MOBILE FINITE ELEMENTS

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CONTENTS

ABSTRACT

A new adaptive grid method is presented which derives its motivation from the Moving Finite Element method (MFE) and makes use of the relationship with MFE to derive its discrete form. A framework is developed which encompasses both the new adaptive method (which we call here the Mobile Element Method (MEM)) and the MFE method. Analysis and results of both methods is presented.

1. INTRODUCTION

We shall study hyperbolic systems of conservation laws of the form

$$\underline{u}_t + \underline{f}_x = 0 \quad , \tag{1.1}$$

for example the Euler equations of Aerodynamics.

Baines & Wathen [1], Mosher [2] have attempted to find approximate solutions of these equations using the Moving Finite Element (MFE) Method of Miller [3], while Harten & Hyman [4], Winkler [5], and others have approached solutions of the equations using moving or adaptive finite difference methods. Many authors have successfully applied adaptive grid methods to scalar equations in 1-D but the generalisation to systems and to higher dimensions has proved a stumbling block. Here we concentrate on a method which is capable of both kinds of generalisation which has been derived directly from a local form of the MFE method. We present first the MFE method in the context of the scalar equation

$$u_t + f_x = 0 \tag{1.2}$$

with particular emphasis on its local formulation (see Baines [7]).

In the case of fixed finite elements an approximation to the function u is represented by

$$v = \sum a_j(t)\alpha_j(x,s) \tag{1.3}$$

where $a_j(t)$ are the amplitudes and $\alpha_j(x,s)$ are the basis functions (see Fig. 1). However, in the Moving Finite Element approximation the nodes are allowed to move with the solution and this nodal movement is built into the basis functions. Instead of (1.3) we have

$$v = \sum a_j(t)\alpha_j(x,s(t)) \tag{1.4}$$

(see Miller & Miller [8], Wathen & Baines [10]) where $a_j(t)$ are the amplitudes and α_j's the basis functions which move with the nodes. The variation of nodal position with time is represented by $s(t)$ (see Fig. 2). The time derivative approximation for the fixed finite element method is found by differentiating (1.3) with respect to time t, which gives

$$\frac{\partial v}{\partial t} = \sum \frac{\partial a_j(t)}{\partial t} \quad \alpha_j(x,s) \quad . \tag{1.5}$$

Following Miller [3], Lynch [9] and Baines [6] the time derivative for the MFE method can be derived from differentiating (1.4) with respect to time t. The result is

$$\frac{\partial v}{\partial t} = \sum (\dot{a}_j - \frac{\partial v}{\partial x} \dot{s}_j)\alpha_j \quad . \tag{1.6}$$

In order to clarify (1.6) an alternative description is presented which explains the transformation which is implicit in differentiating (1.4). First let us consider linear basis functions $\alpha_j(x,s(t))$. In the neighbouring elements of node j

$$\alpha_j = \frac{x-s_{j-1}}{s_j-s_{j-1}} \qquad x \in [s_{j-1},s_j]$$

$$\alpha_j = \frac{(s_{j+1}-x)}{(s_{j+1}-s_j)} \qquad x \in [s_j,s_{j+1}] \quad .$$

(See Fig. 2). The α_j are clearly dimensionless variables, and are such that $0 \leq \alpha_j \leq 1$, i.e. in an element $[s_{j-1},s_j]$,

$$v(x,t) = v_{j-1}\frac{(s_j-x)}{(s_j-s_{j-1})} + v_j\frac{(x-s_{j-1})}{(s_j-s_{j-1})} \quad .$$

Let
$$\xi = \frac{x-s_{j-1}}{(s_j-s_{j-1})}$$

$$1-\xi = \frac{(s_j-x)}{(s_j-s_{j-1})} \quad ,$$

so that
$$v = v_{j-1}(1-\xi) + v_j\,\xi$$

and $1-\xi$, ξ are now the local basis functions [c.f. Baines (6)]

$$\phi_{k1} = 1-\xi, \qquad \phi_{k2} = \xi \quad .$$

Now if the grid moves with a speed $\dot{x}$ then the local basis functions move with the grid (Fig. 3?) with no variations in the dimensionless variable ξ. Therefore the total derivative of ξ with respect to time t is zero. Thus

$$\frac{D}{Dt}\xi = 0$$

where $\frac{D}{Dt}$ denotes the mobile derivative moving with the local frame of reference, and therefore

$$\frac{D\alpha_j}{Dt} = 0 \quad . \tag{1.7}$$

Now instead of taking the partial derivative of (1.4) with respect to time t we take the <u>total</u> derivative $\left(\frac{D}{Dt}\right)$ of (1.4) to give

$$\frac{Dv}{Dt} = \sum_j \left(\frac{D\dot{a}_j}{Dt}\alpha_j + a_j\frac{D\alpha_j}{Dt}\right) = \sum \frac{Da_j}{Dt}\,\alpha_j \quad , \tag{1.8}$$

by (1.7).

Now

$$\frac{\partial v}{\partial t} = \frac{Dv}{Dt} - \frac{Dx}{Dt}\frac{\partial v}{\partial x} \quad , \tag{1.9}$$

and, assuming an isoparametric mapping for x of the form

$$x = \sum_j x_j \alpha_j(\xi) \quad , \tag{1.10}$$

$$\frac{Dx}{Dt} = \sum_j \frac{Dx_j}{Dt} \alpha_j(\xi) \quad . \tag{1.11}$$

Using (1.11), (1.8) in (1.9) we have

$$\frac{\partial v}{\partial t} = \sum_j \frac{Da_j}{Dt} \alpha_j - \sum_j \dot{x}_j \alpha_j \frac{\partial v}{\partial x} \quad . \tag{1.12}$$

Clearly (1.12) is identical to (1.6) with $s_j = x_j$. But now the transformation is absolutely clear, and we see that the MFE method relies on (1.9) to build in the mobile operator. Although we have considered linear basis functions the result is also clear for higher dimensions and higher order basis functions. We note that the identity (1.9) can be used to rewrite (1.2) as

$$\frac{Du}{Dt} - \dot{x}\frac{\partial u}{\partial x} + f_x = 0. \tag{1.13}$$

We shall be returning to this form of the equation throughout this analysis.

Now following Baines [6], [7], we briefly describe the local MFE procedure in 1-D. We start with (1.12) which may be written in the local element form

$$v_t = \sum_k \sum_{i=1}^{2} (\dot{a}_{ki} - (v_x)_k \dot{s}_{ki})\phi_{ki}$$

$$= \sum_k \sum_{i=1}^{2} \frac{w_{ki}\phi_{ki}}{\Delta s_k}$$

where

$$w_{ki}/\Delta s_k = \dot{a}_{ki} - (v_x)_k \dot{s}_{ki} \tag{1.14}$$

and $\frac{Da_k}{Dt}$ is denoted by $\dot{a}_k$. For linear basis functions $(v_x)_k$ is constant in each element. The local MFE method proceeds to minimise the L_2 norm of the residual, i.e.

$$\| v_t + f_x \|_2 = \| \sum \frac{w_{ki}\phi_{ki}}{\Delta s_k} + f(a_{ki}\phi_{ki})_x \| \tag{1.15}$$

over w_{ki}. This results in matrix systems

$$C_k \underline{w}_k = \underline{b}_k \qquad \forall k \tag{1.16}$$

where

$$C_k = \begin{pmatrix} \langle \phi_{k1}, \phi_{k1} \rangle & \langle \phi_{k1}, \phi_{k2} \rangle \\ \langle \phi_{k2}, \phi_{k1} \rangle & \langle \phi_{k2}, \phi_{k2} \rangle \end{pmatrix} \tag{1.17}$$

$$\underline{w}_k = \begin{pmatrix} w_{k1}/\Delta s_k \\ w_{k2}/\Delta s_k \end{pmatrix}, \quad \underline{b}_k = \begin{bmatrix} -\langle \phi_{k1}, f_x \rangle \\ -\langle \phi_{k2}, f_x \rangle \end{bmatrix} = \begin{bmatrix} b_{k1} \\ b_{k2} \end{bmatrix}$$

Also using (1.14) we have two equations at each node connecting the element information in (1.16) to the speeds $(\dot{a}_k, \dot{s}_k)$, namely,

$$\begin{pmatrix} 1 & -\Delta_k/\Delta s_k \\ 1 & -\Delta_{k+1}/\Delta s_{k+1} \end{pmatrix} \begin{pmatrix} \dot{a}_k \\ \dot{s}_k \end{pmatrix} = \begin{pmatrix} w_{2k}/\Delta s_k \\ w_{1,k+1}/\Delta s_{k+1} \end{pmatrix} \tag{1.18}$$

which can be written as

$$M_k \dot{\underline{y}}_k = \underline{w}_k \quad , \tag{1.19}$$

where

$$M_k = \begin{pmatrix} 1 & -\Delta_k \Delta s_k \\ 1 & -\Delta_{k+1}/\Delta s_{k+1} \end{pmatrix}, \quad \dot{\underline{y}}_k = \begin{pmatrix} \dot{a}_k \\ \dot{s}_k \end{pmatrix}, \quad \underline{w}_k = \begin{pmatrix} w_{2k}/\Delta s_k \\ w_{1k+1}/\Delta s_{k+1} \end{pmatrix},$$

$$\Delta_k = a_k - a_{k-1}, \qquad \Delta s_k = s_k - s_{k-1}$$

and $\Delta_k = a_k - a_{k-1}$. The local MFE procedure is to solve (1.17) for $\underline{w}_k$ and, following Baines [11],

$$\begin{pmatrix} w_{1k} \\ w_{2k} \end{pmatrix} = \begin{pmatrix} 4b_{k1} - 2b_{k2} \\ -2b_{k1} + 4b_{k2} \end{pmatrix} \tag{1.20}$$

Then using (1.18) we can solve for $\dot{a}_k$ and $\dot{s}_k$ in the form

$$\left.\begin{aligned} \dot{a}_k &= \frac{(\Delta_{k+1}/\Delta s_{k+1})(w_{2k}/\Delta s_k) - (\Delta_k/\Delta s_k)(w_{1k+1}/\Delta s_{k+1})}{(\Delta_{k+1}/\Delta s_{k+1} - \Delta_k/\Delta s_k)} \\ \dot{s}_k &= \frac{(w_{2k}/\Delta s_k - w_{1k+1}/\Delta s_{k+1})}{(\Delta_{k+1}/\Delta s_{k+1} - \Delta_k/\Delta s_k)} = \dot{s}_{MFE} \quad \text{say} \end{aligned}\right\} \tag{1.21}$$

provided that $\dfrac{\Delta_{k+1}}{\Delta s_{k+1}} - \dfrac{\Delta_k}{\Delta s_k} \neq 0$.

We now evaluate the discrete form of $\underline{w}_k$ for the hyperbolic equation (1.2) using the definition of $\underline{b}_k$ in (1.17). Integrating by parts,

$$\underline{b}_k = - \begin{pmatrix} \int_{s_{k-1}}^{s_k} \phi_{k1} \dfrac{\partial f}{\partial x}\, dx \\ \int_{s_{k-1}}^{s_k} \phi_{k2} \dfrac{\partial f}{\partial x}\, dx \end{pmatrix} \tag{1.22}$$

$$= \begin{pmatrix} f_{k-1} - \hat{f} \\ -f_k + \hat{f} \end{pmatrix} \tag{1.23}$$

where

$$\hat{f} = \frac{1}{\Delta s_k} \int_{s_{k-1}}^{s_k} f\, dx \tag{1.24}$$

and by (1.20)

$$\underline{w} = \begin{pmatrix} w_{k1} \\ w_{k2} \end{pmatrix} = \begin{pmatrix} 4f_{k-1} - 6\hat{f} + 2f_k \\ -4f_k + 6\hat{f} - 2f_{k-1} \end{pmatrix} \qquad (1.25)$$

If the Trapezoidal rule is used to approximate the integral $\hat{f}$ of (1.24) then

$$\hat{f} = \frac{(f_k + f_{k-1})}{2} \qquad (1.26)$$

and, from (1.25),

$$\begin{pmatrix} w_{1\,k} \\ w_{2\,k} \end{pmatrix} = \begin{pmatrix} f_k - f_{k-1} \\ f_k - f_{k-1} \end{pmatrix} \qquad (1.27)$$

where $f_k = f(u_k)$. However, if Simpsons rule is used to approximate $\hat{f}$, then

$$\hat{f} = (f_{k-1} + 4f_{k-\frac{1}{2}} + f_k)/6 \qquad (1.28)$$

where

$$f_{k-\frac{1}{2}} = f\left[\frac{u_k + u_{k-1}}{2}\right] \qquad (1.29)$$

and (1.25) gives

$$\begin{pmatrix} w_{1k} \\ w_{2k} \end{pmatrix} = \begin{pmatrix} 3f_{k-1} - 4f_{k-\frac{1}{2}} + f_k \\ -3f_k + 4f_{k-\frac{1}{2}} - f_{k-1} \end{pmatrix} \qquad (1.30)$$

We note that, in the Trapezoidal case using (1.26) and (1.27), w_{1k} may be interpreted as a first order forward difference approximation to the derivative $-\frac{\partial f}{\partial x}$ at the point $k-1$, while w_{2k} may be interpreted as a first order backward difference approximation to $-\frac{\partial f}{\partial x}$ at k, i.e. using (1.27)

$$\begin{aligned} w_{1k} &= -\Delta s\left(\frac{\partial f}{\partial x}\right)_{k-1} + O(\Delta s^2) \\ w_{2k} &= -\Delta s\left(\frac{\partial f}{\partial x}\right)_{k} + O(\Delta s^2) \end{aligned} \qquad (1.31)$$

Similarly, in the Simpsons rule case using (1.28) and (1.30), w_{1k} represents a second order forward difference approximation to $-\frac{\partial f}{\partial x}$ at $k-1$ and w_{2k} represents a second order backward difference approximation to $-\frac{\partial f}{\partial x}$ at k, i.e. using (1.30)

$$\begin{aligned} w_{1k} &= -\Delta s\left(\frac{\partial f}{\partial x}\right)_{k-1} + O(\Delta s)^3 \\ w_{2k} &= -\Delta s\left(\frac{\partial f}{\partial x}\right)_{k} + O(\Delta s)^3 \quad . \end{aligned} \tag{1.32}$$

2. THE MOBILE ELEMENT METHOD (MEM)

To introduce this method we first rewrite equations (1.18) in the form

$$\begin{pmatrix} \sqrt{\Delta s_k} \\ \sqrt{\Delta s_{k+1}} \end{pmatrix} \dot{a}_k = \begin{pmatrix} (w_{2k} + \Delta_k \dot{s})/\sqrt{\Delta s_k} \\ (w_{1k+1} + \Delta_{k+1}\dot{s})/\sqrt{\Delta s_{k+1}} \end{pmatrix} \tag{2.1}$$

The square root factors are introduced in order to maintain a conservative scheme. This is an overdetermined system for $\dot{a}_k$, to solve for $\dot{a}_k$ in terms of $\dot{s}_k$ as a parameter, we may minimise the norm

$$\| D^{\frac{1}{2}} L\, \dot{a}_k - D^{-\frac{1}{2}} \overline{w} \|_2^2 \tag{2.2}$$

with respect to $\dot{a}_k$, where

$$D = \begin{pmatrix} \Delta s_k & 0 \\ 0 & \Delta s_{k+1} \end{pmatrix}, \quad L = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \overline{w} = \begin{pmatrix} w_{2k} + \Delta_k \dot{s} \\ w_{1k+1} + \Delta_{k+1}\dot{s} \end{pmatrix} \tag{2.3}$$

This gives

$$\dot{a}_k = \frac{(w_{2k} + w_{1k+1})}{(\Delta s_k + \Delta s_{k+1})} + \frac{(\Delta_{k+1} + \Delta_k)}{(\Delta s_k + \Delta s_{k+1})}\dot{s}_k \quad . \tag{2.4}$$

Having found $\dot{a}_k$ in terms of $\dot{s}_k$, it now remains to choose the $\dot{s}_k$. Before doing this, however, let us consider equation (2.4) in more detail.

