\alpha}^K)$ such that $\underline{\alpha}^K + \underline{\delta\alpha}^K$ lies in U. The conditional gradient method generates the controls $\underline{\alpha}^K$ such that $\underline{\alpha}^{K+1} = \underline{\alpha}^K + \theta^K \underline{\delta\alpha}^K$ where $\theta^K \in [0,1]$, and such that either $\overline{P}(\underline{\alpha}^{K+1}) > \overline{P}(\underline{\alpha}^K)$ or $\underline{\alpha}^{K+1}$ satisfies necessary conditions (2.10). In practice the iteration is terminated when the measure $M(\underline{\alpha}^K)$ is less than a given positive tolerance, where $M(\underline{\alpha})$ is given by

$$M(\underline{\alpha}) = \max_{\underline{\beta} \in U} \langle \underline{\nabla}\overline{P}(\underline{\alpha}), \underline{\beta} - \underline{\alpha} \rangle \quad , \tag{3.1}$$

and $\underline{\alpha}^K$ is then accepted as a good solution to the optimal control problem. The solutions of the state problem (2.1), (2.3) and adjoint problem (2.8) are approximated at each step of the method by the trapezoidal rule [14], the state equation being integrated forward in time and the adjoint equation being integrated in the reverse time direction.

The numerical optimisation algorithm is obtained by replacing all integrations in the following algorithm by the corresponding discrete approximation.

Algorithm

Step 0 : Choose $\underline{\alpha}^0 \in U \quad (\underline{\alpha}^0(t) \not\equiv 0)$

Choose $\theta \in (0,1]$

$E(\underline{\alpha}^{-1}) := 0$

$k := 0$

Step 1 : Solve problem (2.1), (2.3) for η with $\underline{\alpha} = \underline{\alpha}^k$

Step 2 : Solve problem (2.8) for λ with $\underline{\alpha} = \underline{\alpha}^k$ and with η from step 1.

Step 3 : Evaluate $\overline{P}(\underline{\alpha}^k)$ using (2.4)

Step 4 : If $\overline{P}(\underline{\alpha}^k) < \overline{P}(\underline{\alpha}^{k-1})$ then $\theta := \phi\theta$ where $\phi \in (0,1)$

Step 5 : Evaluate $\underline{\nabla}\overline{P}(\underline{\alpha}^k)$ using (2.9)

Step 6 : Evaluate $\underline{\beta}^k$, $M(\underline{\alpha}^k)$ where

$$M(\alpha^k) = \langle \underline{\nabla}\overline{P}(\underline{\alpha}^k), \underline{\beta}^k - \underline{\alpha}^k \rangle = \max_{\beta \in U} \langle \underline{\nabla}\overline{P}(\underline{\alpha}^k), \underline{\beta} - \underline{\alpha}^k \rangle$$

Step 7 : If $M(\underline{\alpha}^k) < tol$, then STOP.

Step 8 : $\underline{\alpha}^{k+1} = \underline{\alpha}^k + \theta(\underline{\beta}^k - \underline{\alpha}^k)$; $k = k + 1$

Step 9 : Go to *Step 1.*

Details of the numerical integration schemes are given in Appendix II.

3.2 Ebb Generation Schemes

So far it has been assumed that we are modelling generation schemes which provide power on both the flood and the ebb tide. Schemes suggested for the Severn Estuary have mainly been those generating only on the ebb tide [11].

To model such schemes we take $P(y) = 0$ for $y \geqq 0$. In order to analyse the system in this case it is necessary to relax condition (ii) of §2.3. If we make instead the natural assumption that $P(0) = R(0) = 0$ and that $\alpha_2(t) \not\equiv 0$, so that some sluicing takes place during the cycle, then existence of unique periodic solutions to (2.1) and continuity condition (2.5) still hold. It is also noted that for ebb schemes, the control $\alpha_1(t)$ has no effect during periods of positive head difference and for convenience we set $\alpha_1(t)$ to zero during such periods.

4. RESULTS

Numerical results are described for a problem which approximates the Severn estuary, where the non-linearities model both the effects of drying out of sand banks and the variation of turbine efficiency with head difference. We give two examples. The first uses a tidal period of about 12 hours and a constant tariff function. The second example uses the half lunar cycle as the tidal period, where the tides have changing amplitude from day to day.

4.1 Half Day tides

We take the following data as an approximation to a Severn estuary scheme:

$$S(\eta) = 4.6 \times 10^8 + 2.6 \times 10^7 \eta \ m^2 \quad ;$$

$$P(y) = \begin{cases} 290\ (1 + \tanh(10(y - 1.7)) & y \geq 0, \\ -P\ (-y) & y < 0; \end{cases}$$

$$R(y) = 216 \sqrt{2g|y|}\ \mathrm{sgn}(y)$$

$$e(y) = \begin{cases} g(y) & g(y) \geq 0, \\ 0 & g(y) < 0, \quad \text{where} \end{cases}$$

$$g(y) = 0.14 + 0.68 \tanh\ (0.7(|y| - 1.7));$$

$$C(t) = 1.0;$$

$$T = 44600\ s;$$

$$f(t) = F_0 \cos(2\pi t/T), \quad K_1 = 140, \ K_2 = 160$$

where F_0 is the tidal amplitude in metres. It is noted that although $e(y)$ is not differentiable at the zeros of $g(y)$, these zeros lie in the interval where $P(y)$ is effectively also zero, and hence we can ignore the lack of differentiability of e at these two points. The same comment applies to $P(y)$ which has a small jump discontinuity at $y = 0$. From a mathematical point of view we can always replace P, e by smooth approximations as accurate as required; from a computational point of view we may leave P, e as defined.

The spring and neap tidal amplitudes are taken to be $F_0 = 5.2\text{m}$ and $F_0 = 2.65\text{m}$ respectively, and the results are computed with 800 time steps, taking the initial controls as $\alpha_1^0 \equiv 0.1$, $\alpha_2^0 \equiv 0$. Table 1 illustrates the convergence of the iteration method for a spring tide. Table 2 gives the results of the same calculation for 100% efficient turbines $(e(y) \equiv 1)$. Table 3 shows the best average power obtained at springs and neaps, using firstly 100% efficient turbines and then the variable efficiency machines. It is noted that in the case of 100% efficient turbines, the two-way scheme is always superior to the ebb scheme, whereas for the variable efficiency machines the situation is reversed. This is rather a surprising result since we would expect a two way scheme to be at least as effective as an ebb scheme for our model. It is found, however, in the case of the variable efficiency model, that if the two way scheme is run with the best ebb scheme controls, then the algorithm terminates without updating the controls. We conclude that for the variable efficiency model, the ebb scheme is the best and that the control strategy computed by the algorithm for the two way scheme is only a local maximum.

Figures 1a,b, 2a,b give the main flow parameters for the best computed ebb and two way schemes respectively, calculated using the variable efficiency model with a spring tide cycle. It is seen from Figure 2b that almost all of the power from the two way scheme is being generated on the flood tide so that our algorithm has sought out a one way scheme, but one generating in the wrong direction! For comparison, Figures 3a,b, 4a,b, give the main flow parameters for the best computed ebb and two way schemes respectively, using 100% efficient machines with a spring tide cycle.

4.2 Half lunar cycles

The second example we examine is a Severn estuary model, where the average profit functional (2.4) is calculated over a 14 day Spring-Neap-Spring cycle. We take the variable efficiency turbine model, as before, operating in ebb generating mode. The tidal elevation $f(t)$ is given by

$$f(t) = \left(2.65 \cos^2 \left(\frac{\pi t}{T}\right) + 2.25\right) \cos\left(54 \frac{\pi t}{T}\right) ,$$

where T is the half lunar period, and, for the purposes of comparison, we take two tariff functions $C(t)$. The first tariff function is based on winter rate electricity prices as follows

Weekdays	0030 - 0730	1.37p unit
	0730 - 2000	6.05p/unit
	2000 - 0030	2.43p/unit
Weekends		2.43p/unit .

The second tariff function is $C(t) \equiv 1$, so that we are merely seeking to maximise average power rather than revenue. Table 4 gives the best computed values of the profit functional/revenue for the two cost functionals, assuming the high spring tide is at 7.00 a.m. on a Sunday. As expected it is seen that the effect of maximising the revenue is to reduce the average power slightly. In this example a 2% increase in revenue is achieved by rescheduling the power output, which decreases in average value by only 0.8%. Figures 5a,b, 6a,b show the main flow parameters for the winter tariff and constant tariff models respectively. If we examine the instantaneous power curve in 5b for the winter rate model we see that the high tariff periods are in phase with the best generating times for the Spring-Neap period given by the power curve in 6b. On the second half of the cycle from Neap to Spring we see that the power is generated both during high tariff periods and low tariff periods due to the changed phase of the tides at this part of the cycle. The power output at low tariff periods has been depressed in order to increase output during the high tariff periods. It is noted that our example is not perhaps the best to illustrate this point and that at certain times of the year the phase of the tides will be such that an optimal average power strategy will be producing a lot of energy at low tariff periods. Under these conditions we conclude that an optimal revenue policy will reschedule power production to give a higher profit.

5. CONCLUSIONS

In this report we examine a non-linear ordinary differential equation model of a tidal power generation scheme and, using the theory of optimal control, develop techniques for determining the maximum revenue derivable from the scheme. The model incorporates the non-linear effects due to non-rectangular estuarine geometry, and also due to flow through turbines and sluices. Further additions to previous models allow for variation of turbine efficiency with head difference and the inclusion of a tariff function to weight the power integral.

The optimal profit problem is formulated as a constrained optimal control problem and necessary conditions for the optimum are given. A conditional gradient algorithm is described for determining the optimal control strategy and numerical results are presented.