We have already observed the form of the w's for two types of approximate integration in the case of the hyperbolic equation (1.2); from equations (1.31) and (1.32), for a uniform grid, we have

$$\frac{w_{2k} + w_{1k+1}}{\Delta s_k + \Delta s_{k+1}} = -\frac{\partial f}{\partial x} + O(\Delta s^2) \tag{2.5}$$

and since †

$$\frac{\Delta_{k+1} + \Delta_k}{\Delta s_{k+1} + \Delta s_k} = \frac{\partial u}{\partial x} + O(\Delta s^2) \tag{2.6}$$

then (2.4) represents a discrete analogue of equation (1.13). Now if we solve the hyperbolic equation (1.2) using characteristics, then from (1.13) we require that $\frac{Du}{Dt} = 0$. In the corresponding discrete version (2.4) we should require that $\dot{a}_k = 0$ so that the natural choice for $\dot{s}_k$, which corresponds most closely to characteristic description, is

$$\begin{aligned} \dot{s} &= -\frac{(w_{k2} + w_{1k+1})}{(\Delta_{k+1} + \Delta_k)} = \frac{\partial f}{\partial x} \Big/ \frac{\partial u}{\partial x} + O(\Delta s^2) \\ &= \dot{s}_{MEM} \quad \text{say} \end{aligned} \tag{2.7}$$

and $\dot{a}_k = 0.$

This might be referred to as a Lagrangian treatment, but a clear difference arises later when we go to systems. We note here, for future reference when we come to systems, that the same result is achieved if we think of (2.7) as minimising $\|\dot{a}_k\|^2$.

One final point to note here is that the $\dot{a}_k$ of the MFE method (1.21) can be recovered by substituting into (2.4) the $\dot{s}_{MFE}$ of (1.21). We shall return to this point again in section 5 where we look at the truncation error of the MFE method.

† (If $u_x = 0$ see Parallelism Appendix 2).

3. MEM FOR SYSTEMS OF EQUATIONS

We now consider systems of hyperbolic equations and seek an approximate solution using a single moving grid. First we consider the MFE method, see Wathen & Baines [10], Baines [6], Baines & Wathen [1]. For simplicity we demonstrate the analysis for a two component system (although there is an immediate generalisation to n components). For a two component system (1.1) may be written explicitly as

$$\left.\begin{aligned} u_t^{(1)} + f^{(1)}(u^{(1)},u^{(2)})_x &= 0 \\ u_t^{(2)} + f^{(2)}(u^{(1)},u^{(2)})_x &= 0 \end{aligned}\right\} \qquad (3.1)$$

As before, an elementwise minimisation over $w_{ki}^{(m)}$, $(m = 1,2)$, gives

$$c_k^{(m)} \underline{w}_k^{(m)} = \underline{b}_k^{(m)} \qquad (m = 1,2) \qquad (3.2)$$

(c.f. (1.17)) with obvious generalisations of c_k, $\underline{w}_k$ and $\underline{b}_k$.

However, for a single grid we have an overdetermined system to solve for $\dot{\underline{y}}$, i.e.

$$D^{\frac{1}{2}} M_k \underline{y}_k = D^{-\frac{1}{2}} \underline{w}_k \qquad (3.3)$$

(c.f. (1.18)), where now

$$D^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\Delta s_k} & 0 & 0 & 0 \\ 0 & \sqrt{\Delta s_{k+1}} & 0 & 0 \\ 0 & 0 & \sqrt{\Delta s_k} & 0 \\ 0 & 0 & 0 & \sqrt{\Delta s_{k+1}} \end{bmatrix}, \quad M_k = \begin{bmatrix} 1 & 0 & -\Delta_k^{(1)}/\Delta s_k \\ 1 & 0 & -\Delta_{k+1}^{(1)}/\Delta s_{k+1} \\ 0 & 1 & -\Delta_k^{(2)}/\Delta s_k \\ 0 & 1 & -\Delta_{k+1}^{(2)}/\Delta s_{k+1} \end{bmatrix} \qquad (3.4)$$

$$\dot{\underline{y}}_k = \begin{bmatrix} \dot{a}_k^{(1)} \\ \dot{a}_k^{(2)} \\ \dot{s}_k \end{bmatrix} , \qquad \underline{w}_k = \begin{bmatrix} w_{2k}^{(1)} \\ w_{1k+1}^{(1)} \\ w_{2k}^{(2)} \\ w_{1k+1}^{(2)} \end{bmatrix}$$

Again, the square root factors are introduced to give a conservative scheme. Now the local MFE (Baines [7]) proceeds to minimise

$$\| D^{\frac{1}{2}} M_k \, \dot{\underline{y}}_k - D^{-\frac{1}{2}} \underline{w}_k \|_2^2 \tag{3.5}$$

with respect to $\dot{\underline{y}}_k$, which results in the matrix system

$$M_k^T D M_k \dot{\underline{y}}_k = M_k^T \underline{w}_k \tag{3.6}$$

where

$$M^T D M = \begin{bmatrix} \Delta s_k + \Delta s_{k+1} & 0 & -(\Delta_k^{(1)} + \Delta_{k+1}^{(1)}) \\ 0 & \Delta s_k + \Delta s_{k+1} & -(\Delta_k^{(2)} + \Delta_{k+1}^{(2)}) \\ -(\Delta_k^{(1)} + \Delta_{k+1}^{(1)}) & -(\Delta_k^{(2)} + \Delta_{k+1}^{(2)}) & \sum\limits_{L=1}^{2} \dfrac{(\Delta_k^{(L)})^2}{\Delta s_k} + \dfrac{(\Delta_{k+1}^{(L)})^2}{\Delta s_{k+1}} \end{bmatrix} \tag{3.7}$$

$$M^T \underline{w} = \begin{bmatrix} w_{2k}^{(1)} + w_{1k+1}^{(1)} \\ w_{2k}^{(2)} + w_{k1k+1}^{(2)} \\ -\sum\limits_{L=1}^{2} \left(\dfrac{\Delta_k^{(L)} w_{2k}^{(L)}}{\Delta s_k} + \dfrac{\Delta_{k+1}^{(L)} w_{1k+1}^{(L)}}{\Delta s_{k+1}} \right) \end{bmatrix} \tag{3.8}$$

Solving (3.6) for $\dot{\underline{y}}_k$ in terms of $\underline{w}_k$ and $\dot{\underline{s}}_k$ gives

$$\dot{a}_k^{(L)} = \frac{(w_{2k}^{(L)} + w_{1k+1}^{(L)})}{(\Delta s_k + \Delta s_{k+1})} + \frac{(\Delta_k^{(L)} + \Delta_{k+1}^{(L)})}{(\Delta s_k + \Delta s_{k+1})} \dot{s}_k \tag{3.9}$$

$$\dot{s}_k = \frac{\sum_{L=1}^{2} \left[\frac{w_{2k}^{(L)}}{\Delta s_k} - \frac{w_{1k+1}^{(L)}}{\Delta s_{k+1}} \right] \left[\frac{\Delta_{k+1}^{(L)}}{\Delta s_{k+1}} - \frac{\Delta_k^{(L)}}{\Delta s_{k1}} \right]}{\sum_{L=1}^{2} \left(\frac{\Delta_{k+1}^{(L)}}{\Delta s_{k+1}} - \frac{\Delta_k^{(L)}}{\Delta s_k} \right)^2} . \tag{3.10}$$

Comparing (3.9) and (3.10) with the corresponding scalar MFE equation (1.21), we note that $\dot{s}$ given by (3.10) can be thought of as being derived from minimising

$$\sum_L \left\| \left(\frac{\Delta_{k+1}^{(L)}}{\Delta s_{k+1}} - \frac{\Delta_k^{(L)}}{\Delta s_k} \right) \dot{s} - \left(\frac{w_{2k}^{(L)}}{\Delta s_k} - \frac{w_{1k+1}^{(L)}}{\Delta s_{k+1}} \right) \right\|^2$$

with respect to $\dot{s}$.

Now we follow the MEM procedure given in section 2 and starting from (3.2) and (3.3) we regroup terms involving $\dot{s}$ in (3.3) on the right hand side of (3.3). Then we have a different over-determined system to solve, namely,

$$D^{\frac{1}{2}} L \underline{\dot{a}} = D^{-\frac{1}{2}} \underline{\overline{w}} \tag{3.11}$$

where $D^{\frac{1}{2}}$ is defined in (3.4) and

$$\underline{\dot{a}} = \begin{bmatrix} \dot{a}_k^{(1)} \\ \dot{a}_k^{(2)} \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \underline{\overline{w}} = \begin{bmatrix} w_{2k}^{(1)} + \Delta_k^{(1)} \dot{s} \\ w_{1k+1}^{(1)} + \Delta_{k+1}^{(1)} \dot{s} \\ w_{2k}^{(2)} + \Delta_k^{(2)} \dot{s} \\ w_{1k+1}^{(2)} + \Delta_{k+1}^{(2)} \dot{s} \end{bmatrix} \tag{3.12}$$

Minimisation of the L_2 norm of this matrix system with respect to $\underline{\dot{a}}$ results in the normal equations

$$L^T D L \, \underline{\dot{a}} = L^T \, \underline{\overline{w}} \quad . \tag{3.13}$$

This gives

$$\dot{a}_k^{(1)} = \frac{\left(w_{2k}^{(1)} + w_{1k+1}^{(1)}\right)}{(\Delta s_k + \Delta s_{k+1})} + \frac{\left(\Delta_k^{(1)} + \Delta_{k+1}^{(1)}\right)\dot{s}_k}{(\Delta s_k + \Delta s_{k+1})} \tag{3.14}$$

$$\dot{a}_k^{(2)} = \frac{\left(w_{2k}^{(2)} + w_{1k+1}^{(2)}\right)}{(\Delta s_k + \Delta s_{k+1})} + \frac{\left(\Delta_k^{(2)} + \Delta_{k+1}^{(2)}\right)\dot{s}_k}{(\Delta s_k + \Delta s_{k+1})} \tag{3.15}$$

Now, however, we cannot choose a unique $\dot{s}$ such that $\dot{a}_k^{(1)}, \dot{a}_k^{(2)} = 0$, but instead we can proceed by minimising

$$\| \dot{a}_k^{(L)} \|^2 = \sum_L (\dot{a}_k^{(L)})^2 \tag{3.16}$$

(c.f. (2.7)) with respect to $\dot{s}_k$. That is, we minimise

$$\left\| \begin{pmatrix} \Delta_k^{(1)} + \Delta_{k+1}^{(1)} \\ \Delta_k^{(2)} + \Delta_{k+1}^{(2)} \end{pmatrix} \dot{s}_k + \begin{pmatrix} w_{2k}^{(1)} + w_{1k+1}^{(1)} \\ w_{2k}^{(2)} + w_{1k+1}^{(2)} \end{pmatrix} \right\|_2^2 \tag{3.17}$$

with respect to $\dot{s}_k$, which results in

$$\dot{s}_k = - \frac{\sum_{L=1}^{2} (\Delta_k^{(L)} + \Delta_{k+1}^{(L)})(w_{2k}^{(L)} + w_{1k+1}^{(L)})}{\sum_{L=1}^{2} (\Delta_k^{(L)} + \Delta_{k+1}^{(L)})^2} . \tag{3.18}$$

Again we recall the observations of section 1 (c.f. (1.31) (1.32)) & see that equations (3.14) and (3.15) are a discrete analogue of the differential equation system (see also (2.5),(2.6))

$$\frac{Du^{(L)}}{Dt} = - \frac{\partial f^{(L)}}{\partial x} + \dot{s} \frac{\partial u^{(L)}}{\partial x} \quad (L = 1,2) \tag{3.19}$$

(c.f. (1.1)), and that the minimisation procedure is the discrete analogue of minimising

$$\sum_L \left\| \frac{Du^{(L)}}{Dt} \right\|^2 = \sum_L \left\| -\frac{\partial f^{(L)}}{\partial x} + \dot{s}\,\frac{\partial u^{(L)}}{\partial x} \right\|^2 \tag{3.20}$$

with respect to $\dot{s}$. We emphasise that the left hand side of (3.20) is the sum of the squares of the mobile derivatives $\frac{Du^{(L)}}{Dt}$, and in minimising this quantity we obtain the best mean path, i.e. the one which is nearest to a global mean "characteristic". This minimisation also aids stability since the sum of the total time derivatives have been minimised (also see below).

We note that the differential form of $\dot{s}$ obtained from minimising (3.20) and corresponding to (3.18) is

$$\dot{s} = \frac{\sum_L \frac{\partial f^{(L)}}{\partial x}\,\frac{\partial u^{(L)}}{\partial x}}{\sum_L \left(\frac{\partial u^{(L)}}{\partial x}\,\frac{\partial u^{(L)}}{\partial x}\right)} \tag{3.21}$$

Further, if we take the dot product of (3.19) with $\frac{\partial u^{(L)}}{\partial x}$, then in vector notation $\left(\frac{\partial \underline{u}}{\partial x} = \left\{\frac{\partial u^{(L)}}{\partial x}\right\},\ \frac{D\underline{u}}{Dt} = \left\{\frac{Du^{(L)}}{Dt}\right\}\right)$

$$\frac{\partial \underline{u}}{\partial x}\cdot\frac{D\underline{u}}{Dt} = -\frac{\partial \underline{u}}{\partial x}\cdot\frac{\partial \underline{f}}{\partial x} + \dot{s}\,\frac{\partial \underline{u}}{\partial x}\cdot\frac{\partial \underline{u}}{\partial x}$$

and by (3.21)

$$\frac{\partial \underline{u}}{\partial x}\cdot\frac{D\underline{u}}{Dt} = 0 \tag{3.22}$$

Clearly the discrete analogue can also be achieved i.e.

$$\sum_{L=1}^{2} (\Delta_k^{(L)} + \Delta_{k+1}^{(L)})\,\dot{a}_{(k)}^{(L)} = 0 \tag{3.23}$$

Equation (3.22) llustrates that we may eliminate one of the unknown mobile operators $\dot{a}_k^{(L)}$ from the equations, i.e. we may obtain one $\dot{a}_k^{(L)}$ using a linear combination of the other operators. [This result is consistent with applying the moving grid method to one equation]. For example, suppose we take the Euler equations: we may choose

$$\frac{D\rho}{Dt} = 0 \quad , \qquad \dot{s} = \frac{\partial m}{\partial x} \Big/ \frac{\partial \rho}{\partial x}$$

where ρ is the density and m the momentum.

Although this elimination is clearly possible this approach effectively chooses a specific $\dot{s}$ tied to the density equation only and has lost the good stability property of (3.21) (see also Baines & Wathen [1]).

Returning now to equations (3.22) and (3.23), and the previously described Mobile Element Method (MEM), (3.23) might also be used as a constraint condition for the local MFE method of Baines [7]; the $\dot{s}_{MFE}$ would then clearly be attempting to match (3.21).