The results indicate that when turbine efficiency drops off rapidly as head decreases then an ebb generating scheme performs better than a two way scheme. It is also seen that over a half lunar cycle we can improve the revenue from a tidal power scheme by rescheduling generation over high tariff periods. This rescheduling produces a decrease in total energy output; this decrease is, however, only marginal.

We conclude that the application of optimal control techniques to the tidal power problem where the flow equations are non-linear is quite feasible and is an attractive method for systematically computing flow control policies, even in complicated situations where we are seeking to maximise some weighted integral of the power output.

APPENDIX I

ANALYSIS OF STATE EQUATIONS

We sketch here some theoretical results on the existence of periodic solutions to the state problem (2.1), (2.3) and the continuity of such solutions with respect to the control. These results are required in establishing the existence of the gradient $\underline{\nabla}\overline{P}$ (2.9). For convenience we briefly restate the problem and the assumptions made in §2.

Problem 1 : Find a continuous and differentiable function $\eta(t)$ such that

$$S(\eta)\dot{\eta} = \alpha_1(t)K_1P(f-\eta) + \alpha_2(t)K_2R(f-\eta) \tag{A1}$$

$$\eta(0) = \eta(T) \quad , \tag{A2}$$

where

(i) $\alpha_1(t)$, $\alpha_2(t)$ are measurable on $[0,T]$ and satisfy $0 \leqq \alpha_1, \; \alpha_2 \leqq 1$.

(ii) $f(t)$ is continuous on $[0,T]$, $P(y)$, $R(y)$ are continuous everywhere.

(iii) P', R' exist everywhere and satisfy $0 < \underline{K} \leqq P'(y), \; R'(y) \leqq \overline{K} < \infty$.

(iv) $S(y)$ is continuous everywhere and satisfies $0 < \sigma \leqq S(y) \leqq \sum < \infty$.

Proposition 1 : For each $\underline{\alpha}$ such that

$$\int_0^T \alpha_1(t) + \alpha_2(t)dt > 0,$$

there is a unique continuous solution $\eta(t)$ to Problem 1.

Proof : We introduce a new volume variable $w(t)$ defined by

$$w = W(\eta) = \int_0^\eta S(y)\, dy \quad , \tag{A3}$$

so that equation (A1) becomes

$$\dot{w} = \alpha_1 K_1 P(f - W^{-1}(w)) + \alpha_2 K_2 R(f - W^{-1}(w)) \quad , \qquad (A4)$$

where W is a one-to-one function and $W^{-1}(\ )$ is its inverse. Conditions (i)-(iv) then guarantee the existence of a unique continuous solution $w(t)$ to the initial value problem (A4) with $w(0) = w_0$ for every $w_0 \in \mathbb{R}$ [15]. If now $w_1(t)$, $w_2(t)$ represent two solutions of (A4), corresponding to two different initial conditions then (A4) implies

$$(\dot{w}_1-\dot{w}_2)(w_1-w_2) = \alpha_1 K_1 [P(f-W^{-1}(w_1)) - P(f-W^{-1}(w_2))](w_1-w_2)$$

$$+ \alpha_2 K_2 [R(f-W^{-1}(w_1)) - R(f-W^{-1}(w_2))](w_1-w_2) \ . \qquad (A5)$$

By using the mean value theorem, conditions (iii), (iv) and noting that $\frac{d\eta}{dw} = \frac{1}{S(\eta)}$, equation (A5) gives the inequality

$$\tfrac{1}{2} \frac{d}{dt} (w_1 - w_2)^2 \leqq q(w_1 - w_2)^2 \ (\alpha_1 + \alpha_2), \qquad \text{where}$$

$$q = \frac{\underline{K} \min (K_1, K_2)}{\Sigma} \quad ,$$

and hence, by applying Growall's Lemma [13], we arrive at the inequality

$$|w_1(t) - w_2(t)| \leqq e^{-q\int_0^t \alpha_1+\alpha_2 \, dt} |w_1(0) - w_2(0)| \ . \qquad (A6)$$

If the mapping $G : \mathbb{R} \to \mathbb{R}$ is defined by

$$Gu = V$$

where $w(0) = u \in \mathbb{R}$, $w(t)$ satisfies (A4) and $w(T) = V \in \mathbb{R}$, then inequality (A6) implies that G is a contraction on $\mathbb{R}$ [10] and, therefore, that there is a unique fixed point $V^* \in \mathbb{R}$ such that $GV^* = V^*$. This implies that there is a unique initial value $w(0) = V^*$ such that $w(t)$ satisfies the periodic initial value problem of (A4) with $w(0) = V^* = w(T)$, and since there is a one-one mapping between w and η via (A3), this completes the proof.

We now prove the inequality (2.5), which shows that the solution to (A1), (A2) depends continuously on the control $\underline{\alpha}$. This requires two intermediate results, namely the continuity of the corresponding initial value problem with respect to the control, and inequality (A6). We first prove a lemma.

Lemma 1

For any two solutions w_1, w_2 of (A4) satisfying

$$\dot{w}_1 = \alpha_1 K_1 P(f-W^{-1}(w_1)) + \alpha_2 K_2 R(f-W^{-1}(w_1)) , \qquad w_1(0) = w_0 \tag{A7}$$

$$\dot{w}_2 = \beta_1 K_1 P(f-W^{-1}(w_2)) + \beta_2 K_2 R(f-W^{-1}(w_2)), \qquad w_2(0) = w_0 , \tag{A8}$$

where $\underline{\alpha}$, $\underline{\beta}$ are non-trivial admissible controls, there exists a constant $L < \infty$ such that

$$|w_1(t) - w_2(t)| \leq Le^{Lt} \int_0^t |\underline{\alpha} - \underline{\beta}| \, dt \tag{A9}$$

where we denote $|\underline{\alpha}| = |\alpha_1| + |\alpha_2| = \alpha_1 + \alpha_2$.

Proof : Subtracting (A8) from (A7), integrating from 0 to t, and using the mean value theorem for a scalar function of two variables,

$$|w_1(t) - w_2(t)| \leq L \int |w_1(s) - w_2(s)| + |\underline{\alpha} - \underline{\beta}| \, ds \tag{A10}$$

where

$$L = \max_{t \in [0,1]} \left\{ K_1 |P(f-W^{-1}(v))|, \; K_2 |R(f-W^{-1}(v))|, \; \frac{K_1 \overline{K}}{\sigma} , \; \frac{K_2 \overline{K}}{\sigma} \right\} ,$$

and $v(t)$ is a function intermediate between $w_1(t)$, $w_2(t)$. Application of Gronwall's lemma to (A10) then gives inequality (A9), completing the proof of Lemma 1.

Proposition 2 : For any two (periodic) solutions w_1, w_2 of (A4) satisfying

$$\dot{w}_1 = \alpha_1 K_1 P(f - W^{-1}(w_1)) + \alpha_2 K_2 R(f - W^{-1}(w_1)), \quad w_1(0) = w_1(T)$$

$$\dot{w}_2 = \beta_1 K_1 P(f - W^{-1}(w_2)) + \beta_2 K_2 R(f - W^{-1}(w_2)), \quad w_2(0) = w_2(T)$$

where $\underline{\alpha}$, $\underline{\beta}$ are non-trivial admissible controls, there exists a constant $C_\alpha < \infty$ such that

$$w_1(t) - w_2(t) \leq C_\alpha \|\underline{\beta} - \underline{\alpha}\|_1 \tag{A11}$$

Proof : Let $z(t)$ be the unique solution to the initial value problem

$$\dot{z} = \alpha_1 K_1 P(f - W^{-1}(z)) + \alpha_2 K_2 R(f - W^{-1}(z)) \quad ,$$

$$z(0) = w_2(0) \quad ,$$

then

$$|w_1(t) - w_2(t)| \leq |w_1(t) - z(t)| + |z(t) - w_2(t)| \quad . \tag{A12}$$

From inequality (A6) we have

$$|w_1(t) - z(t)| \leq |w_1(0) - w_2(0)| \, e^{-q\int_0^t |\underline{\alpha}| ds} \tag{A13}$$

for some $q > 0$, and from Lemma 1 we have

$$|z(t) - w_2(t)| \leq M \|\beta - \alpha\|_1 \tag{A14}$$

for some $M < \infty$. Then putting $t = T$ in (A12) and using (A13), (A14) gives

$$|w_1(0) - w_2(0)| \leq e^{-q\int_0^T |\underline{\alpha}| ds} |w_1(0) - w_2(0)| + M \|\beta - \alpha\|_1 \tag{A15}$$

and hence

$$|w_1(t) - w_2(t)| \leq \left(\frac{M}{1 - e^{-q\|\underline{\alpha}\|_1}}\right) \|\underline{\beta} - \underline{\alpha}\|_1 \quad ,$$

which proves proposition 2 with $C_\alpha = \dfrac{M}{1 - e^{-q\|\alpha\|_1}}$. If we use (A3) then we also have $|\eta_\alpha(t) - \eta_\beta(t)| \leqq \frac{C_\alpha}{\sigma}||\beta - \alpha||_1$ where $w_1 = W(\eta_\alpha)$, $w_2 = W(\eta_\beta)$, and hence inequality (2.5) is proved.

APPENDIX II

NUMERICAL SOLUTION OF STATE AND ADJOINT EQUATIONS

The numerical solution of periodic initial value problem (2.1), (2.3) is obtained by discretisation of (2.1) using the trapezoidal method, followed by repeated numerical integration of the equation over the interval [0,T] until the initial value η_0 and the final value η_N differ by less than some small tolerance. This process converges to the unique periodic solution of the difference equation, which satisfies a contraction property analogous to that satisfied by the differential equation (see inequality (A6)). Numerical solution of the adjoint problem (2.8) is obtained by the same method, but with the integration being performed in the reverse time direction.

The difference approximation to the state problem (2.1), (2.3) is given by

$$\eta^{n+1} - \eta^{n} = \frac{\Delta t}{2}\left\{\frac{\alpha_1^n K_1 P(\Delta H)^n + \alpha_2^n K_2 R(\Delta H^n)}{S(\eta^n)} + \frac{\alpha_1^{n+1} K_1 P(\Delta H^{n+1}) + \alpha_2 K_2 R(\Delta H^{n+1})}{S(\eta^{n+1})}\right\}$$

$$n = 0,1,\ldots, N-1 \quad , \tag{B1}$$

where $N\Delta t = T$, η^n is the approximation to $\eta(n\Delta t)$, α_1^n, α_2^n approximate $\alpha_1(n\Delta t)$, $\alpha_2(n\Delta t)$ and $\Delta H^n = f^n - \eta^n$ with $f^n = f(n\Delta t)$. At each time step the non-linear algebraic equation (B1) is solved for η^{n+1} using simple iteration [10], which is convergent provided

$$\bar{K}\,\frac{\Delta t}{\sigma}\max\{K_1, K_2\} < 2 \quad .$$

The adjoint problem (2.8) is similarly approximated by the difference scheme

$$\lambda^n - \lambda^{n-1} = \frac{\Delta t}{2}\left\{\frac{[\alpha_1^{n-1} K_1 P'(\Delta H^{n-1}) + \alpha_2^{n-1} K_2 R'(\Delta H^{n-1})]}{S(\eta^{n-1})}\lambda^{n-1} + \frac{[\alpha_1^n K_1 P'(\Delta H^n) + \alpha_2^n K_2 R'(\Delta H^n)]\lambda^n}{S(\eta^n)}\right\}$$

$$+ \frac{\Delta t}{2}\left\{C^n \alpha_1^n \frac{\partial}{\partial y}(P(y)e(y)y)\bigg|_{y=\Delta H^n} + C^{n-1}\alpha_1^{n-1}\frac{\partial}{\partial y}(P(y)e(y)y)\bigg|_{y=\Delta H^{n-1}}\right\} , \tag{B2}$$

$$n = N, N-1, \ldots, 1.$$

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Table 1

NO. OF MESH INTERVALS N = 800
NO. OF ITERATIONS = 30
TIDAL PERIOD = 44600.0000 (SECS)
NUMBER OF TURBINES = 140.
NUMBER OF SLUICES = 160.
TIDAL AMLITUDE = 5.20 (METRE)

2-WAY GENERATION SCHEME

I	IS THE ITERATION NUMBER
THETA	IS THE GRADIENT STEP
P	IS THE AVERAGE POWER (GW)
E	IS THE WEIGHTED AVERAGE POWER (GW)
MAX(DE)	IS THE MAX 1st ORDER CORRECTION
DEDK1	IS THE RATE OF CHANGE OF E WITH RESPECT TO THE NUMBER OF TURBINES (MW/TURBINE)
DEDK2	IS THE RATE OF CHANGE OF E WITH RESPECT TO THE NUMBER OF SLUICES (MW/SLUICE)

I	THETA	P (GW)	E	MAX(DE)	DEDK1 (MW)	DEDK2 (MW)
1	.400000	.183398	.183398	1.881666	1.311849	.000000
2	.400000	.752904	.752904	1.027240	4.469769	-.383718
3	.400000	1.129943	1.129943	.641123	6.031324	-.782996
4	.400000	1.440386	1.440386	.447207	7.296991	-.343261
5	.400000	1.566526	1.566526	.284486	7.633584	-.436533
6	.400000	1.725808	1.725808	.215181	8.293428	.290939
7	.400000	1.752123	1.752123	.127569	8.309947	.018689
8	.400000	1.841169	1.841169	.117253	8.675766	.797215
9	.400000	1.827562	1.827562	.059600	8.603427	.288605
10	.200000	1.853331	1.853331	.052479	8.687318	.582559
11	.200000	1.847639	1.847639	.039665	8.664132	.400890
12	.100000	1.853580	1.853580	.034738	8.674560	.457899
13	.100000	1.858172	1.858172	.030710	8.675026	.494925
14	.100000	1.861808	1.861808	.027406	8.679015	.517835
15	.100000	1.864911	1.864911	.024912	8.679003	.583519
16	.100000	1.867278	1.867278	.023296	8.684313	.513239
17	.100000	1.871829	1.871829	.023728	8.689947	.708593
18	.100000	1.866891	1.866891	.024870	8.718908	.518400
19	.050000	1.871381	1.871381	.018782	8.711178	.596634
20	.050000	1.872361	1.872361	.017788	8.711259	.605375

EBB-GENERATION SCHEME

I	THETA	P (GW)	E	MAX(DE)	DEDK1 (MW)	DEDK2 (MW)
1	.400000	.058134	.058134	.952609	.356472	.000000
2	.400000	.624428	.624428	1.175036	3.871781	.122147
3	.400000	1.122561	1.122561	.789040	6.417999	.248671
4	.400000	1.453488	1.453488	.516707	7.918096	.526716
5	.400000	1.670021	1.670021	.334316	8.887080	.845688
6	.400000	1.809961	1.809961	.213756	9.421441	1.150937
7	.400000	1.899554	1.899554	.135166	9.781540	1.425146
8	.400000	1.956253	1.956253	.084475	10.037382	1.648918
9	.400000	1.991571	1.991571	.052427	10.173289	1.797339
10	.400000	2.013633	2.013633	.032354	10.169703	1.955795
11	.400000	2.027291	2.027291	.019756	10.234621	1.977432

2-WAY GENERATION SCHEME

I	IS THE ITERATION NUMBER
THETA	IS THE GRADIENT STEP
P	IS THE AVERAGE POWER (GW)
E	IS THE WEIGHTED AVERAGE POWER (GW)
MAX(DE)	IS THE MAX 1st ORDER CORRECTION
DEDK1	IS THE RATE OF CHANGE OF E WITH RESPECT TO THE NUMBER OF TURBINES (MW/TURBINE)
DEDK2	IS THE RATE OF CHANGE OF E WITH RESPECT TO THE NUMBER OF SLUICES (MW/SLUICE)

I	THETA	P (GW)	E	MAX(DE)	DEDK1 (MW)	DEDK2 (MW)
1	.400000	.249025	.249025	2.349504	1.780490	.000000
2	.400000	1.225664	1.225664	1.527126	8.167264	.182180
3	.400000	1.831946	1.831946	.934587	11.377437	.231899
4	.400000	2.209484	2.209484	.587486	13.254940	.598246
5	.400000	2.446988	2.446988	.367434	14.427261	.871570
6	.400000	2.595406	2.595406	.229000	15.220180	1.157873
7	.400000	2.688385	2.688385	.142328	15.663140	1.303731
8	.400000	2.745038	2.745038	.094876	16.279548	1.666556
9	.400000	2.776578	2.776578	.072692	16.205564	1.582623
10	.400000	2.794978	2.794978	.066073	16.717534	1.837976
11	.400000	2.801237	2.801237	.054909	16.468902	1.755227
12	.400000	2.811846	2.811846	.036294	16.726167	1.830217
13	.400000	2.821865	2.821865	.023174	16.662490	1.854919
14	.400000	2.829470	2.829470	.015429	16.813442	1.931725
15	.400000	2.834721	2.834721	.009638	16.791676	1.944783
16	.400000	2.838543	2.838543	.006228	16.760694	1.877815
17	.400000	2.840172	2.840172	.008925	16.915299	2.117155
18	.400000	2.839202	2.839202	.014004	17.017565	2.078285
19	.200000	2.840330	2.840330	.005189	16.794946	1.917712
20	.200000	2.841310	2.841310	.004055	16.854792	1.994095
21	.200000	2.842085	2.842085	.003407	16.864815	1.993767
22	.200000	2.842725	2.842725	.002802	16.883719	2.030097

EBB-GENERATION SCHEME

I	THETA	P (GW)	E	MAX(DE)	DEDK1 (MW)	DEDK2 (MW)
1	.400000	.085983	.085983	1.947723	.453425	.000000
2	.400000	.834618	.834618	1.448209	4.953437	.249605
3	.400000	1.434981	1.434981	.961665	7.398434	.600311
4	.400000	1.829611	1.829611	.627794	8.834159	.855828
5	.400000	2.089847	2.089847	.411738	9.789083	1.278097
6	.400000	2.259740	2.259740	.266169	10.464594	1.693853
7	.400000	2.369143	2.369143	.168426	10.834047	1.874232
8	.400000	2.438436	2.438436	.107256	11.018170	2.377448
9	.400000	2.481782	2.481782	.067049	11.170259	2.578622
10	.400000	2.508899	2.508899	.041451	11.323565	2.732949
11	.400000	2.525644	2.525644	.025711	11.281888	2.928949
12	.400000	2.536008	2.536008	.015691	11.281973	3.047904
13	.400000	2.542289	2.542289	.009460	11.391733	3.075091
14	.400000	2.546079	2.546079	.005772	11.393118	3.109135
15	.400000	2.548410	2.548410	.003543	11.342307	3.332869
16	.400000	2.549840	2.549840	.002192	11.325615	3.464336

Average power over a half day cycle (GW).				
Spring tidal amplitude=5.2 m Neap tidal amplitude=2.65m				
			Ebb	Two way
100% efficient turbines	Springs		2.55	2.84
	Neaps		0.95	1.18
			Ebb	Two way
Variable efficiency m/c	Springs		2.03	1.88
	Neaps		0.61	0.60

Table 3.

Spring-Neap-Spring cycle .		
	Average power (GW)	Total revenue over a 14 day period ($ M)
C(t)=1	1.30	15.5
C based on Winter tariff.	1.29	15.8

Table 4.