We can use (3.22) to obtain $\left\| \frac{D\underline{u}}{Dt} \right\|^2$. Take the dot product of (3.19) with $\frac{Du^{(L)}}{Dt}$ giving

$$\left\| \frac{D\underline{u}}{Dt} \right\|^2 = \frac{D\underline{u}}{Dt} \cdot \frac{D\underline{u}}{Dt} = -\frac{\partial \underline{f}}{\partial x} \cdot \frac{D\underline{u}}{Dt} + \dot{s}\, \frac{\partial \underline{u}}{\partial x} \cdot \frac{D\underline{u}}{Dt} \qquad (3.24)$$

which, by (3.22),

$$= -\frac{\partial \underline{f}}{\partial x} \cdot \frac{D\underline{u}}{Dt}$$

$$= -\frac{\partial \underline{f}}{\partial x} \cdot \left(-\frac{\partial \underline{f}}{\partial x} + \dot{s}\, \frac{\partial \underline{u}}{\partial x} \right)$$

using (3.21)

$$= \frac{\partial \underline{f}}{\partial x} \cdot \frac{\partial \underline{f}}{\partial x} - \frac{\left(\frac{\partial \underline{f}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x} \right)^2}{\left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x} \right)} \quad ,$$

and therefore

$$\left\|\frac{D\underline{u}}{Dt}\right\|^2 = \left\|\frac{\partial\underline{f}}{\partial x}\right\|^2 (1 - \cos^2\theta)$$

$$\text{where} \quad \cos^2\theta = \frac{(\underline{f}_x \cdot \underline{u}_x)^2}{(\underline{u}_x \cdot \underline{u}_x)(\underline{f}_x \cdot \underline{f}_x)} \leq 1 \quad .$$

For $\dot{s} \neq 0$,

$$\left\|\frac{D\underline{u}}{Dt}\right\|^2 < \left\|\frac{\partial\underline{f}}{\partial x}\right\|^2 = \left\|\frac{\partial\underline{u}}{\partial t}\right\|^2 \quad ,$$

and this is a further condition consistent with the scalar case, and implies stability of the mobile operators in the steady state.

Shock Speed

So far we have not discussed details of shock capturing for the MEM. Basically it will use the same procedure as that used in MFE (another advantage of staying close to MFE in the formulation).

The minimisation over $\dot{s}$ described above (3.20) can also be applied to the system of Rankine-Hugoniot shock jump equations giving the "best" speed for the conditions on $\dot{s}$. For a system with a unique shock speed S, each equation demands that

$$S[a^{(L)}] = [f^{(L)}] \qquad (L = 1,\ldots,n) \tag{3.25}$$

where

$$[a^{(L)}] = a_i^{(L)} - a_{i-1}^{(L)} \quad .$$

In the MFE (Wathen & Baines (10== a simple average of the equations (3.25) is used to obtain S,

i.e.
$$S_{MFE} = \frac{1}{n} \sum_{L=1}^{n} \frac{[f^{(L)}]}{[a^{(L)}]}$$

Following the MEM procedure (c.f. (3.21)) then at the shock S_{MEM} is chosen to minimise

$$\|S[a^{(L)}] - [f^{(L)}]\|_2^2 \qquad (3.26)$$

with respect to S, which gives

$$S_{MEM} = \frac{\sum_L [f^{(L)}][a^{(L)}]}{\sum_L [a^{(L)}][a^L]} \qquad (3.27)$$

which determines a unique shock speed compatible with the grid speed of (3.21).

Finally, we end this section on a similar "note" to the end of section 2. Comparing equations (3.14) (3.15) (3.18) with (3.9) (3.10) we see that the equations for $\dot{a}_k$ in terms of $\dot{s}_k$ are identical for both the MFE and MEM methods. We can easily recover MFE from MEM by using $\dot{s}_{MFE}$ in place of $\dot{s}_{MEM}$. We have already seen that the same is true for the scalar case (c.f. §2).

4.(A) A SIMPLE LINEARISED STABILITY ANALYSIS

Let us suppose we have a linear system of decomposed characteristic equations,

$$\left.\begin{aligned} v_t^{(1)} + \lambda^{(1)} v_x^{(1)} &= 0 \\ v_t^{(2)} + \lambda^{(2)} v_x^{(2)} &= 0 \quad . \end{aligned}\right\} \qquad (4.1)$$

If we apply MFE or MEM using a single grid to this system then, as $\lambda v^{(L)}$ is linear

$$\left.\begin{bmatrix} w_{1k}^{(L)} \\ w_{2k}^{(L)} \end{bmatrix} = - \begin{bmatrix} \lambda^{(L)}(v_k^{(L)} - v_{k-1}^{(L)}) \\ \lambda^{(L)}(v_k^{(L)} - v_{k-1}^{(L)}) \end{bmatrix}\right\} \qquad (4.2)$$

and by (3.9) (3.14) (3.15) we see that for this linearised case

$$\dot{v}_k^{(L)} = (\dot{s}_k - \lambda^{(L)}) \left(\frac{v_k^{(L)} - v_{k-1}^{(L)} + v_{k+1}^{(L)} - v_k^{(L)}}{(\Delta s_k + \Delta s_{k+1})}\right)$$

We make the following observations (which clearly apply to both the MFE and MEM schemes).

For a uniform grid,

$$\dot{v}_k^{(L)} = (\dot{s}_k - \lambda^{(L)}) \frac{(v_{k+1}^{(L)} - v_{k-1}^{(L)})}{2\Delta s} \tag{4.4}$$

which is decoupled between odd and even k. Further, for forward Euler time stepping this is a classic unconditionally unstable scheme (any instability will be carried upstream with speed $\dot{s}_k$). We give the form of $\dot{s}_{MEM}$ and $\dot{s}_{MFE}$ for this linearised decomposed case.

$$\dot{s}_{MEM} = \frac{\sum_L \lambda^{(L)} \left[\frac{v_{k+1}^{(L)} - v_{k-1}^{(L)}}{(\Delta s_k + \Delta s_{k+1})}\right]^2}{\sum_L \left[\frac{v_{k+1}^{(L)} - v_{k-1}^{(L)}}{(\Delta s_k + \Delta s_{k+1})}\right]^2} \tag{4.5}$$

$$= \frac{\sum_L \lambda^{(L)} (v_x^{(L)})^2}{\sum_L (v_x^{(L)})^2}$$

$$\dot{s}_{MFE} = \frac{\sum_L \lambda^{(L)} \left[\frac{(v_{k+1}^{(L)} - v_k^{(L)})}{(\Delta s_{k+1})} - \frac{(v_k^{(L)} - v_{k-1}^{(L)})}{(\Delta s_k)}\right]^2}{\sum_L \left[\frac{(v_{k+1}^{(L)} - v_k^{(L)})}{(\Delta s_{k+1})} - \frac{(v_k^{(L)} - v_{k-1}^{(L)})}{(\Delta s_k)}\right]^2} \tag{4.6}$$

$$= \frac{\sum \lambda^{(L)} (v_{xx}^{(L)})^2}{\sum (v_{xx}^{(L)})^2}$$

Clearly, for $\lambda^{(L)} = \dot{s}_k + O(\Delta s)$, then according to (4.4) both schemes may remain stable in this linear case. However, the analysis of MEM in section 3 is valid for general non-linear non-decomposed systems, so that in general MEM should be more stable than any similar single grid method for non-linear problems.

In an attempt to circumvent any possible instability using a single grid, an established fixed grid method may be applied to stabilise the system. This is outlined below. Consider the differential equation in the form

$$\frac{D\underline{u}}{Dt} = -\frac{\partial \underline{f}}{\partial x} + \dot{s}\frac{\partial \underline{u}}{\partial x} \tag{4.8}$$

and write it as

$$\frac{D\underline{u}}{Dt} = -\frac{\partial \underline{\bar{f}}}{\partial x} \tag{4.9}$$

where

$$\underline{\bar{f}} = \underline{f} - \dot{s}\underline{u} \tag{4.10}$$

and the matrix

$$\frac{\partial \bar{f}_i}{\partial u_j} = \frac{\partial f_i}{\partial u_j} - \dot{s}\,\delta_{ij} \tag{4.11}$$

We may now think of (4.9) as the fixed grid equation

$$\frac{\partial \underline{u}}{\partial t} + \frac{\partial \underline{\bar{f}}}{\partial x} = 0 \tag{4.12}$$

which may be solved using for example a suitable finite difference TVD scheme, but now the updated stabilised solution will be carried upstream to $x_i^n + \dot{s}_i \Delta t$. Since one of the motivations of this work is to avoid characteristic decompositions, this idea was tried using the Davis [12] TVD scheme with appropriate modifications for moving grid effects. Preliminary results show no clear improvement over the fixed grid version, but the investigation is proceeding and details of this work will be presented in a later report.

4.(B) RELATIONSHIP WITH CHARACTERISTIC WAVE SPEED

Consider the decomposed system

$$\frac{\partial v_i}{\partial t} + \lambda_i \frac{\partial v_i}{\partial x} = 0 \qquad i = 1...n \tag{4.13}$$

To apply the single grid MEM procedure to this system first rewrite (4.13) as (c.f. (1.9))

$$\frac{Dv_i}{Dt} + \lambda_i \frac{\partial v_i}{\partial x} - \dot{s} \frac{\partial v_i}{\partial x} = 0$$

Now minimise

$$\sum_i \left(\frac{Dv_i}{Dt}\right)^2 = \sum_i \left(\lambda_i \frac{\partial v_i}{\partial x} - \dot{s} \frac{\partial v_i}{\partial x}\right)^2 \tag{4.14}$$

over $\dot{s}$, which gives

$$\dot{s} = \frac{\sum_i \lambda_i \frac{\partial v_i}{\partial x} \frac{\partial v_i}{\partial x}}{\sum_i \frac{\partial v_i}{\partial x} \frac{\partial v_i}{\partial x}} \tag{4.15}$$

Note that

(i) we have minimised the sum of squares of the total derivatives of the Riemann invariants.

(ii) we have obtained the grid speed $\dot{s}$ as a linear sum of the wavespeeds with a weighting such that the speed will favour the eigenvalue corresponding to Riemann invariant with the largest gradient, which is a useful property to have when shocks may occur.

Now the general MEM has been applied to the non-decomposed system, and we have already noted that it ensures that the sum of squares of the total derivatives of the conservative variables is minimised. Let us turn now to the grid speed $\dot{s}$ (3.21) and write this as

$$\dot{s} = \frac{\underset{\sim}{u}_x^T \; A \; \underset{\sim}{u}_x}{\underset{\sim}{u}_x^T \; \underset{\sim}{u}_x} \tag{4.16}$$

where

$$A_{ij} = \frac{\partial f_i}{\partial u_j} \tag{4.17}$$

For a linear characteristic decomposition we can write

$$\underline{u} = R\underline{v} \tag{4.18}$$

where R is the matrix of the eigenvectors of A, and v the Riemann invariants. From (4.18)

$$\underset{\sim}{u}_x = R \; \underset{\sim}{v}_x \tag{4.19}$$

and, using (4.16), (4.19), we have

$$\dot{s} = \frac{\underset{\sim}{v}_x^T \; R^T \; A \; R \; \underset{\sim}{v}_x}{\underset{\sim}{v}_x^T \; R^T \; R \; \underset{\sim}{v}_x} \tag{4.20}$$

By the definition of R

$$R^{-1} \; A \; R = \Lambda = \mathrm{diag} \; (\lambda_i) \tag{4.21}$$

so that we can diagonalise A and write

$$\dot{s} = \frac{\underset{\sim}{v}_x^T \; R^T \; R \; \Lambda \; \underset{\sim}{v}_x}{\underset{\sim}{v}_x^T \; R^T \; R \; \underset{\sim}{v}_x} \tag{4.22}$$

Now if we suppose that a particular characteristic i, say, has a Riemann invariant with a large gradient then approximately

$$\underset{\sim}{v}_x \triangleq \begin{pmatrix} 0 \\ \vdots \\ (v_i)_x \\ \vdots \\ 0 \end{pmatrix} \tag{4.23}$$

so that by (4.22), (4.23)

$$\dot{s} \simeq \lambda_i \tag{4.24}$$

We conclude from this argument that the grid speed of the MEM picks out the characteristic which is likely to shock and moves the nodes with a corresponding approximate characteristic speed.

5. SYSTEMS OF EQUATIONS WITH SEPARATE GRIDS

The MFE method may be applied to systems of equations with a separate grid for each component of the system (see Wathen & Baines [10], Baines [6], [7]) and the MFE method retains its simple structure as in the scalar case (c.f. (1.14)-(1.21)), but with a suffix L added to indicate the L^{th} component of the system. We have

$$C^{(L)}\underline{w}^{(L)} = \underline{b}^{(L)} \tag{5.1}$$

and inverting $C^{(L)}$ we find as in Baines [11] that

$$\begin{bmatrix} w_{1k}^{(L)} \\ w_{2k}^{(L)} \end{bmatrix} = \begin{bmatrix} 4b_{1k}^{(L)} - 2b_{2k}^{(L)} \\ -2b_{1k}^{(L)} + 4b_{2k}^{(L)} \end{bmatrix} \tag{5.2}$$

and as before (c.f. (1.18))

$$\begin{bmatrix} 1 & -\Delta_k^{(L)}/\Delta s_k^{(L)} \\ 1 & -\Delta_{k+1}^{(L)}/\Delta s_{k+1}^{(L)} \end{bmatrix} \begin{bmatrix} \dot{a}_k^{(L)} \\ \dot{s}_k^{(L)} \end{bmatrix} = \begin{bmatrix} w_{2k}^{(L)}/\Delta s_k^{(L)} \\ w_{1k+1}^{(L)}/\Delta s_{k+1}^{(L)} \end{bmatrix} \tag{5.3}$$

i.e. $M^{(L)}\dot{\underline{y}}^{(L)} = \underline{w}^{(L)}$.

We can solve (5.3) to obtain

$$\dot{a}_k^{(L)} = \frac{(\Delta_{k+1}^{(L)}/\Delta s_{k+1}^{(L)})(w_{2k}^{(L)}/\Delta s_k^{(L)}) - (\Delta_k^{(L)}/\Delta s_k^{(L)})(w_{1k+1}^{(L)}/\Delta s_{k+1}^{(L)})}{\Delta_{k+1}^{(L)}/\Delta s_{k+1}^{(L)} - \Delta_k^{(L)}/\Delta s_k^{(L)}}$$

$$\dot{s}_k^{(L)} = \frac{w_{2k}^{(L)}/\Delta s_k^{(L)} - w_{1k+1}^{(L)}/\Delta s_{k+1}^{(L)}}{\Delta_{k+1}^{(L)}/\Delta s_{k+1}^{(L)} - \Delta_k^{(L)}/\Delta s_k^{(L)}} \tag{5.4}$$

As $f^{(L)}$ may be a function of $v^{(L)}$ $(L = 1,\dots,n)$, the main difficulty with this approach is the quadrature in $\underline{b}^{(L)}$ where

$$\underline{b}^{(L)} = \begin{bmatrix} \langle\phi_{1k}^{(L)}, \frac{\partial f^{(L)}}{\partial x}(v^{(1)}\dots v^{(n)})\rangle \\ \langle\phi_{2k}^{(L)}, \frac{\partial f^{(L)}}{\partial x}(v^{(1)}\dots v^{(n)})\rangle \end{bmatrix} \tag{5.5}$$

Since

$$a^{(L)}(x,t) = \sum_j a_j^{(L)} \alpha_j(x, s^{(L)}(t)) \tag{5.6}$$

we have values of $v^{(P)}(s^{(L)}_{(t)},t)$ for all P,L, but on different grids.

Before considering the MEM for separate grids it is necessary to consider the transformation derived in section 1 (1.12) and the consequence of its application to separate grids, both for MFE and MEM. First reconsider a system of equations written as

$$\frac{\partial v^{(L)}}{\partial t} + \frac{\partial f^{(L)}}{\partial x} = 0 \qquad (L = 1,\ldots,n) \tag{5.7}$$

By applying the MFE method to approximate (5.7) with n separate grids, using (1.12) n times, we see that n discrete mobile operators have been employed. Using (1.12) with suffix (L) to replace $\frac{\partial v^{(L)}}{\partial t}$ we have

$$\frac{\partial v^{(L)}}{\partial t} = \sum_j \frac{Da_j^{(L)}}{Dt} \alpha_j^{(L)} - \sum_j \dot{x}_j^{(L)} \alpha_j^{(L)} \frac{\partial v^{(L)}}{\partial x} \tag{5.8}$$

But in each application of (5.8) we have, by (1.10),

$$x = \sum_j x_j^{(L)} \alpha_j^{(L)} \tag{5.9}$$

so that the separate grids method relies on a non-unique mapping of x on to each grid moving with its own component of the system and, by (4.9),

$$\frac{Dx}{Dt} = \sum_j \frac{Dx_j^{(L)}}{Dt} \alpha_j^{(L)} \tag{5.10}$$

which is non-unique.

For separate grids MEM may be derived from MFE with a similar analysis to the scalar case (c.f. section 2). Instead of solving (5.3) for $\dot{y}^{(L)}$, multiply by $(D^{(L)})^{\frac{1}{2}}$ and group the terms involving $\dot{s}^{(L)}$ to the right hand side of (4.3) and minimise the overdetermined system with respect to $\dot{a}_k^{(L)}$ with $\dot{s}^{(L)}$ as a parameter, giving

$$\dot{a}_k^{(L)} = \frac{w_{2k}^{(L)} + w_{1k+1}^{(L)}}{(\Delta s_k^{(L)} + \Delta s_{k+1}^{(L)})} + \dot{s}^{(L)} \frac{(\Delta_k^{(L)} + \Delta_{k+1}^{(L)})}{(\Delta s_k^{(L)} + \Delta s_{k+1}^{(L)})} \tag{5.11}$$

Now using the identity

$$\frac{\partial v^{(L)}}{\partial t} = \frac{Dv^{(L)}}{Dt} - \frac{Dx}{Dt}\frac{\partial v^{(L)}}{\partial x}$$

we rewrite (5.7) as

$$\frac{Dv^{(L)}}{Dt} = -\frac{\partial f^{(L)}}{\partial x} + \frac{Dx}{Dt}\frac{\partial v^{(L)}}{\partial x} \tag{5.12}$$

and as in the scalar case (c.f. section 2) we may identify (5.11) as the discrete analogue of (5.12), but with $\frac{Dx}{Dt}$ being a non-unique grid speed. The final part of the MEM procedure is to minimise $\sum_L |\dot{a}_k^{(L)}|^2$ with respect to $\dot{s}^{(L)}$ to obtain

$$\left.\begin{aligned} \dot{s}_{k_{MEM}}^{(L)} &= -\frac{(w_{2k}^{(L)} + w_{1k+1}^{(L)})}{(\Delta_k^{(L)} + \Delta_{k+1}^{(L)})} \\ \dot{a}_{k_{MEM}}^{(L)} &= 0 \end{aligned}\right. \tag{5.13}$$

Finally we note that (5.13) represents the discrete analogue of solving (5.12) by using the non-unique $\frac{Dx}{Dt}$ to cancel the forcing term in (5.12) and set

$$\left.\begin{aligned} \frac{Dv^{(L)}}{Dt} &= 0 \\ \frac{Dx}{Dt} &= \frac{\partial f^{(L)}}{\partial x} \Big/ \frac{\partial v^{(L)}}{\partial x} \end{aligned}\right\} \tag{5.14}$$

for each component of the system.

6. TRUNCATION ERROR ANALYSIS FOR THE MFE METHOD AND FURTHER OBSERVATIONS

We have already observed in section 2 that for the scalar equation (1.2), the MFE approximation may be written as

$$\dot{a}_k = \frac{(w_{2k} + w_{1k+1})}{(\Delta s_k + \Delta s_{k+1})} + \dot{s}_{MFE} \frac{(\Delta_k + \Delta_{k+1})}{(\Delta s_k + \Delta s_{k+1})} \tag{6.1}$$

$$\dot{s}_{MFE} = \frac{(w_{2k}/\Delta s_k - w_{1k+1}/\Delta s_{k+1})}{(\Delta_{k+1}/\Delta s_{k+1} - \Delta_k/\Delta s_k)} \tag{6.2}$$

and for a uniform grid we have observed in section 2 that equation (2.4) (and therefore (6.1)) represents a discrete analogue of (1.13) to order $(\Delta s)^2$. That is, the spatial truncation error of (6.1) is

$$\tau = L(u) - L_h(u) =$$

$$\frac{Du}{Dt} + \frac{\partial f}{\partial x} - \dot{x}\frac{\partial u}{\partial x} - \left[\frac{Da_k}{Dt} - \frac{(w_{2k} + w_{1k+1})}{(\Delta s_k + \Delta s_{k+1})} - \dot{s}_{MFE}\frac{(\Delta_k + \Delta_{k+1})}{(\Delta s_k + \Delta s_{k+1})}\right]\Bigg|_{a=u} \tag{6.3}$$

and, using the discrete form of w from (1.30) and m,

$$\tau = -(\dot{x} - \dot{s}_{MFE})\frac{\partial u}{\partial x} + O(\Delta s)^2 \quad . \tag{6.4}$$

Therefore it remains to examine the form of (6.2).

Let us substitute the exact solution into the discrete approximation for $\dot{s}_{MFE}$, assuming a uniform grid and using Simpsons rule for the quadrature.

$$\dot{s}_{MFE} = \frac{[4(f_{k+\frac{1}{2}} + f_{k-\frac{1}{2}}) - (f_{k-1} + 6f_k + f_{k+1})]}{(\Delta_{k+1} - \Delta_k)} \tag{6.5}$$

where

$$h = \Delta s, \quad f_{k+\frac{1}{2}} = f\left(\frac{u_{k+1} + u_k}{2}\right), \quad f_k = f(u_k) \tag{6.6}$$

Now

$$\left.\begin{aligned} u_k &= u_{k+\frac{1}{2}} - \frac{h}{2}(u_x)_{k+\frac{1}{2}} + \left(\frac{h}{2}\right)^2 \tfrac{1}{2}(u_{xx})_{k+\frac{1}{2}} + O(h^3) \\ u_{k+1} &= u_{k+\frac{1}{2}} + \frac{h}{2}(u_x)_{k+\frac{1}{2}} + \left(\frac{h}{2}\right)^2 \tfrac{1}{2}(u_{xx})_{k+\frac{1}{2}} + O(h^3) \end{aligned}\right\} \qquad (6.7)$$

from which we obtain

$$\frac{u_k + u_{k+1}}{2} = u_{k+\frac{1}{2}} + \frac{h^2}{8}(u_{xx})_{k+\frac{1}{2}} + O(h^4) \qquad (6.8)$$

and

$$f\left(\frac{u_k + u_{k+1}}{2}\right) = f\left(u_{k+\frac{1}{2}} + \frac{h^2}{8}(u_{xx})_{k+\frac{1}{2}}\right) = f(\theta_{k+\frac{1}{2}}) \text{ , say} \qquad (6.9)$$

Now expanding $f(\theta)$ about $u_{k+\frac{1}{2}}$ we have

$$f(\theta_{k+\frac{1}{2}}) = f(u_{k+\frac{1}{2}}) + \frac{\partial f}{\partial u}\frac{h^2}{8}(u_{xx})_{k+\frac{1}{2}} + O(h^4) \qquad (6.10)$$

Similarly

$$f(\theta_{k-\frac{1}{2}}) = f(u_{k-\frac{1}{2}}) + \frac{\partial f}{\partial u}\frac{h^2}{8}(u_{xx})_{k-\frac{1}{2}} + O(h^4)$$

Since

$$\left.\begin{aligned} f(u_{k+\frac{1}{2}}) + f(u_{k-\frac{1}{2}}) &= 2f(u_k) + \frac{h^2}{4}(f_{xx})_k + O(h^4) \text{ ,} \\ f(u_{k+1}) + f(u_{k-1}) &= 2f(u_k) + h^2(f_{xx})_k + O(h^4) \text{ ,} \end{aligned}\right. \qquad (6.11)$$

and

$$\frac{\partial f}{\partial u}\frac{h^2}{8}(u_{xx})_{k+\frac{1}{2}} + \frac{\partial f}{\partial u}\frac{h^2}{8}(u_{xx})_{k-\frac{1}{2}} = \frac{\partial f}{\partial u}\frac{h^2}{4}(u_{xx})_k + O(h^4),$$

then the numerator of (6.5) can be written as

$$4\left(2f(u_k) + \frac{h^2}{4}(f_{xx})_k + \frac{\partial f}{\partial u}\frac{h^2}{4}(u_{xx})_k\right) - 8f(u_k) - h^2(f_{xx})_k + O(h^4)$$

$$= (u_{xx})_k \frac{\partial f}{\partial u}k\ h^2 + O(h^4) \text{ .} \qquad (6.12)$$

Since

$$\Delta_{k+1} - \Delta_k = u_{xx}h^2 + O(h^4)$$

we have

$$\dot{s}_{MFE} = \frac{\partial f}{\partial u} + O(h^2) \qquad (6.13)$$

Now choose $\dot{x} = \frac{\partial f}{\partial u}$

and we conclude that the local MFE scheme applied to the 1-D scalar wave equation may be identified as a subtle Lagrangian scheme or chracteristic method, with a spatial truncation error which is second-order accurate on a uniform grid for a general flux function f. By equation (6.1) we thus see that for MFE

$$\dot{a}_k = O(h^2) \qquad \dot{s}_{MFE} = \frac{\partial f}{\partial u} + O(h^2) \tag{6.14}$$

while for MEM (c.f. (2.7))

$$\dot{a}_k = 0 \qquad \dot{s}_{MEM} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial u}{\partial x}} + O(h^2) \tag{6.15}$$

Further Observations

In [11] Baines has shown that in an element, MFE satisfies a consistent entropy property, namely that

$$\frac{Dm}{Dt} = -m^2 \frac{\partial^2 f}{\partial v^2} \tag{6.16}$$

where m is the gradient of v in an element. Also that the velocity of the mid-point of the element segment of the solution satisfies

$$\dot{a}\cos\theta - \dot{s}\sin\theta = -\sin\theta \frac{\partial f}{\partial v} \tag{6.17}$$

Reconsider the differential equation (1.13) which we rewrite here as

$$\frac{Du}{Dt} = -\frac{\partial f}{\partial x} + \dot{s}\frac{\partial u}{\partial x} \tag{6.18}$$

we can write this as

$$\dot{u} - \dot{s}\, u_x = -f_u\, u_x \tag{6.19}$$

and, with

$$\tan\theta = u_x = m ,$$

we obtain (6.17). Further consider

$$\frac{D}{Dt}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \dot{s}\ \frac{\partial^2 u}{\partial x^2} \qquad (6.20)$$

but with

$$\frac{\partial u}{\partial t} = -\frac{\partial f}{\partial x} \quad .$$

Then (6.20) gives

$$\frac{D}{Dt}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 f}{\partial x^2} + \dot{s}\ \frac{\partial^2 u}{\partial x^2} \quad .$$

Now

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}\,(u_x)^2 + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2} \quad ,$$

so finally (6.20) gives

$$\frac{D}{Dt}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \left(\dot{s} - \frac{\partial f}{\partial u}\right)\ \frac{\partial^2 u}{\partial x^2} \quad . \qquad (6.21)$$

With

$$\dot{s} = \frac{\partial f}{\partial u} \quad , \qquad m = \frac{\partial u}{\partial x}$$

we may obtain (6.16) from (6.21), so that both of the local element properties of MFE are direct consequences of a Lagrangian characteristic scheme, which is consistent with the above truncation error analysis. Further all characteristic schemes clearly contain the entropy property (6.16).

7. MEM DERIVED FROM THE GLOBAL MFE

Starting from the usual MFE viewpoint, we approximate the time derivative by (c.f. (1.6))

$$\begin{aligned} v_t &= \sum_j \left(\dot{a}_j - \frac{\partial v}{\partial x}\,\dot{s}_j\right)\alpha_j \\ &= \sum_j \left(\dot{a}_j\alpha_j + \dot{s}_j\beta_j\right) \end{aligned} \qquad (7.1)$$

with

$$\beta_j = -\frac{\partial v}{\partial x}\alpha_j$$

The global MFE procedure is to minimise

$$\left\|\sum(\dot{a}_j\alpha_j + \dot{s}_j\beta_j) + \frac{\partial f}{\partial x}\right\|_2^2 \tag{7.2}$$

with respect to $\dot{a}_j$ and $\dot{s}_j$.

MEM may be derived as a global method by first minimising (7.2) with respect to $\dot{a}_j$ only regarding $\dot{s}$ as a parameter. This results in a system of equations for $\dot{a}_j$ and $\dot{s}_j$ involving inner products of α_j and β_j. Typically the j'th row is

$$\begin{aligned}&\langle\alpha_j,\alpha_{j-1}\rangle\dot{a}_{j-1} + \langle\alpha_j,\beta_{j-1}\rangle\dot{s}_{j-1} + \\ &\langle\alpha_j,\alpha_j\rangle\dot{a}_j + \langle\alpha_j,\beta_j\rangle\dot{s}_j + \\ &\langle\alpha_j,\alpha_{j+1}\rangle\dot{a}_{j+1} + \langle\alpha_j,\beta_{j+1}\rangle\dot{s}_{j+1} = -\langle\alpha_j, \frac{\partial f}{\partial x}\rangle\end{aligned} \tag{7.3}$$

Now we follow the MEM procedure by grouping all terms involving $\dot{s}$ on the right hand side of (7.3) to obtain a system of equations in the form

$$\begin{aligned}&\langle\alpha_j,\alpha_{j-1}\rangle\dot{a}_{j-1} + \langle\alpha_j,\alpha_j\rangle\dot{a}_j + \langle\alpha_j,\alpha_{j+1}\rangle\dot{a}_j = \\ &-\langle\alpha_j, \frac{\partial f}{\partial x}\rangle \quad -(\langle\alpha_j,\beta_{j-1}\rangle\dot{s}_{j-1} + \langle\alpha_j,\beta_j\rangle\dot{s}_j + \langle\alpha_j,\beta_{j+1}\rangle\dot{s}_{j+1})\end{aligned} \tag{7.4}$$

which we denote by

$$\bar{A}\,\underline{\dot{a}} = \underline{g} - Z\underline{\dot{s}} \tag{7.5}$$

where $\bar{A}$, Z are square tridiagonal matrices with entries

$$\bar{A}_{jj} = \langle\alpha_j,\alpha_j\rangle, \quad \bar{A}_{j,j-1} = \langle\alpha_j,\alpha_{j-1}\rangle, \quad \bar{A}_{j,k} = 0 \begin{Bmatrix} k > j+1 \\ k < j-1 \end{Bmatrix}$$

$$Z_{jj} = \langle \alpha_j , \beta_j \rangle , \quad Z_{j,j-1} = \langle \alpha_j , \beta_{j-1} \rangle$$

$$Z_{j,j+1} = \langle \alpha_j , \beta_{j+1} \rangle , \qquad Z_{jk} = 0 \quad \begin{cases} k < j-1 \\ k > j+1 \end{cases} \tag{7.6}$$
$$Z_{j+1,j} = \langle \alpha_{j+1} , \beta_j \rangle$$

and

$$g_j = - \langle \alpha_j , \frac{\partial f}{\partial x} \rangle$$

Now MEM proceeds to minimise

$$\| \bar{A} \dot{a} \|^2 = \| \underline{g} - Z \underline{\dot{s}} \|^2 \tag{7.7}$$

with respect to $\dot{s}_k$. This results in

$$Z^T Z \underline{\dot{s}} = Z^T \underline{g}$$

and since Z is square

$$\underline{\dot{s}} = (Z^T Z)^{-1} Z^T \underline{g} = Z^{-1} \underline{g} \quad . \tag{7.8}$$

Using (7.8) and (7.5) we see that for this case

$$\| \bar{A} \underline{\dot{a}} \|^2 = 0 \qquad \text{and} \quad \underline{\dot{a}} = 0 \quad .$$

This may be compared with the local element approach for the scalar equation : indeed for the scalar case this may be thought of as an implicit characteristic method.