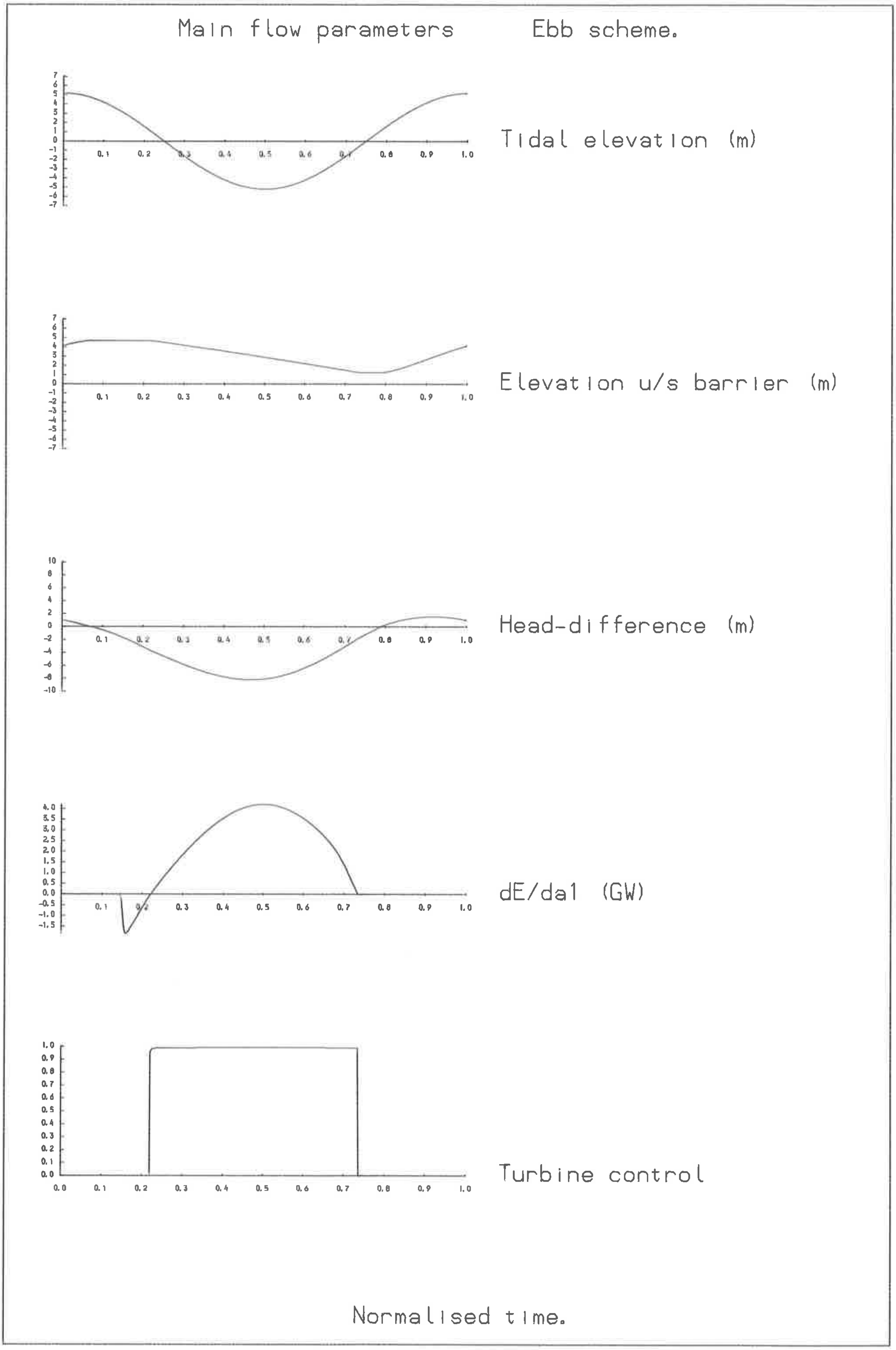
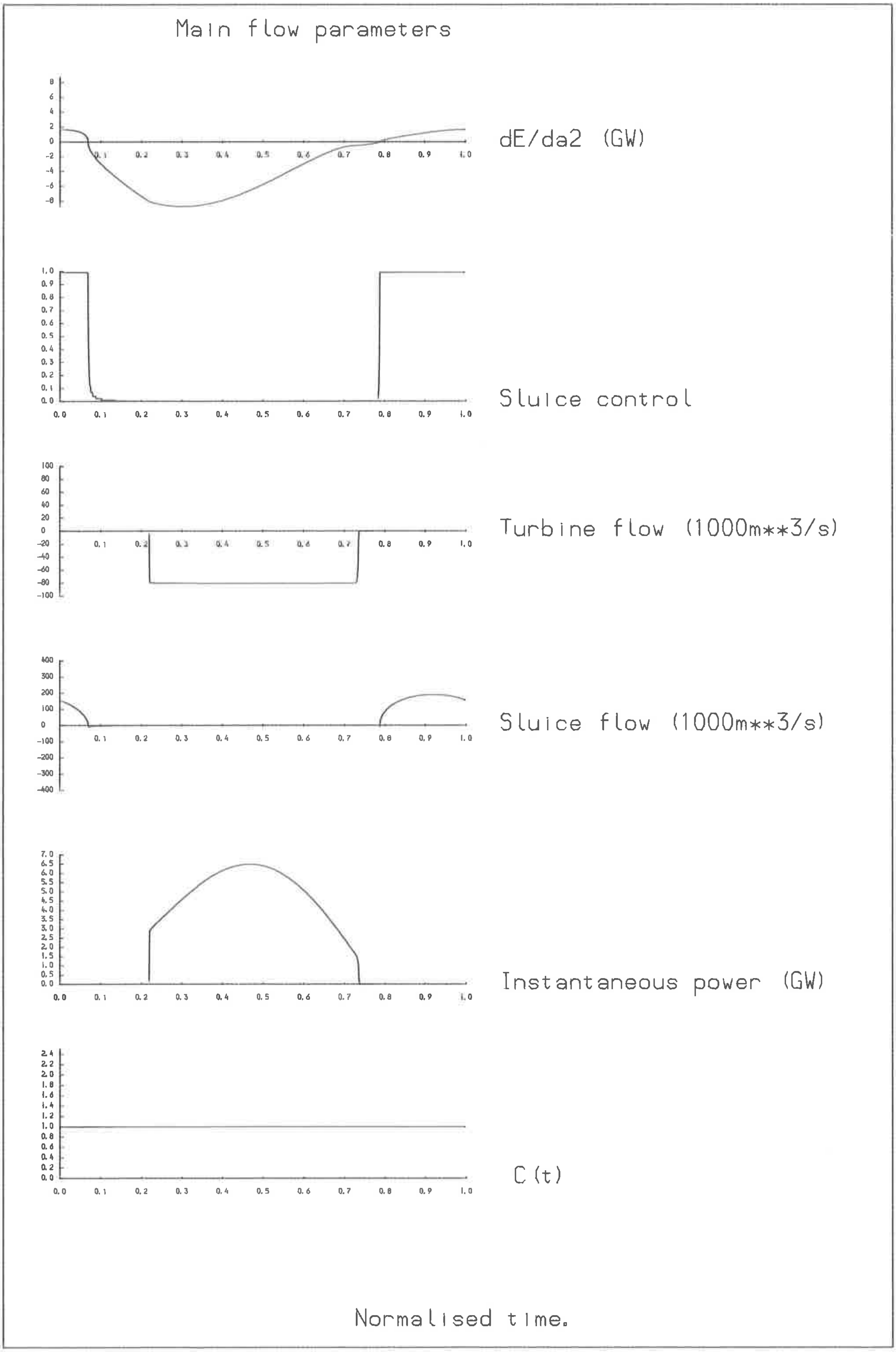
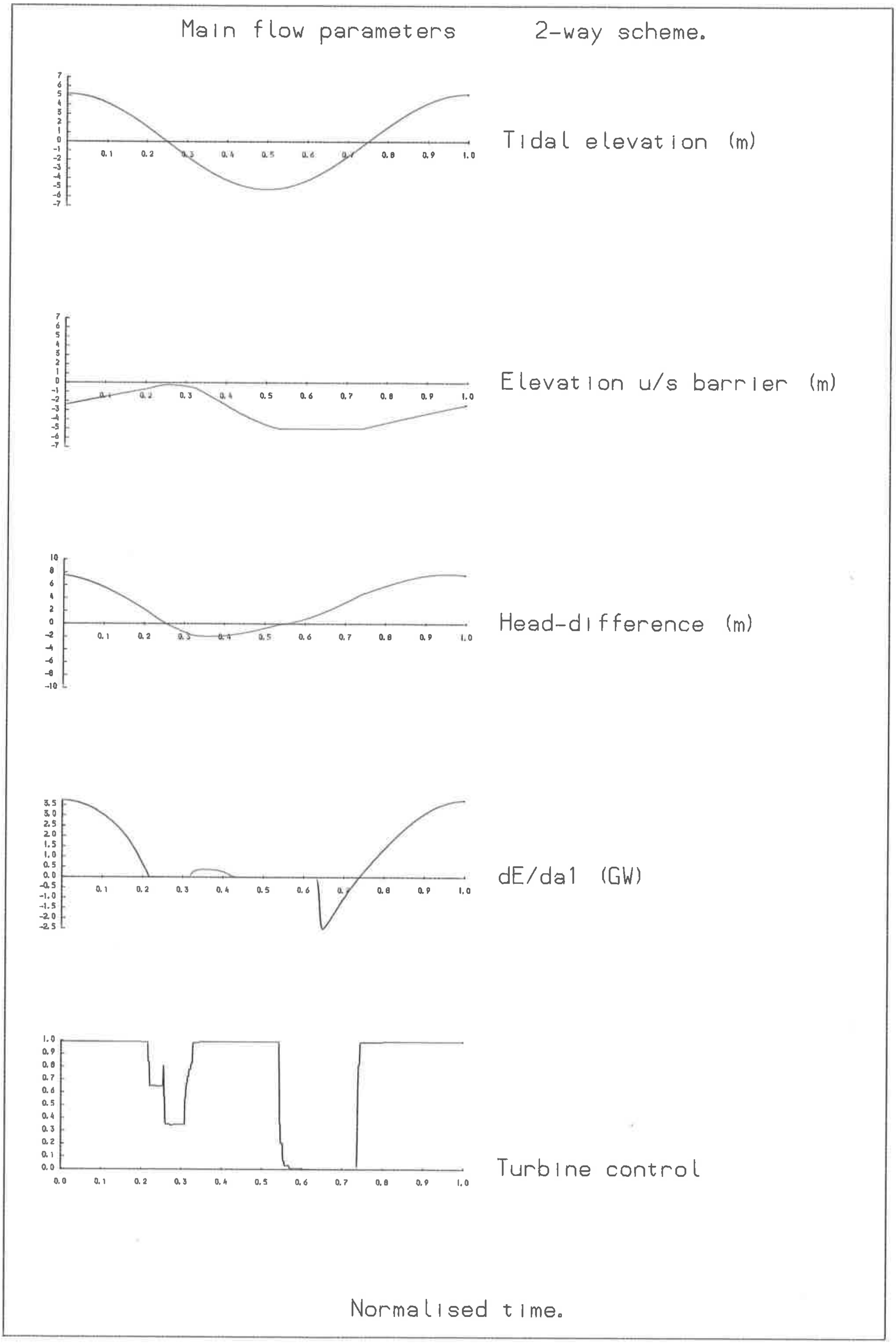