Systems

For systems MEM first minimises

$$\sum_L \| \dot{a}_j^{(L)} \alpha_j + \dot{s}_j \beta_j^{(L)} + \frac{\partial f}{\partial x}^{(L)} \|^2 \tag{7.9}$$

with respect to $\dot{a}_j^{(L)}$. After taking all terms involving $\dot{s}$ to the right hand side this results in a system

$$\bar{A}^{(L)} \underline{\dot{a}}^{(L)} = \underline{g}^{(L)} - Z^{(L)} \underline{\dot{s}} \tag{7.10}$$

Then MEM proceeds to minimise

$$\sum_L \|\bar{A}^{(L)} \underline{\dot{a}}^{(L)}\|_2^2 \tag{7.11}$$

with respect to $\dot{s}$, i.e. equivalently minimising

$$\sum_L \|\underline{g}^{(L)} - Z^{(L)} \underline{\dot{s}}\|_2^2 \tag{7.12}$$

over the $\underline{\dot{s}}$ which results in

$$\sum_L Z^{(L)T} \underline{g}^{(L)} = \sum_L Z^{(L)T} Z^{(L)} \underline{\dot{s}} \tag{7.13}$$

For separate grids we have as in the scalar case,

$$\underline{\dot{s}}^{(L)} = Z^{(L)-1} \underline{g}^{(L)} \tag{7.14}$$

for each L, and by (7.10)

$$\|\bar{A}^{(L)} \underline{\dot{a}}^{(L)}\|^2 = 0 \tag{7.15}$$

As this global MEM approach is not being pursued further at present we give no further discussion here.

8. 2-D GENERALISATION

The discrete form of the MEM in 2-D will be given in a later report. As in 1-D the analogy with the differential case (i.e. analytic mobile operator) is followed. Here the general procedure in 2-D is given for the differential continuous case, together with further analysis and observations.

Starting with the scalar equation

$$u_t + f_x + g_y = 0 \tag{8.1}$$

this is written in the mobile operator form

$$\frac{Du}{Dt} + f_x - \dot{x}u_x + g_y - \dot{y}u_y = 0 \tag{8.2}$$

As in 1-D the scalar MEM proceeds along the lines of a characteristic method with

$$\left.\begin{aligned} \dot{x} &= f_x/u_x \\ \dot{y} &= g_y/u_y \\ \dot{u} &= 0 \end{aligned}\right\} \tag{8.3}$$

Systems with a Single Grid

We start with the mobile operator form

$$\frac{Du}{Dt} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} - \dot{x}\frac{\partial u}{\partial x} - \dot{y}\frac{\partial u}{\partial y} = 0 \quad . \tag{8.4}$$

As in the 1-D procedure we minimise

$$\left\| \frac{Du}{Dt} \right\|_2^2 \tag{8.5}$$

with respect to $\dot{x}$ and $\dot{y}$, which is equivalent to minimising

$$\left\| \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} - \dot{x}\frac{\partial u}{\partial x} - \dot{y}\frac{\partial u}{\partial y} \right\|_2^2 \tag{8.6}$$

with respect to $\dot{x}$ and $\dot{y}$. This gives two equations for $\dot{x}$ and $\dot{y}$, i.e.

$$\frac{\partial \underline{f}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x} + \frac{\partial \underline{g}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial x} = \dot{x}\, \frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x} + \dot{y}\, \frac{\partial \underline{u}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial x}$$

$$\frac{\partial \underline{f}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial y} + \frac{\partial \underline{g}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial y} = \dot{x}\, \frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial y} + \dot{y}\, \frac{\partial \underline{u}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial y} \qquad (8.7)$$

Then, solving for $\dot{x}$, $\dot{y}$,

$$\dot{x} = \frac{\left(\frac{\partial \underline{f}}{\partial x} + \frac{\partial \underline{g}}{\partial y}\right) \cdot \left[\frac{\partial \underline{u}}{\partial x}\left(\frac{\partial \underline{u}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial y}\right) - \frac{\partial \underline{u}}{\partial y}\left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial y}\right)\right]}{\left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x}\right)\left(\frac{\partial \underline{u}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial y}\right) - \left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial y}\right)^2}$$

$$\dot{y} = \frac{\left(\frac{\partial \underline{f}}{\partial x} + \frac{\partial \underline{g}}{\partial y}\right) \cdot \left[\frac{\partial \underline{u}}{\partial y}\left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x}\right) - \frac{\partial \underline{u}}{\partial x}\left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial y}\right)\right]}{\left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x}\right)\left(\frac{\partial \underline{u}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial y}\right) - \left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial y}\right)^2} \qquad (8.8)$$

<u>Parallelism</u>

If the denominator in (8.8) is zero then

$$\left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x}\right)\left(\frac{\partial \underline{u}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial y}\right) = \left(\frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial y}\right)^2 \qquad (8.9)$$

If also

$$\left.\begin{aligned} \frac{\partial \underline{u}}{\partial x} \cdot \frac{\partial \underline{u}}{\partial x} &= 0 \qquad \text{(a)} \\ \frac{\partial \underline{u}}{\partial y} \cdot \frac{\partial \underline{u}}{\partial y} &= 0 \qquad \text{(b)} \end{aligned}\right\} \qquad (8.10)$$

then $\dot{x}$, $\dot{y}$ are chosen as the average of the $\dot{x}$, $\dot{y}$ velocity components of the surrounding nodes. If (8.10) does not hold then (8.9) implies

$$\frac{\partial \underline{u}}{\partial x} = \lambda \frac{\partial \underline{u}}{\partial y} \qquad (8.11)$$

where λ is an arbitrary constant. In this case instead of minimising (8.6), we minimise

$$\left\| \frac{\partial \underline{f}}{\partial x} - \dot{x} \frac{\partial \underline{u}}{\partial x} \right\|^2 + \left\| \frac{\partial \underline{g}}{\partial y} - \dot{y} \frac{\partial \underline{u}}{\partial y} \right\|^2 \tag{8.12}$$

with respect to $\dot{x}$ and $\dot{y}$.

Further, suppose that for some I

$$\frac{\partial u_I}{\partial x} \neq 0 \qquad \frac{\partial u_J}{\partial x} = 0 \qquad J \neq I$$

$$\frac{\partial u_I}{\partial y} \neq 0 \qquad \frac{\partial u_J}{\partial y} = 0 \qquad J \neq I$$

Then the procedure given in (8.12) will reduce to the scalar case (8.3). If either (8.10a) or (8.10b) holds we choose the corresponding component speed to be an average of the neighbours as before.

Orthogonality

Returning to (8.4) and taking the dot product of (8.4) with $\frac{\partial \underline{u}}{\partial x}$, $\frac{\partial \underline{u}}{\partial y}$ respectively and using (8.7) gives

$$\left.\begin{aligned} \frac{D\underline{u}}{Dt} \cdot \frac{\partial \underline{u}}{\partial x} &= 0 \\ \frac{D\underline{u}}{Dt} \cdot \frac{\partial \underline{u}}{\partial y} &= 0 \end{aligned}\right\} \tag{8.13}$$

From (8.13) we see that it is possible to solve for two components of the of the unknown $\frac{D\underline{u}^{(L)}}{Dt}$ time derivatives in terms of a linear combination of the other time derivatives. In particular, if we have a two-component system to solve, then by (8.13)

$$\begin{bmatrix} \frac{\partial u^{(1)}}{\partial x} & \frac{\partial u^{(2)}}{\partial x} \\ \frac{\partial u^{(1)}}{\partial y} & \frac{\partial u^{(2)}}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{Du^{(1)}}{Dt} \\ \frac{Du^{(2)}}{Dt} \end{bmatrix} = 0 \tag{8.14}$$

and, for

$$D = \underline{\nabla} u^{(1)} \times \underline{\nabla} u^{(2)} \neq 0 \tag{8.15}$$

we have

$$\frac{Du^{(1)}}{Dt} = 0 \ , \qquad \frac{Du^{(2)}}{Dt} = 0 \ . \tag{8.16}$$

We note that for this case the equations of minimisation (8.7) may be written as

$$\begin{bmatrix} \frac{\partial u^{(1)}}{\partial x} & \frac{\partial u^{(2)}}{\partial x} \\ \frac{\partial u^{(1)}}{\partial y} & \frac{\partial u^{(2)}}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial f^{(1)}}{\partial x} + \frac{\partial g^{(1)}}{\partial y} \\ \frac{\partial f^{(2)}}{\partial x} + \frac{\partial g^{(2)}}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u^{(1)}}{\partial x} & \frac{\partial u^{(2)}}{\partial x} \\ \frac{\partial u^{(1)}}{\partial y} & \frac{\partial u^{(2)}}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial u^{(1)}}{\partial x} & \frac{\partial u^{(1)}}{\partial y} \\ \frac{\partial u^{(2)}}{\partial x} & \frac{\partial u^{(2)}}{\partial y} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \tag{8.17}$$

and, if (8.15) holds, (8.17) may be inverted to give

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} \frac{\partial u^{(2)}}{\partial y} \left(\frac{\partial f^{(1)}}{\partial x} + \frac{\partial g^{(1)}}{\partial y} \right) - \frac{\partial u^{(1)}}{\partial y} \left(\frac{\partial f^{(2)}}{\partial x} + \frac{\partial g^{(2)}}{\partial y} \right) \\ \frac{\partial u^{(1)}}{\partial x} \left(\frac{\partial f^{(2)}}{\partial x} + \frac{\partial g^{(2)}}{\partial y} \right) - \frac{\partial u^{(2)}}{\partial x} \left(\frac{\partial f^{(1)}}{\partial x} + \frac{\partial g^{(1)}}{\partial y} \right) \end{bmatrix} \tag{8.18}$$

For the inviscid Burgers equations in 2-D,

i.e.

$$u_t + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = 0$$
$$v_t + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = 0 \tag{8.19}$$

then

$$\begin{bmatrix} \frac{\partial f^{(1)}}{\partial x} + \frac{\partial g^{(1)}}{\partial y} \\ \frac{\partial f^{(2)}}{\partial x} + \frac{\partial g^{(2)}}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{8.20}$$

and therefore, by (8.17), (8.20),

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} . \tag{8.21}$$

In this case the grid nodes will move along the stream lines. Now consider the dot product of (8.4) with $\frac{D\underline{u}}{Dt}$, i.e.

$$\frac{D\underline{u}}{Dt} \cdot \frac{D\underline{u}}{Dt} = - \frac{\partial \underline{f}}{\partial x} \cdot \frac{D\underline{u}}{Dt} - \frac{\partial \underline{g}}{\partial y} \cdot \frac{D\underline{u}}{Dt} \tag{8.22}$$

where we have used (8.13). This gives

$$\left\| \frac{D\underline{u}}{Dt} \right\|^2 = \left(\frac{\partial \underline{f}}{\partial x} + \frac{\partial \underline{g}}{\partial y} \right) \cdot \left[\left(\frac{\partial \underline{f}}{\partial x} + \frac{\partial \underline{g}}{\partial y} \right) - \left(\dot{x} \frac{\partial \underline{u}}{\partial x} + \dot{y} \frac{\partial \underline{u}}{\partial y} \right) \right] \tag{8.23}$$

Let

$$\left.\begin{aligned} \underline{a} &= \frac{\partial \underline{u}}{\partial x} \\ \underline{b} &= \frac{\partial \underline{u}}{\partial y} \\ \underline{c} &= \frac{\partial \underline{f}}{\partial x} + \frac{\partial \underline{g}}{\partial y} \quad . \end{aligned}\right\} \tag{8.24}$$

Then

$$\left\| \frac{D\underline{u}}{Dt} \right\|^2 = \underline{c} \cdot \underline{c} - \dot{x}\, \underline{c} \cdot \underline{a} + \dot{y}\, \underline{c} \cdot \underline{b} \tag{8.25}$$

and using (8.8) and (8.24)

$$\begin{aligned} \dot{x} &= \frac{(\underline{c}\cdot\underline{a})(\underline{b}\cdot\underline{b}) - (\underline{c}\cdot\underline{b})(\underline{a}\cdot\underline{b})}{(\underline{a}\cdot\underline{a})(\underline{b}\cdot\underline{b}) - (\underline{a}\cdot\underline{b})^2} \\ \dot{y} &= \frac{(\underline{c}\cdot\underline{b})(\underline{a}\cdot\underline{a}) - (\underline{c}\cdot\underline{a}\ \underline{a}\cdot\underline{b})}{(\underline{a}\cdot\underline{a})(\underline{b}\cdot\underline{b}) - (\underline{a}\cdot\underline{b})^2} \quad . \end{aligned} \tag{8.26}$$

(8.25), (8.26) then give

$$\left\| \frac{D\underline{u}}{Dt} \right\|^2 = \underline{c} \cdot \underline{c} - \frac{(\underline{c}\cdot\underline{a}\ \underline{b} - \underline{c}\cdot\underline{b}\ \underline{a})^2}{(\underline{a}\cdot\underline{a}\ \underline{b}\cdot\underline{b} - (\underline{a}\cdot\underline{b})^2)} \quad , \tag{8.27}$$

by the Cauchy-Schwartz inequality

$$(\underline{a}\cdot\underline{b})^2 \leq (\underline{a}\cdot\underline{a})(\underline{b}\cdot\underline{b})$$

then

$$\left\|\frac{Du}{Dt}\right\|^2 \leq \left\|\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right\|^2 = \left\|\frac{\partial u}{\partial t}\right\|^2 \tag{8.28}$$

and again we conclude as in 1-D that this implies stability of the mobile operator in the steady state.

Systems with Separate Grids

As in the 1-D case we may introduce non-unique $\dot{x}$, $\dot{y}$ such that for the L^{th} component of the system

$$\frac{Du}{Dt}^{(L)} + \frac{\partial f}{\partial x}^{(L)} - \dot{x}^{(L)}\frac{\partial u}{\partial x}^{(L)} + \frac{\partial g}{\partial y}^{(L)} - \dot{y}^{(L)}\frac{\partial y}{\partial y}^{(L)} = 0$$

and take

$$\dot{x}^{(L)} = \frac{\partial f}{\partial x}^{(L)} \Big/ \frac{\partial u}{\partial x}^{(L)}$$

$$\dot{y}^{(L)} = \frac{\partial f}{\partial y}^{(L)} \Big/ \frac{\partial u}{\partial y}^{(L)}$$

$$\frac{Du}{Dt}^{(L)} = 0 \quad . \tag{8.29}$$

For the Euler equations in 2-D, we can either take four separate grids, one for each component, or use the method of the single grid minimisation, but applied on two grids where each grid will eliminate two variables. A natural pairing for the Euler equations is to solve for the two momentum components on one grid, and the density and energy on the second grid, the density and energy being the variables which can develop contact discontinuities as well as shocks. This results in

$$\frac{D(\rho u)}{Dt} = 0 \,, \quad \frac{D(\rho v)}{Dt} = 0 \,, \quad \begin{bmatrix} \frac{\partial f}{\partial x}(m_x) + \frac{\partial g}{\partial y}(m_x) \\ \frac{\partial f}{\partial x}(m_y) + \frac{\partial g}{\partial y}(m_y) \end{bmatrix} = \begin{bmatrix} \frac{\partial(\rho u)}{\partial x} & \frac{\partial(\rho u)}{\partial y} \\ \frac{\partial(\rho v)}{\partial x} & \frac{\partial(\rho v)}{\partial y} \end{bmatrix} \begin{bmatrix} \dot{x}^{(1)} \\ \dot{y}^{(1)} \end{bmatrix} \tag{8.30}$$

$$\frac{D\rho}{Dt} = 0 \qquad \frac{DE}{Dt} = 0\,, \qquad \begin{bmatrix} \dfrac{\partial f}{\partial x}(\rho) + \dfrac{\partial g}{\partial y}(\rho) \\ \dfrac{\partial f}{\partial x}(E) + \dfrac{\partial g}{\partial y}(E) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \rho}{\partial x} & \dfrac{\partial \rho}{\partial y} \\ \dfrac{\partial E}{\partial x} & \dfrac{\partial E}{\partial y} \end{bmatrix} \begin{bmatrix} \dot{x}^{(2)} \\ \dot{y}^{(2)} \end{bmatrix} \tag{8.31}$$

and we note that (8.30/31) correspond to minimising

$$\left(\frac{D(\rho u)}{Dt}\right)^2 + \left(\frac{D(\rho v)}{Dt}\right)^2 \qquad \text{w.r.t.} \quad \dot{x}^{(1)}, \dot{y}^{(1)}$$

and

$$\left(\frac{D\rho}{Dt}\right)^2 + \left(\frac{DE}{Dt}\right)^2 \qquad \text{w.r.t.} \quad \dot{x}^{(2)}, \dot{y}^{(2)} \tag{8.32}$$

9. MEM 2

In this section we give a brief account of a second derivation of MEM which results in a gradient weighted scheme, (but quite different from the gradient weighting of Miller [16]). Preliminary results indicate that this version is in fact superior to the first version in two dimensions, at least for a scalar problem. Full details of this scheme will be given in a subsequent report, but an outline is included here.

To illustrate the alternative version, MEM 2, we consider again the solution of the equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{9.1}$$

First we replace the u_t term with the identity (1.9) and rearrange (9.1) so that we have

$$\frac{Du}{Dt} = -\frac{\partial f}{\partial x} + \frac{Dx}{Dt}\frac{\partial u}{\partial x} \quad . \tag{9.2}$$

Now take the square of both sides of (9.2) and integrate over $-\infty \leqq x \leqq \infty$: then

$$\left\|\frac{Du}{Dt}\right\|^2 = \left\|-\frac{\partial f}{\partial x} + \frac{Dx}{Dt}\frac{\partial u}{\partial x}\right\|^2 \quad . \tag{9.3}$$

Now we can see that if we minimise

$$\left\|-\frac{\partial f}{\partial x} + \frac{Dx}{Dt}\frac{\partial u}{\partial x}\right\|^2 \tag{9.4}$$

over all possible $\frac{Dx}{Dt}$, then we actually minimise

$$\left\|\frac{Du}{Dt}\right\|^2 \tag{9.5}$$

over $\frac{Dx}{Dt}$. If we carry out this minimisation in the differential case we obtain

$$\int \left(-\frac{\partial f}{\partial x}\frac{\partial u}{\partial x} + \frac{Dx}{Dt}\left(\frac{\partial u}{\partial x}\right)^2 \right) dx = 0 \tag{9.6}$$

which is clearly satisfied if

$$\frac{Dx}{Dt} = \frac{\partial f}{\partial x} \Big/ \frac{\partial u}{\partial x} \tag{9.7}$$

and (9.7) is the usual expression for the characteristic speed.

Now if we make the usual M.F.E. approximations in (9.4) (see §1), we see that by minimising

$$\left\| -\frac{\partial f}{\partial x} + \sum_j \dot{s}_j \alpha_j \frac{\partial v}{\partial x} \right\|^2 \tag{9.8}$$

over $\dot{s}_j$ an approximate characteristic property is explicitly built into the MFE framework, with $\dot{s}_j$ given by

$$<\beta_i, \sum \dot{s}_j \beta_j> + <\beta_i, \frac{\partial f}{\partial x}> = 0 \tag{9.9}$$

We obtain the $\dot{a}$'s in the usual way by minimising

$$\left\| \sum_j \dot{a}_j \alpha_j + \sum \beta_j \dot{s}_j + \frac{\partial f}{\partial x} \right\|^2 \tag{9.10}$$

over $\dot{a}_j$. The resulting matrix system is

$$A\underline{\dot{y}} = \underline{g} \tag{9.11}$$

where A is block tridiagonal with a typical row

$$\begin{bmatrix} <\alpha_j, \alpha_{j-1}> & <\alpha_j, \beta_{j-1}> \\ 0 & <\beta_j, \beta_{j-1}> \end{bmatrix} \begin{bmatrix} <\alpha_j, \alpha_j> & <\alpha_j, \beta_j> \\ 0 & <\beta_j, \beta_j> \end{bmatrix} \begin{bmatrix} <\alpha_j, \alpha_{j+1}> & <\alpha_j, \beta_{j+1}> \\ 0 & <\beta_j, \beta_{j+1}> \end{bmatrix}$$

corresponding to the minimisation of (9.10) over $\dot{a}_j$ and (9.8) over $\dot{s}_j$ respectively at node j. (Note that unlike the MFE matrix, A is not symmetric). Also

$$\underline{\dot{y}} = \begin{bmatrix} \dot{a}_j \\ \dot{s}_j \end{bmatrix} \qquad\qquad \underline{g} = \begin{bmatrix} \langle \alpha_j , -\frac{\partial f}{\partial x} \rangle \\ \langle \beta_j , -\frac{\partial f}{\partial x} \rangle \end{bmatrix}$$

in the usual MFE notation.

The above equation (9.11) illustrates MEM 2 in the global form written in terms of $\underline{\dot{y}}$ so that it may easily be compared with MFE. There is no need to solve for $\underline{\dot{y}}$: since the second equation only involves $\underline{\dot{s}}$ from (9.9), this may be solved first, then the first equation of (9.11) can be used to solve for $\underline{\dot{a}}$. Both of these solutions then only involve the inversion of symmetric tridiagonal matrices.

As with the MEM of §2 the main advantage of this approach (besides being more explicit) is the corresponding treatment of systems, where we minimise

$$\sum_L \left\| \frac{Du^{(L)}}{Dt} \right\|^2 = \sum_L \int \frac{Du^{(L)}}{Dt} \frac{Du^{(L)}}{Dt} \, dx \qquad (9.12)$$

which results in a similar form for $\dot{x}$ as in §3 (with all the same properties in 1-D and 2-D, c.f. §3,4,8) but within an integral, i.e. minimising (9.12) over $\dot{x}$ gives

$$\sum_L \int \left(\dot{x} \frac{\partial u^{(L)}}{\partial x} \frac{\partial u^{(L)}}{\partial x} - \frac{\partial f^{(L)}}{\partial x} \frac{\partial u^{(L)}}{\partial x} \right) dx = 0 \qquad (9.13)$$

To obtain the discrete form for $\dot{s}$, minimise

$$\sum_P \left\| -\frac{\partial f^{(P)}}{\partial x} - \sum_j \dot{s}_j \beta_j^{(P)} \right\|^2 \qquad (9.14)$$

over $\dot{s}_j$ which results in

$$\sum_P \langle \beta_j^{(P)} , \sum_i \dot{s}_i \beta_i^{(P)} \rangle + \sum_P \langle \beta_j^{(P)} , \frac{\partial f^{(P)}}{\partial x} \rangle = 0 \qquad (9.15)$$

For each $\dot{a}^{(P)}$ we minimise

$$\sum_{P} \left\| \sum \dot{a}_j^{(P)} \alpha_j + \sum \dot{s}_j \beta_j^{(P)} + \frac{\partial f}{\partial x}^{(P)} \right\|^2 \tag{9.16}$$

over $\dot{a}_j^{(P)}$.

In two dimensions to solve

$$u_t + f_x + g_y = 0 \tag{9.17}$$

we use the two dimensional equivalent of (1.9) and write (9.17) in the form

$$\frac{Du}{Dt} = -(f_x + g_y) + \dot{x}\, u_x + \dot{y}\, u_y \tag{9.18}$$

c.f. (8.2), and as in the 1-D case minimise

$$\left\| \frac{Du}{Dt} \right\|^2 = \int \left(\frac{Du}{Dt}\right)^2 dxdy = \int [-(f_x + g_y) + \dot{x}u_x + \dot{y}u_y]^2 dxdy \tag{9.19}$$

but now over $\dot{x}$ and $\dot{y}$. To obtain the discrete form for $\dot{x}$ and $\dot{y}$ we use the MFE approximations for x, y and v, i.e.

$$\begin{aligned} x &= \sum x_j \alpha_j \\ y &= \sum y_j \alpha_j \\ v &= \sum a_j \alpha_j \end{aligned} \tag{9.20}$$

and approximate (9.19) by

$$\| -(f_x + g_y) + \sum \dot{x}_j \alpha_j v_x + \sum \dot{y}_j \alpha_j v_y \|^2 \tag{9.21}$$

which is minimised over $\dot{x}_j$ $\dot{y}_j$ to give

$$\begin{aligned} \langle \beta_j, \sum_k \dot{x}_k \beta_k \rangle + \langle \beta_j, \sum_k \dot{y}_k \gamma_k \rangle &= -\langle \beta_j, f_x + g_y \rangle \\ \langle \gamma_j, \sum_k \dot{x}_k \beta_k \rangle + \langle \gamma_j, \sum_k \dot{y}_k \gamma_k \rangle &= -\langle \gamma_j, f_x + g_y \rangle \end{aligned} \tag{9.22}$$

where $\beta_k = -v_x\alpha_k$, $\gamma_k = -v_y\alpha_k$ and

$$\sum_k \dot{x}_k\beta_k = - \sum_k \dot{x}_k\alpha_k v_x$$

$$\sum_k \dot{y}_k\gamma_k = - \sum_k \dot{y}_k\alpha_k v_y \qquad (9.23)$$

Similarly for systems, the discrete form for $\dot{x}_j$, $\dot{y}_j$ is obtained by minimising

$$\sum_p \|(f_x^{(p)} + g_y^{(p)}) + \sum_k \dot{x}_k\beta_k^{(p)} + \sum_k \dot{y}_k\gamma_k^{(p)}\|^2 \qquad (9.24)$$

over $\dot{x}_j$ and $\dot{y}_j$ respectively, to give

$$\sum_p \langle\beta_j^{(p)}, \sum \dot{x}_k\beta_k^{(p)}\rangle + \sum_p \langle\beta_j^{(p)}, \sum_k \dot{y}_k\gamma_k^{(p)}\rangle = - \sum_p \langle\beta_j^{(p)}, f_x^{(p)} + g_y^{(p)}\rangle$$

$$\sum_p \langle\gamma_j^{(p)}, \sum \dot{x}_k\beta_k^{(p)}\rangle + \sum_p \langle\gamma_j^{(p)}, \sum \dot{y}_k\gamma_k^{(p)}\rangle = - \sum_p \langle\gamma_j^{(p)}, f_x^{(p)} + g_y^{(p)}\rangle \qquad (9.25)$$

The discrete form for $\underline{\dot{a}}$ is obtained in the usual way, by minimising

$$\|\sum \dot{a}_j\alpha_j + \sum \dot{x}_j\beta_j + \sum \dot{y}_j\gamma_j + f_x + g_y\|^2 \qquad (9.26)$$

over $\dot{a}_j$ in the scalar case and minimise

$$\sum_p \|\sum \dot{a}_j^{(p)}\alpha_j + \sum \dot{x}_j\beta_j^{(p)} + \sum \dot{y}_j\gamma_j^{(p)} + f_x^{(p)} + g_y^{(p)}\|^2 \qquad (9.27)$$

over each $\dot{a}_j^{(p)}$ for the system.

Local MEM2

We here derive the discrete form for $\dot{s}$ for the scalar 1-D case. The form for 2-D, for systems and for separate grids, although similarly derived, will be described in the subsequent report. We start with the global norm (9.8), which may be rewritten as

$$\left\| \sum_j \dot{s}_j \alpha_j \frac{\partial v}{\partial x} - \frac{\partial f}{\partial x} \right\|^2 = \left\| \sum_j \sum_{i=1}^{2} z_{ij} \phi_{ij} + \frac{\partial f}{\partial x} \right\|^2 \tag{9.28}$$

where

$$\begin{aligned} \frac{z_{1k}}{\sqrt{\Delta s_k}} &= -\dot{s}_{k-1} \frac{\Delta_k}{\sqrt{\Delta s_k}} \\ \frac{z_{2k}}{\sqrt{\Delta s_k}} &= -\dot{s}_k \frac{\Delta_k}{\sqrt{\Delta s_k}} \end{aligned} \tag{9.29}$$

and ϕ_{ij} are the local basis functions, c.f. §1. Adopting the usual local MFE procedure, c.f. §1, with the new variable z in place of w, we minimise (9.28) over z_{ik} $(i=1,2)$ to give

$$C\underline{z} = \underline{b} \tag{9.30}$$

where

$$\underline{z} = \begin{pmatrix} z_{1k} \\ z_{2k} \end{pmatrix}$$

and Δ, C and $\underline{b}$ are defined as in §1. From (9.29) we see that we need to solve an overdetermined system for $\dot{s}_k$, namely,

$$-D^{-\frac{1}{2}} \underline{M} \dot{s}_k = D^{-\frac{1}{2}} \underline{z} \tag{9.31}$$

where D is defined as in 2 (and is required to ensure conservation)

$$M = \begin{pmatrix} \Delta_k \\ \Delta_{k+1} \end{pmatrix} \qquad \underline{z} = \begin{pmatrix} z_{2k} \\ z_{1\ k+1} \end{pmatrix} \tag{9.32}$$

We obtain the discrete form for $\dot{s}_k$ by minimising the residual of over $\dot{s}_k$ to give

$$M^T(D^{-\frac{1}{2}})^T D^{-\frac{1}{2}} \underline{M} \quad \dot{s}_k = - M^T(D^{-\frac{1}{2}})^T D^{-\frac{1}{2}} \underline{z} \tag{9.33}$$

which gives

$$\left(\frac{\Delta_k^2}{\Delta s_k} + \frac{\Delta_{k+1}^2}{\Delta s_{k+1}} \right) \dot{s}_k = - \left(\frac{\Delta_k z_{2k}}{\Delta s_k} + \frac{\Delta_{k+1} z_{1\ k+1}}{\Delta s_{k+1}} \right) \tag{9.34}$$

Comparing (9.34) for $\dot{s}_{MEM2}$ with (2.7) for $\dot{s}_{MEM}$ we see that MEM2 contains a gradient weighting.

To obtain the discrete form $\dot{a}_k$ in terms of $\dot{s}_k$ (now given by (9.34) we follow the procedure described earlier for MEM 1, equations (2.1) to (2.4).