Main flow parameters Ebb scheme.
Tidal elevation (m)
Elevation u/s barrier (m)
Head-difference (m)
dE/da1 (GW)
Turbine control
Normalised time.

Figure 1a



Main flow parameters
dE/da2 (GW)
Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Instantaneous power (GW)
C(t)
Normalised time.

Figure 1b



Main flow parameters 2-way scheme.
Tidal elevation (m)
Elevation u/s barrier (m)
Head-difference (m)
dE/da1 (GW)
Turbine control
Normalised time.

Figure 2a

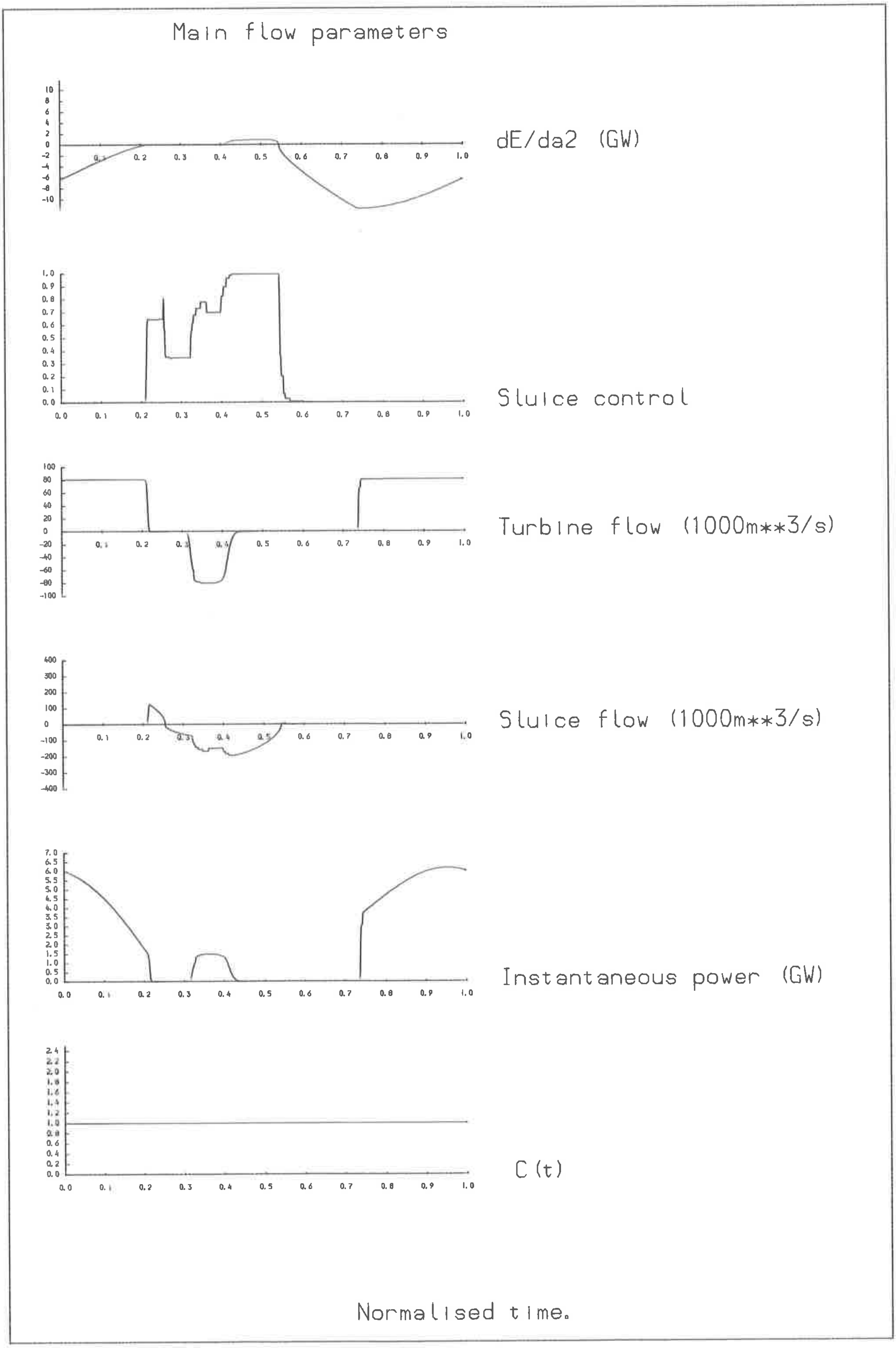

Main flow parameters
dE/da2 (GW)
Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Instantaneous power (GW)
C(t)
Normalised time.

Figure 2b

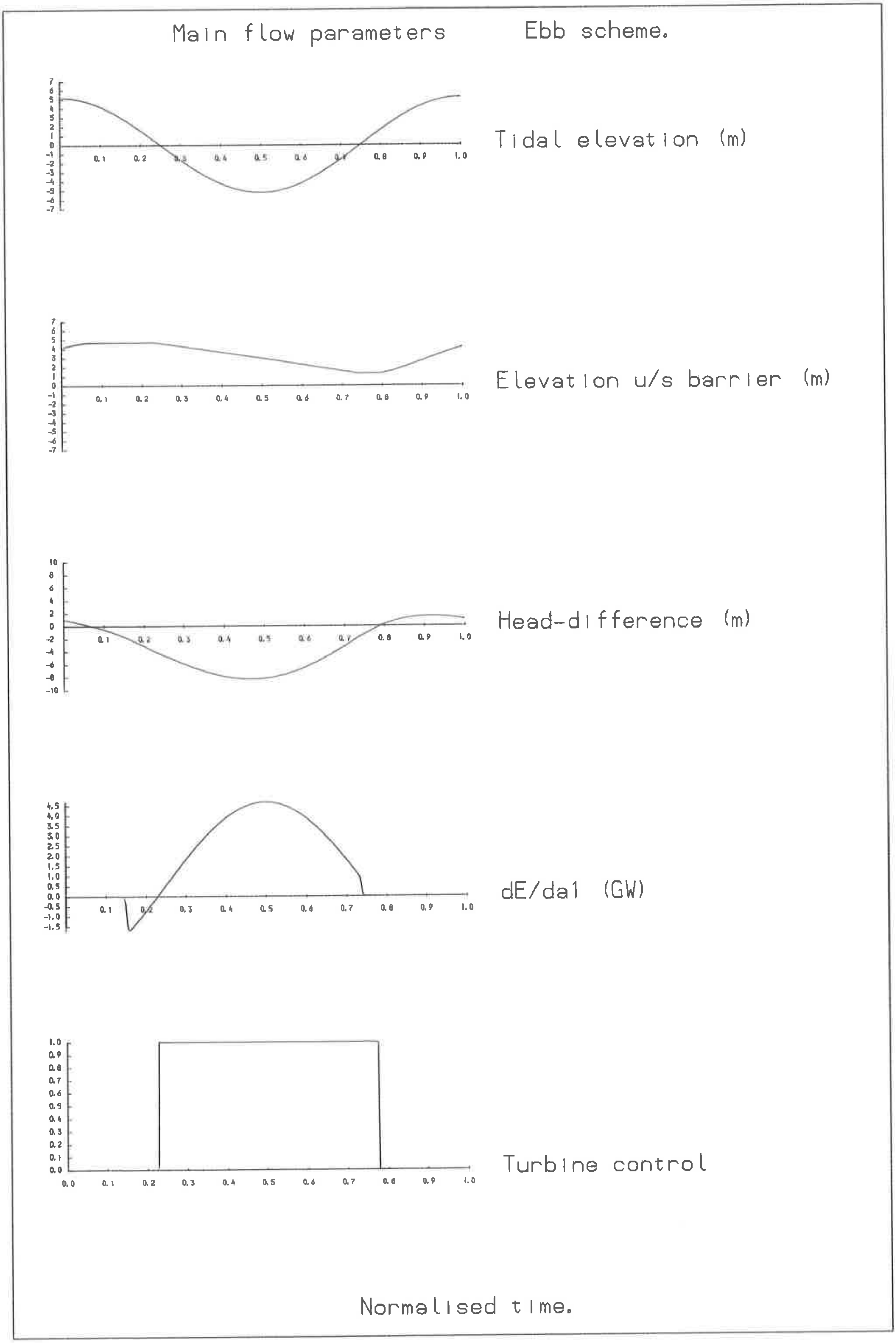

Main flow parameters
Ebb scheme.
Tidal elevation (m)
Elevation u/s barrier (m)
Head-difference (m)
dE/da1 (GW)
Turbine control
Normalised time.