Summary

In contrast to the first MEM (where we minimise (9.10) with respect to $\dot{a}$, then choose $\dot{s}$ to minimise $||\dot{a}||^2$), for MEM 2 we first minimise $\left\|\frac{Du}{Dt}\right\|^2$ over $\dot{s}$ which corresponds approximately to minimising $||\sum \dot{a}_j \alpha_j||^2$ (c.f. 1.8) Then having obtained $\dot{s}$, go back to the usual norm of the residual of the differential equation and minimise over $\dot{a}$ to give the discrete form for $\dot{a}$ in terms of the known $\dot{s}$.

10. RESULTS

All the results presented here are for the shock tube test problem of Sod (13), solving the Euler equations written in the form (1.1) where

$$\underline{u} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} , \qquad \underline{f} = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ u(E + P) \end{bmatrix}$$

The conservative variables ρ , ρu, E are the density, momentum and energy respectively. The pressure and velocity are denoted by P and u respectively. In the figures the solid line is always the exact solution.

(i) Local MFE. Single Grid (§3)

Fig. (6) is the MFE solution run from initial data at time 0.1 (Fig. (5)), to time 0.144 with time step $\Delta t = 0.001$. Note that the nodes are in danger of collision in the expansion.

Fig. (7) is the MFE result run from time 0.1 to time 0.25 with time step 0.001. The nodes have collided and the method attempts to fit a shock.

Fig. (8) is the MFE result again run from time 0.1 to 0.25, now with a time step of 0.001; although a solution is obtained the nodal distribution along the expansion is poor.

(ii) Local MEM. Single Grid (§3)

Figs. (9) and (10) are the MEM solutions, run from an initial time 0.1 and output at 0.144 and 0.25 respectively, both in one time step. The nodes remain well distributed throughout the expansion.

Fig. (11) is the MEM result run from time 0.0072 to 0.144, in one step. If smaller steps are used for this case (i.e. with initial time 0.0072) the method develops expansion shocks. However MFE could not produce a solution at all for this case.

(iii) Separate Grids (§5)

Fig. (12), (13) and (14) are the MEM results using a different grid for each component of the Euler system. Figs. (12) and (13) were run from initial time 0.1 to 0.144, 0.25, respectively. Fig. (14) was run from 0.0072 to 0.144.

(iv) Local MEM 2. Single Grid.

Fig. (15) is the result of the second version of the local MEM (c.f. §9) run from 0.1 to 0.25 in one step.

(v) Local MEM 2. Separate Grids.

Fig. (16) is the MEM 2 result for separate grids run from 0.1 to 0.25 in one step.

CONCLUSION

In this report we have given details of a new method (the Mobile Element Method (MEM)) for systems of conservation laws in any number of dimensions, based on the use of finite elements on a moving grid. The grid motion is regarded as a parameter in the finite element projection and is determined by minimising the mobile derivative of the object function. The basis of comparison is with the method of local Moving Finite Elements, and a number of new results concerning both this method and MEM have been obtained.

Although further testing is required and other formulations are being considered, results are sufficiently good to indicate that the MEM has a solid basis and promises well for further development. In particular further research is needed to investigate the problems with expansion shocks: one reason may be that the scheme is not entropy satisfying for systems.

After this work was done my attention was drawn to related ideas in the papers of Dukowicz [15] and Hyman [14]. However, the MEM is distinct from their work in several ways and was derived without the knowledge of these developments.

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APPENDIX 1

Possible treatment for parabolic equations and forcing terms

If MEM is applied to an equation of form

$$u_t + a(u)u_x = \varepsilon\, u_{xx} \tag{A1}$$

then, using (1.9), (A1) may be written as

$$\frac{Du}{Dt} = \varepsilon\, u_{xx} - a(u)\, u_x + \dot{s}\, u_x \tag{A2}$$

or

$$\frac{Du}{Dt} - \varepsilon\, u_{xx} = -\, a(u)u_x + \dot{s}\, u_x \tag{A3}$$

Equations (A2) and (A3) suggest two approaches for the application of the MEM. In the first approach, from (A2), minimise

$$\left\|\frac{Du}{Dt}\right\|^2 = \|\varepsilon\, u_{xx} - a(u)u_x + \dot{s}\, u_x\|^2 \tag{A4}$$

over $\dot{s}$. In the differential case version simply set

$$\dot{s} = -\,\varepsilon\,\frac{u_{xx}}{u_x} + a(u)$$

$$\frac{Du}{Dt} = 0 \tag{A5}$$

The characteristic is effectively modified to account for the viscous term.

In the second approach, from (A3), we minimise

$$\left\|\frac{Du}{Dt} - \varepsilon\, u_{xx}\right\|^2 = \|-a(u)u_x + u_x\, \dot{s}\,\|^2 \tag{A6}$$

over $\dot{s}$. In the differential case set

$$\dot{s} = a(u) \tag{A7}$$

$$\frac{Du}{Dt} = \varepsilon\, u_{xx}\,.$$

The solution is found along the characteristic (with $\varepsilon = 0$), and although the cell Reynolds number restriction is removed, the limitation on the time step due to the diffusion remains Johnson [17]

To obtain the numerical discretisation for $\frac{Du}{Dt}$ (i.e. $\dot{a}$) in both approaches, the accompanying minimisation is

$$\left\| \frac{Du}{Dt} - \dot{s}\, u_x + a(u)u_x - \varepsilon\, u_{xx} \right\|^2$$

over $\dot{a}$ in the usual way. This latter method does not involve integration of products of β with u_{xx} (see reference above).

For a general forcing term, P, say, we replace εu_{xx} with P in the above.

APPENDIX 2

Parallelism

MEM can suffer from the equivalent of parallelism in MFE (reference (10)), when the coefficient of $\dot{s}$ vanishes. For example, consider the scalar 1-D case (c.f. §2). From (2.7) if the $\dot{s}$ coefficient is zero then

$$\Delta_k + \Delta_{k+1} = 0$$

which is equivalent to

$$\frac{\partial u}{\partial x} = 0 \quad .$$

For this case we minimise (2.1) only over $\dot{a}$, obtaining still a conservative equation for $\dot{a}$. As in the treatment for parallelism in MFE, c.f. Baines (7), the $\dot{s}_j$ with vanishing coefficients are chosen to be means of their non-vanishing neighbours.

FIG. 1 Fixed Finite Elements

$\alpha_k\ (x,s)$ Remains fixed for all time

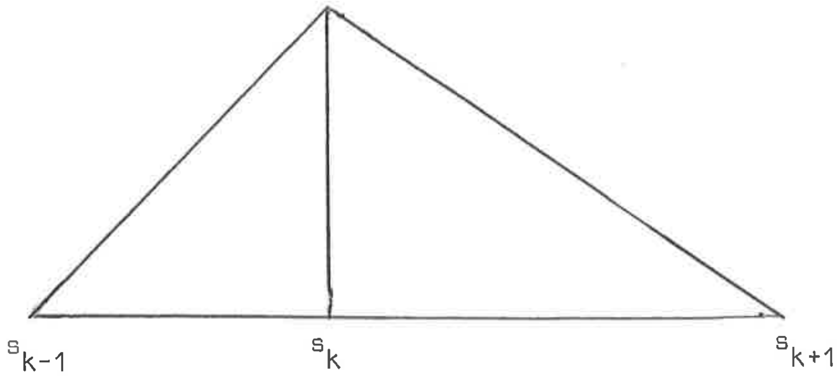


FIG. 2 Moving Finite Elements

$\alpha_k(x,s(t))$ has a nodal variation with time

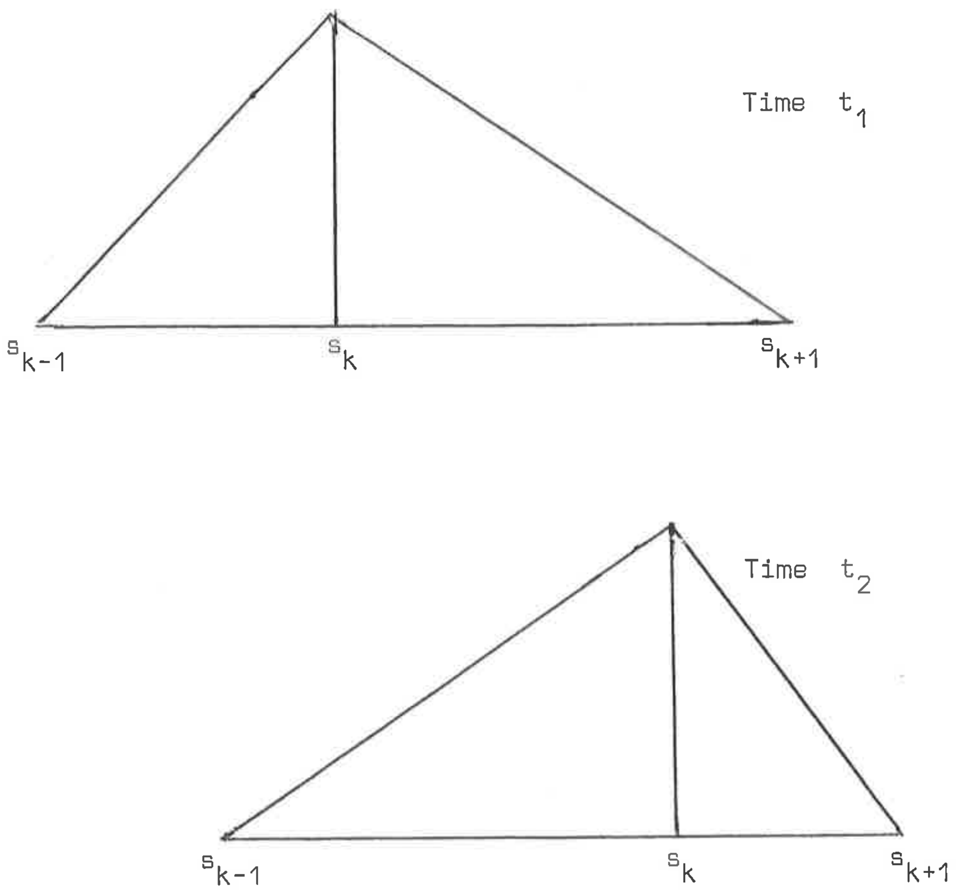
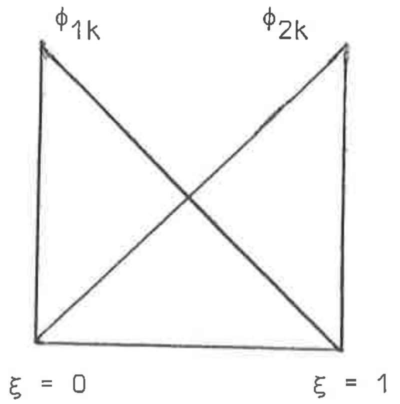


FIG. 3 Dimensionless Basis functions

ϕ_{1k}, ϕ_{2k} drawn in the uniform canonical space

Time t_1



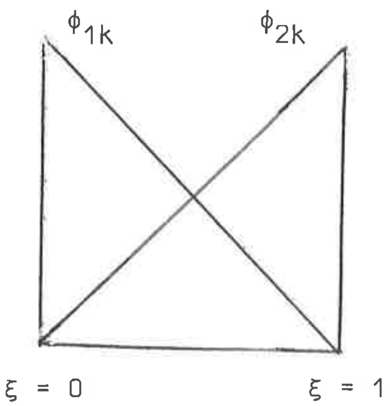
$\xi = 0$ corresponds to physical space co-ordinate s_{k-1}