Figure 3a

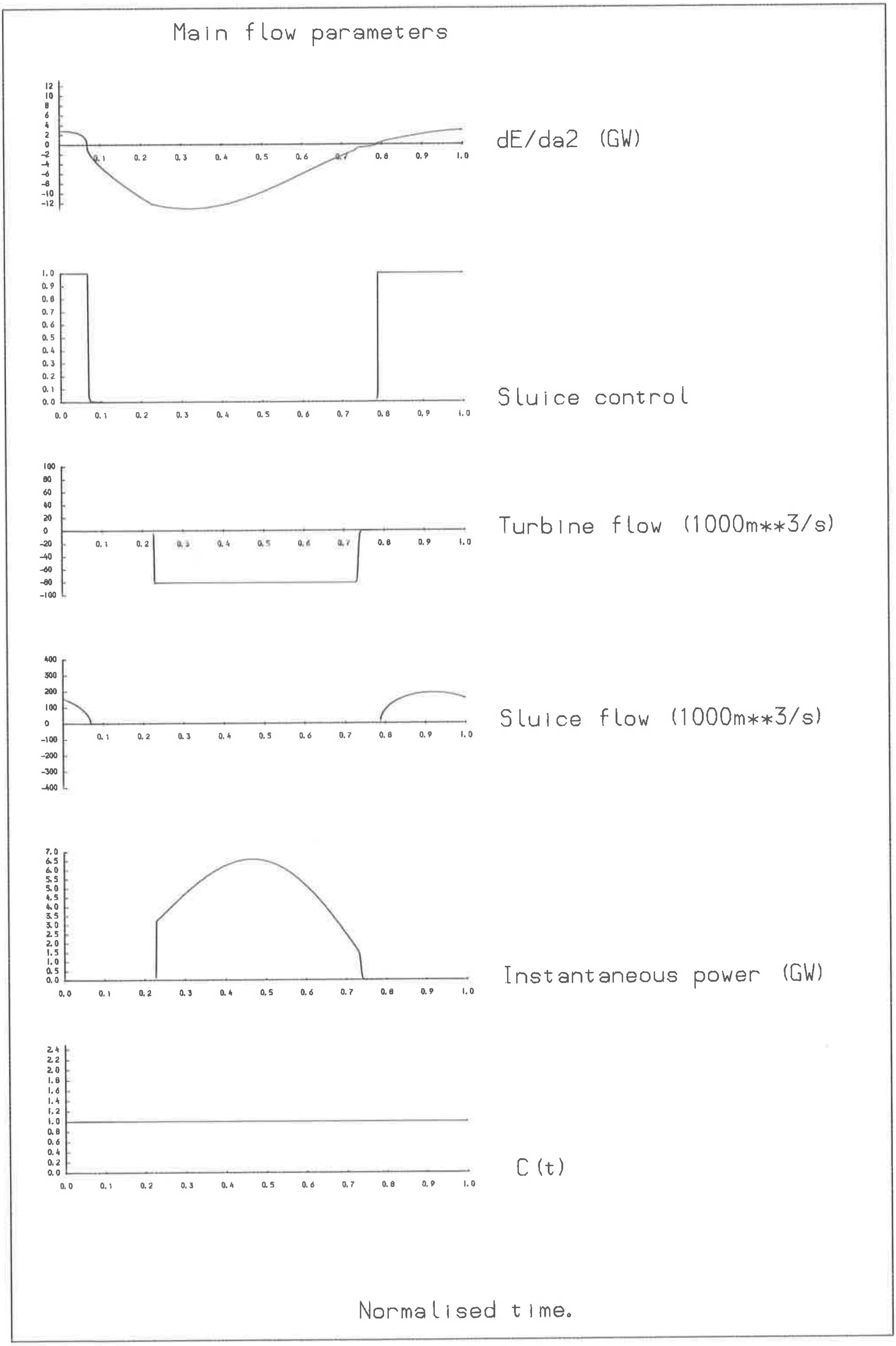

Main flow parameters
dE/da2 (GW)
Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Instantaneous power (GW)
C(t)
Normalised time.

Figure 3b

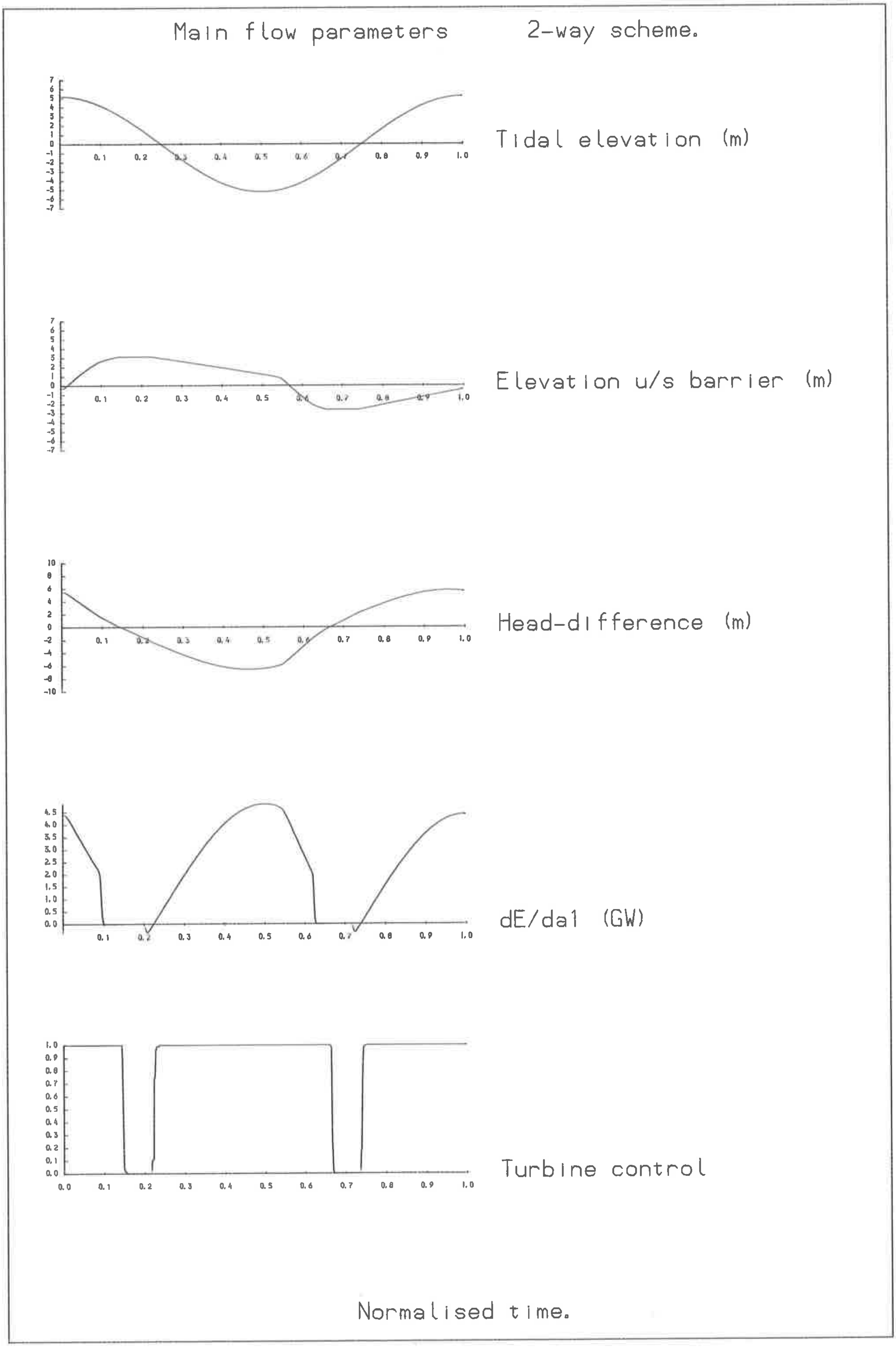

Main flow parameters 2-way scheme.
Tidal elevation (m)
Elevation u/s barrier (m)
Head-difference (m)
dE/da1 (GW)
Turbine control
Normalised time.

Figure 4a

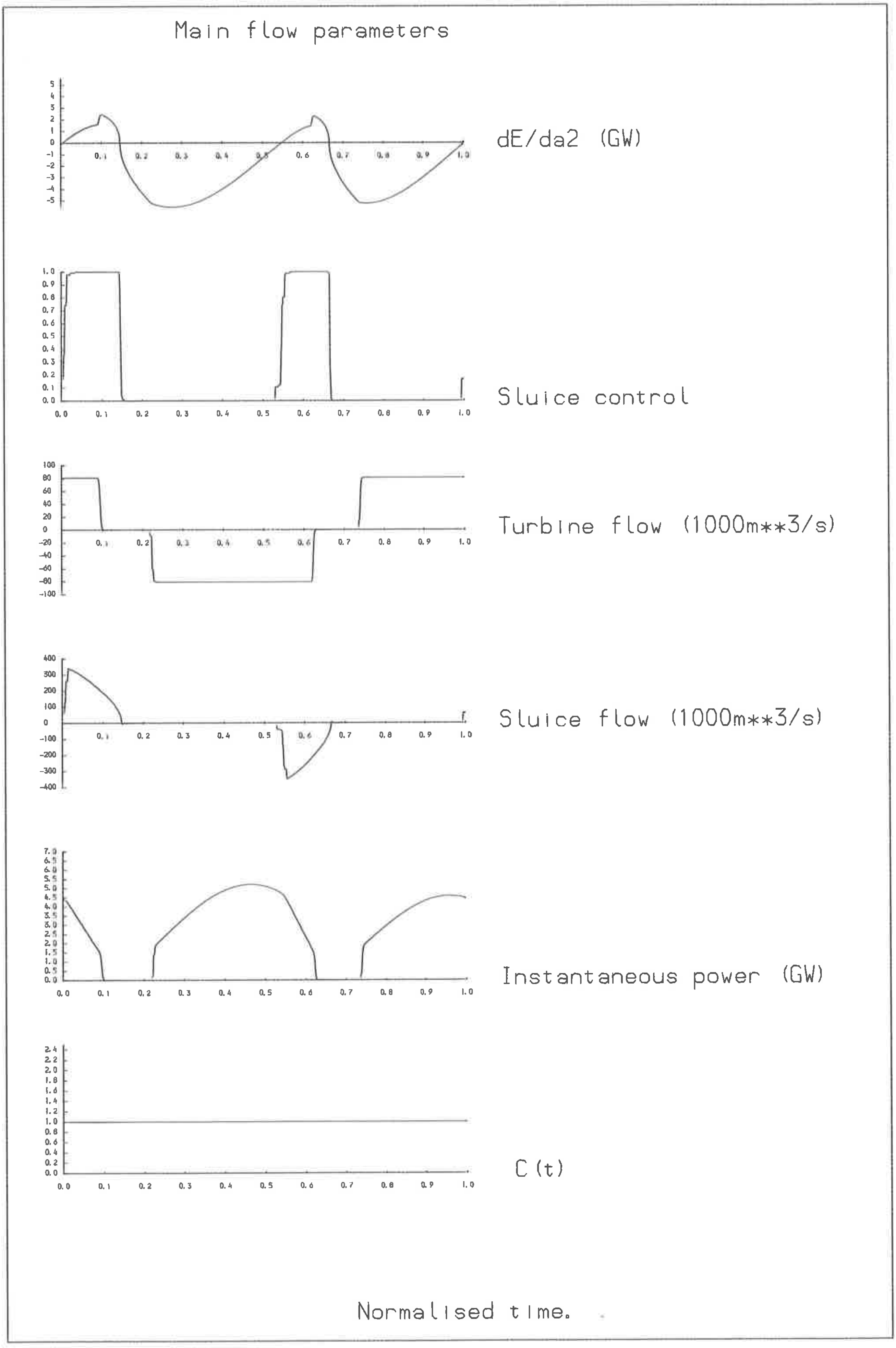

Main flow parameters
dE/da2 (GW)
Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Instantaneous power (GW)
C(t)
Normalised time.

Figure 4b

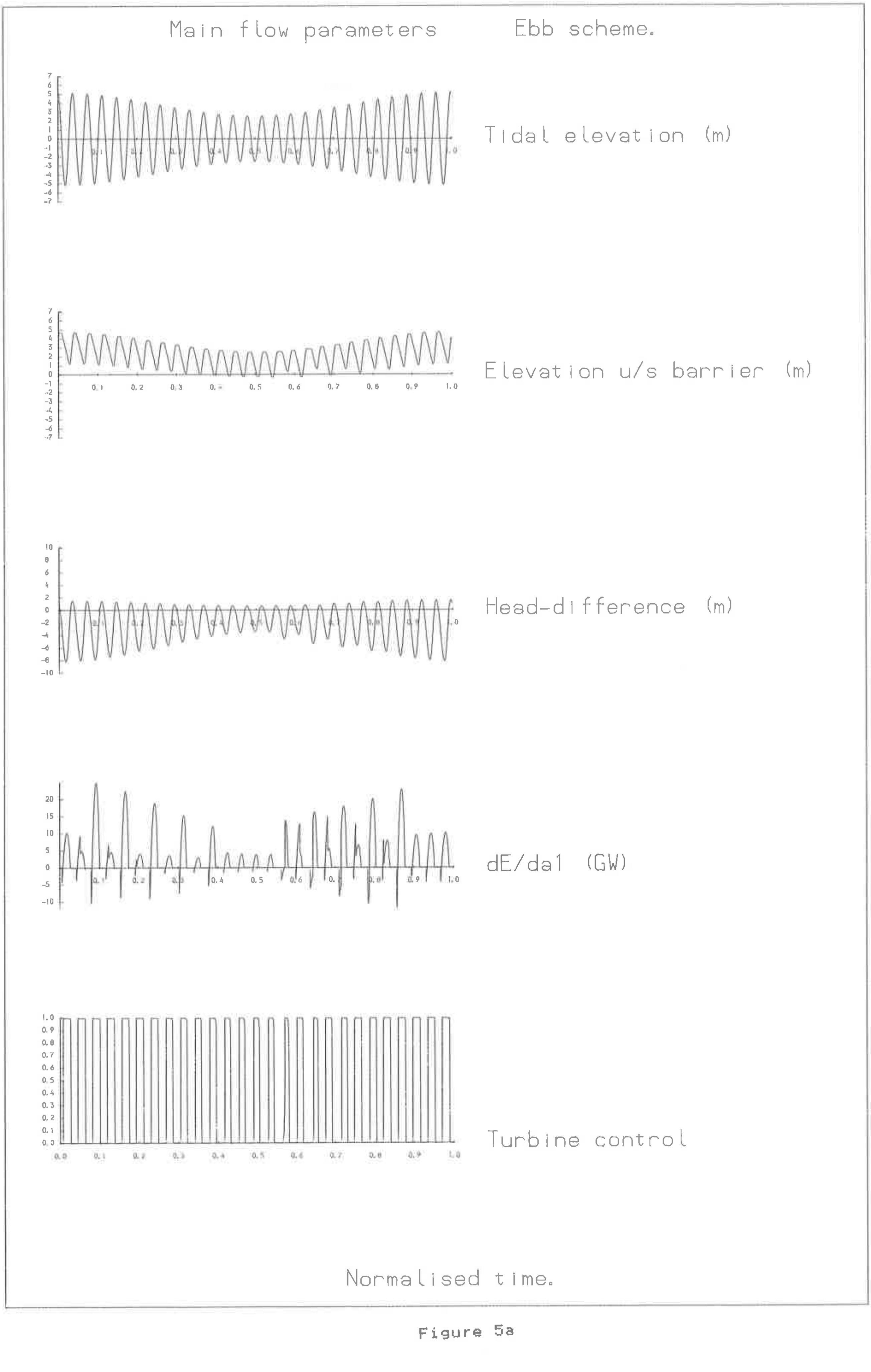

Main flow parameters
Ebb scheme.
Tidal elevation (m)
Elevation u/s barrier (m)
Head-difference (m)
dE/da1 (GW)
Turbine control
Normalised time.

Figure 5a

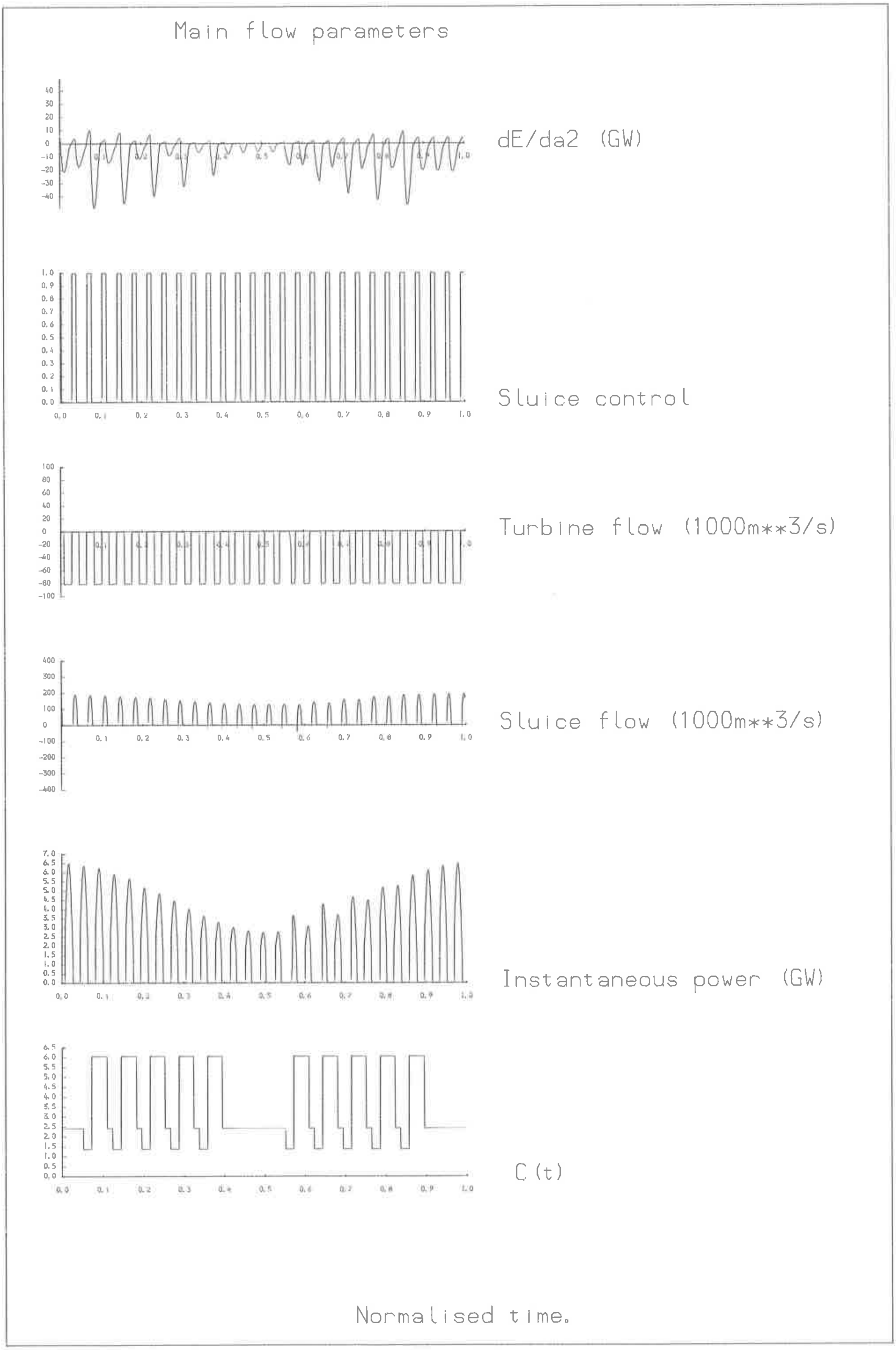

Main flow parameters
dE/da2 (GW)
Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Instantaneous power (GW)
C(t)
Normalised time.

Figure 5b

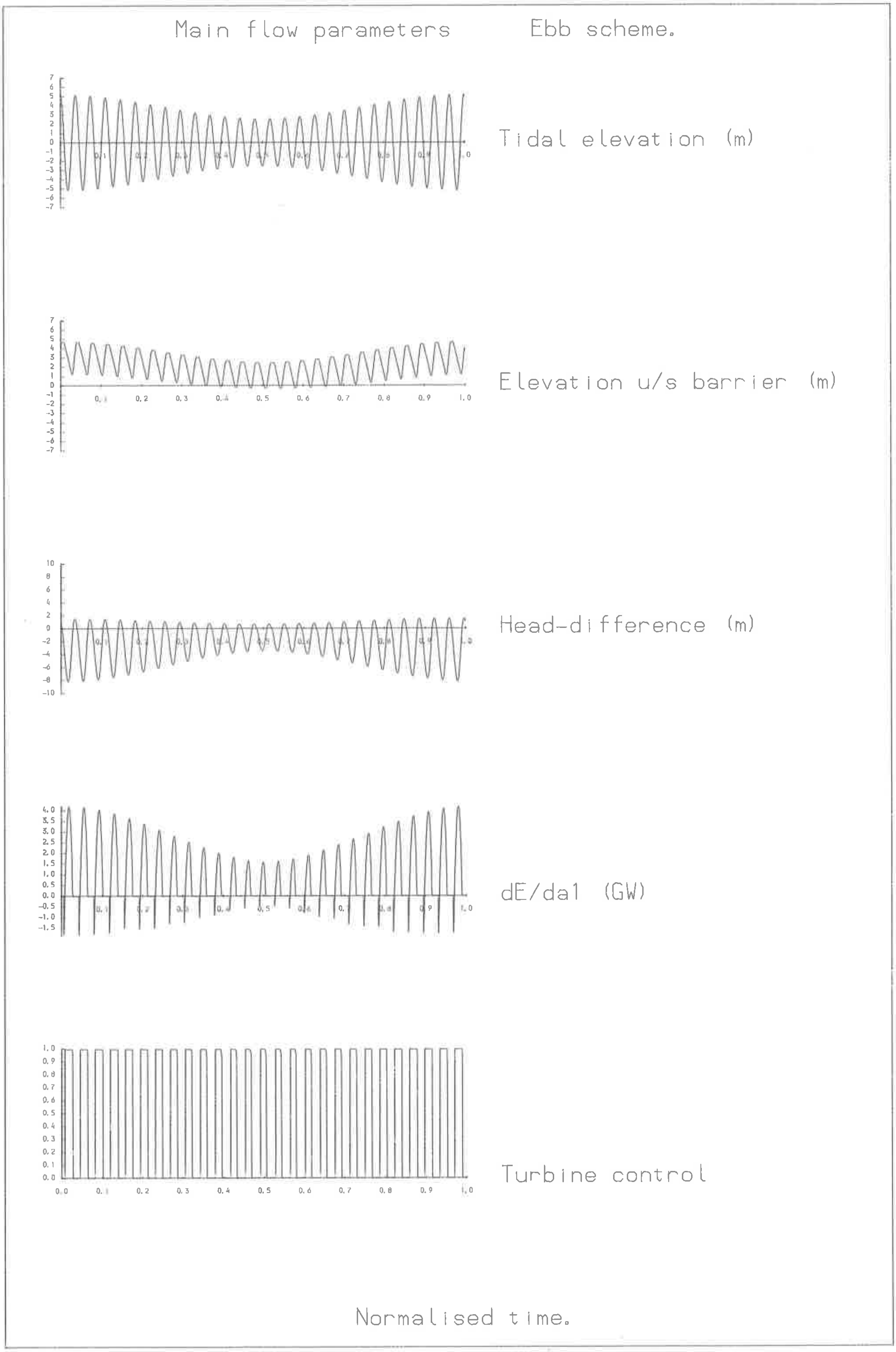

Main flow parameters Ebb scheme.
Tidal elevation (m)
Elevation u/s barrier (m)
Head-difference (m)
dE/da1 (GW)
Turbine control
Normalised time.

Figure 6a

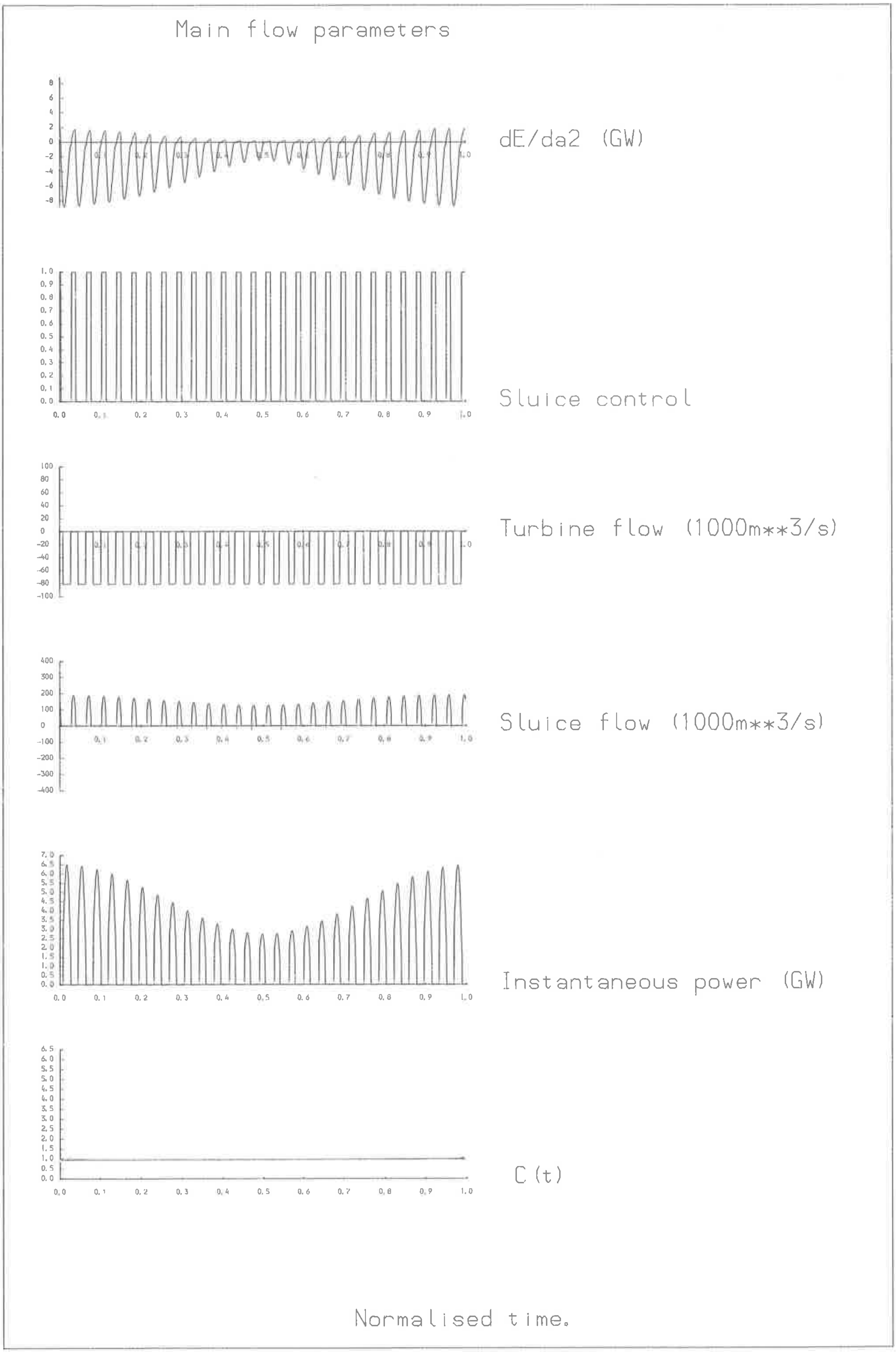

Main flow parameters
dE/da2 (GW)
Sluice control
Turbine flow (1000m**3/s)
Sluice flow (1000m**3/s)
Instantaneous power (GW)
C(t)
Normalised time.

Figure 6b