$\xi = 1$ " " " " s_k

ϕ_{1k}, ϕ_{2k} remain fixed in the dimensionless space

$$\frac{D\phi_{1k}}{Dt} = 0$$

$$\frac{D\phi_{2k}}{Dt} = 0$$



Time t_2

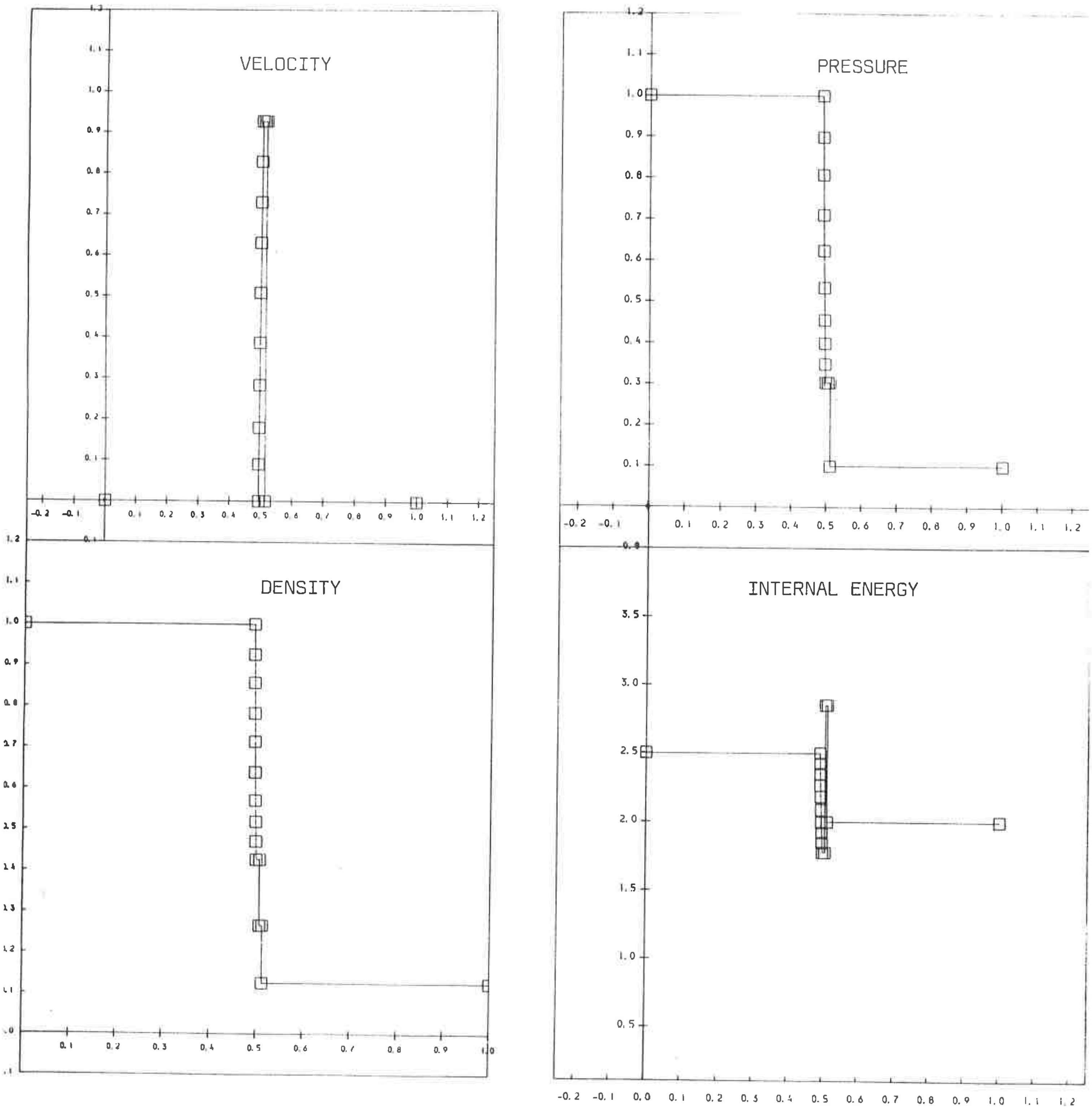


FIG. 4

Exact Initial Data— Time = 0.0072

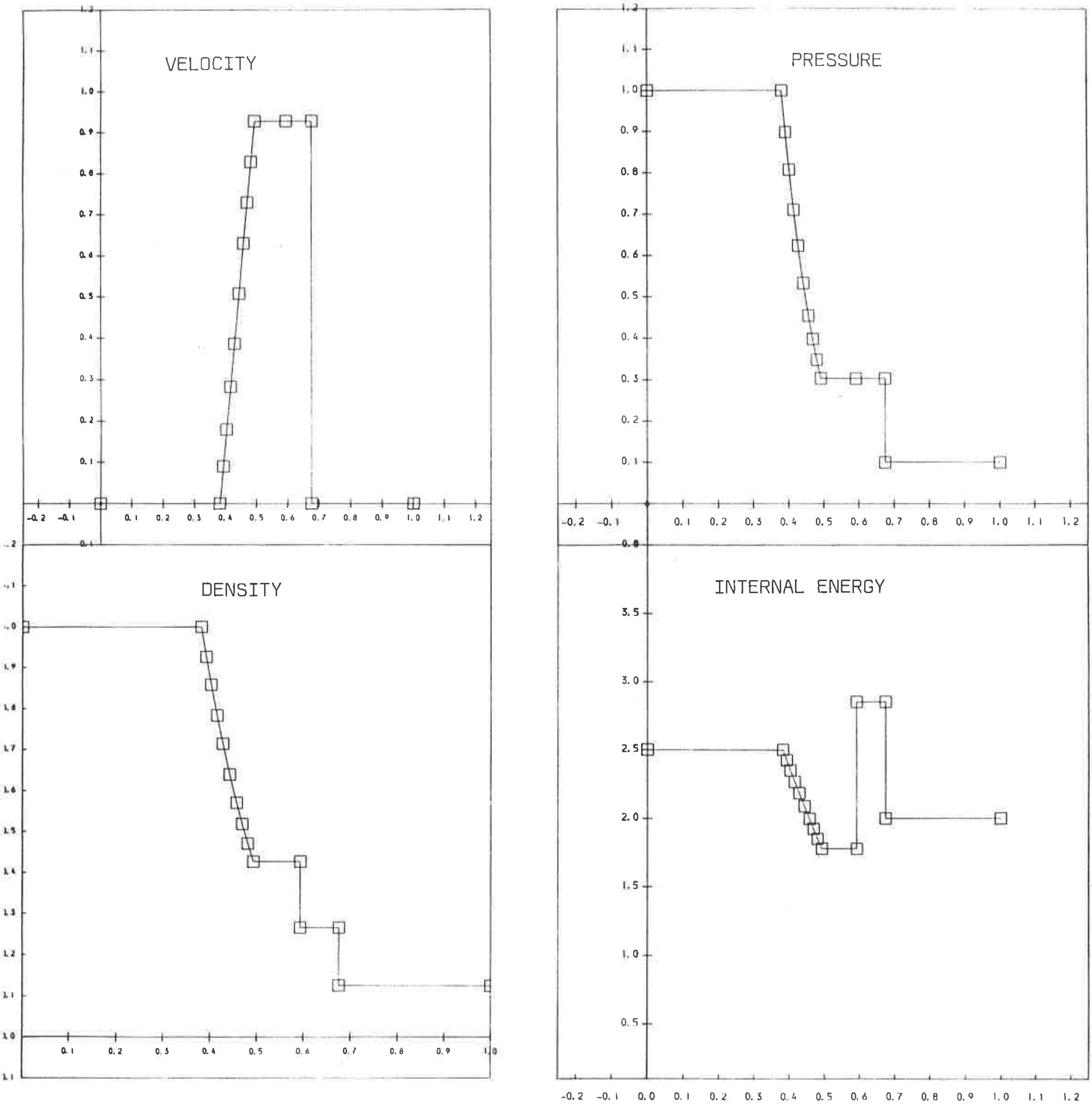


FIG. 5

Exact Initial Data Time = 0.1

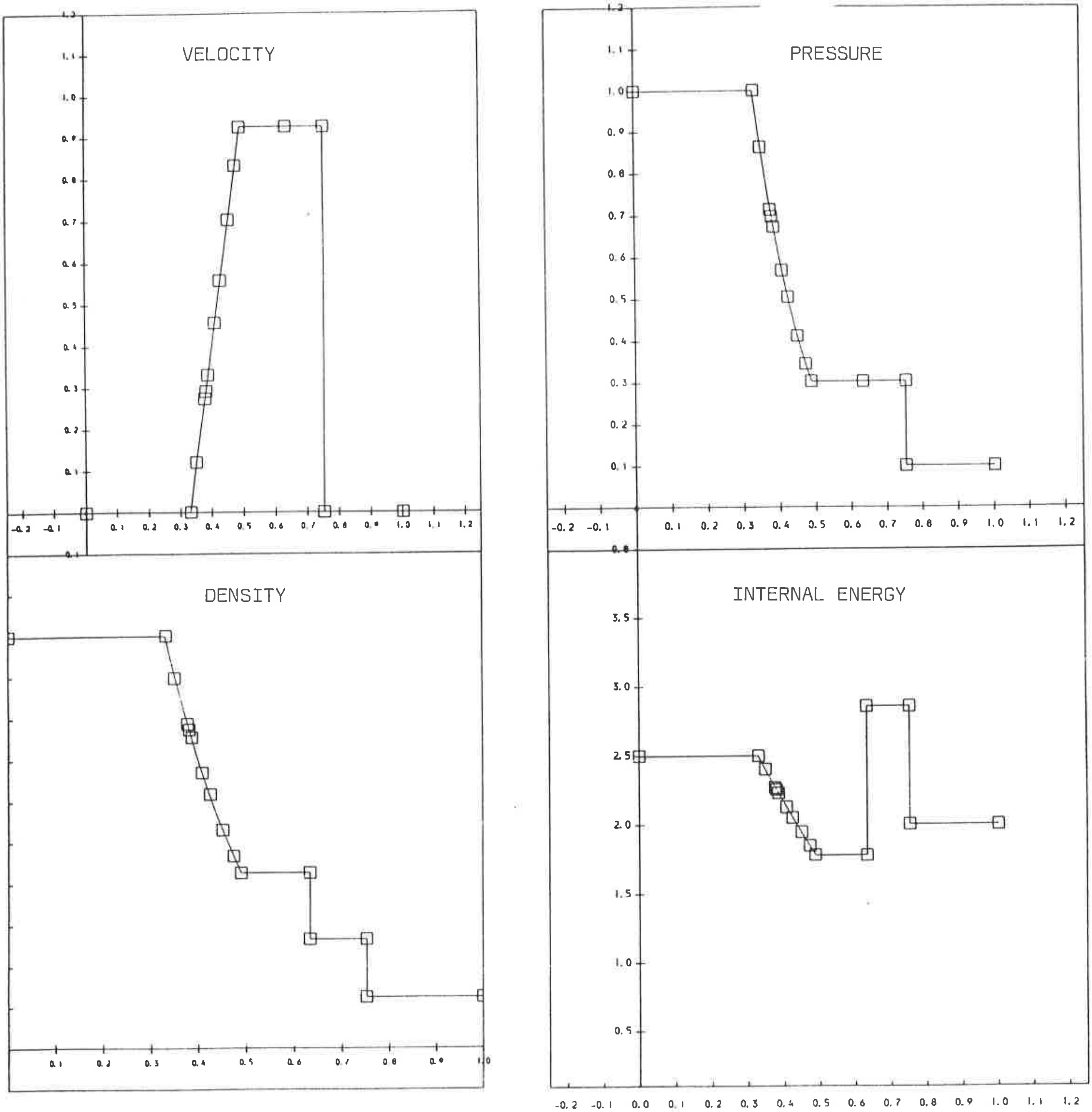


FIG. 6 MFE SINGLE GRID

MFE   Initial time = 0.1
       Output time = 0.144
       Time step = 0.001

—— Exact solution

□ Computed

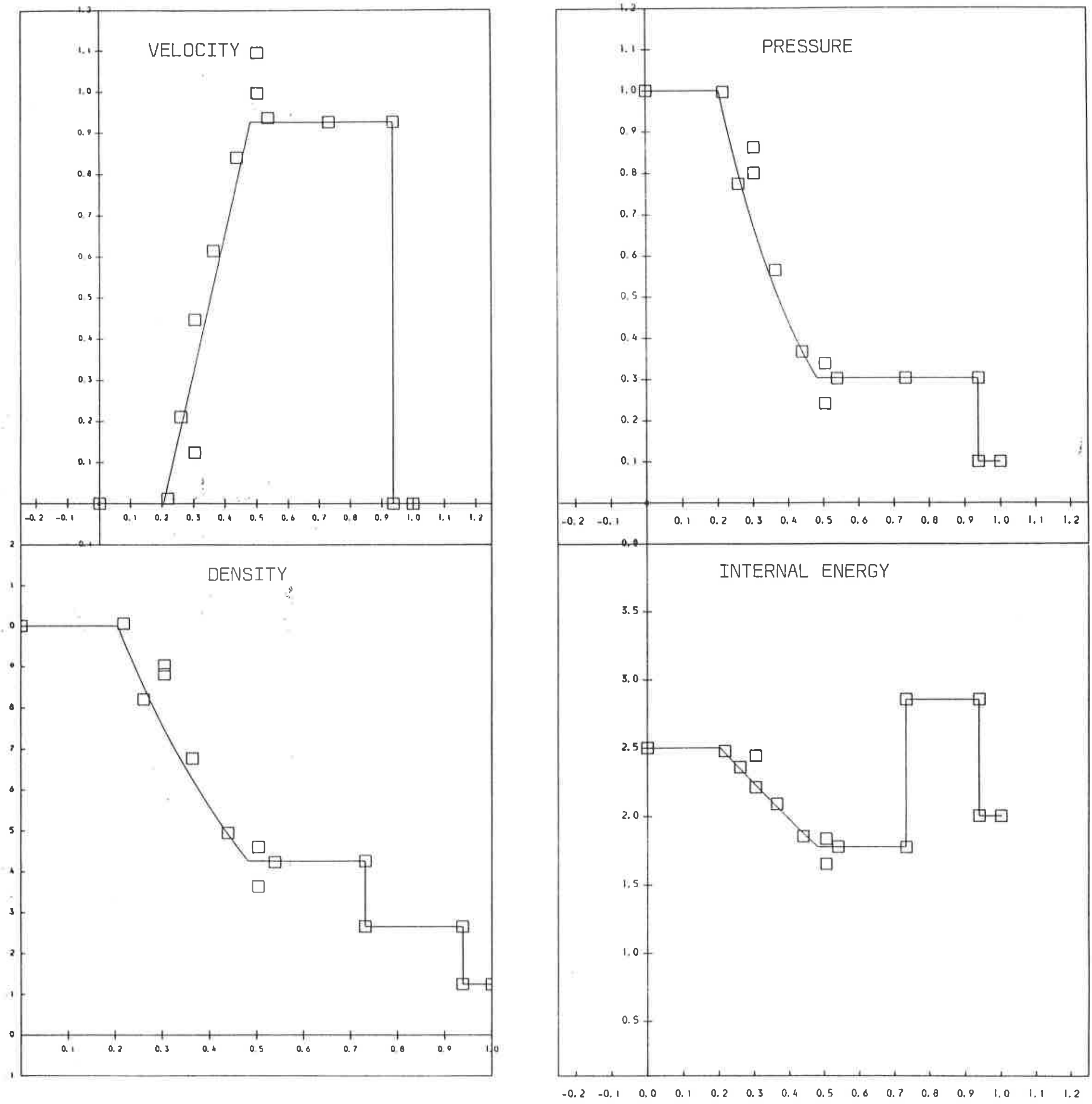


FIG. 7 MFE SINGLE GRID

Initial time = 0.1
Output time = 0.25
Time step = 0.01

Exact solution

Computed

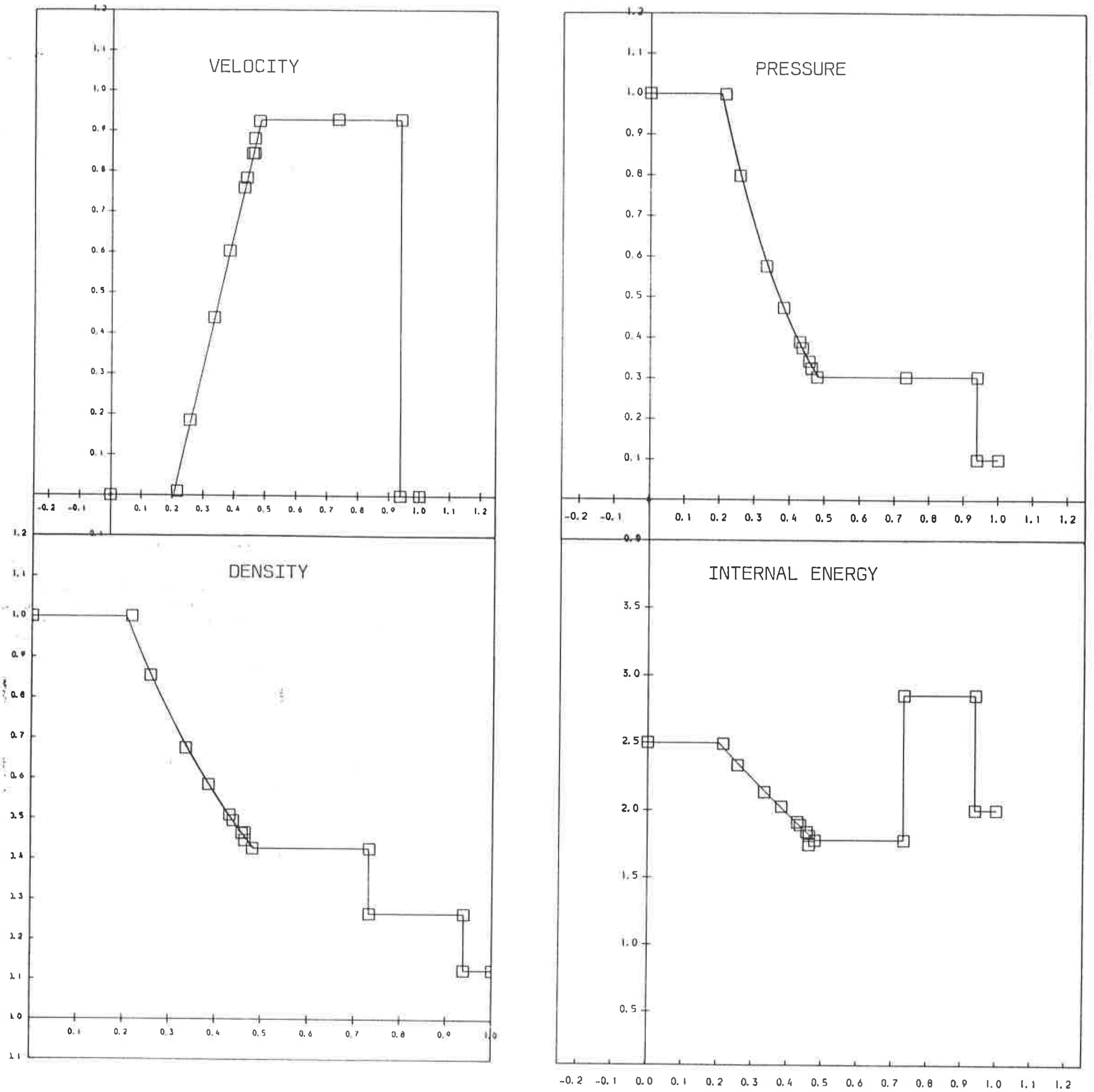


FIG. 8 MFE SINGLE GRID

Exact solution

Computed

Initial time = 0.1
Output time = 0.25
Time step = 0.001

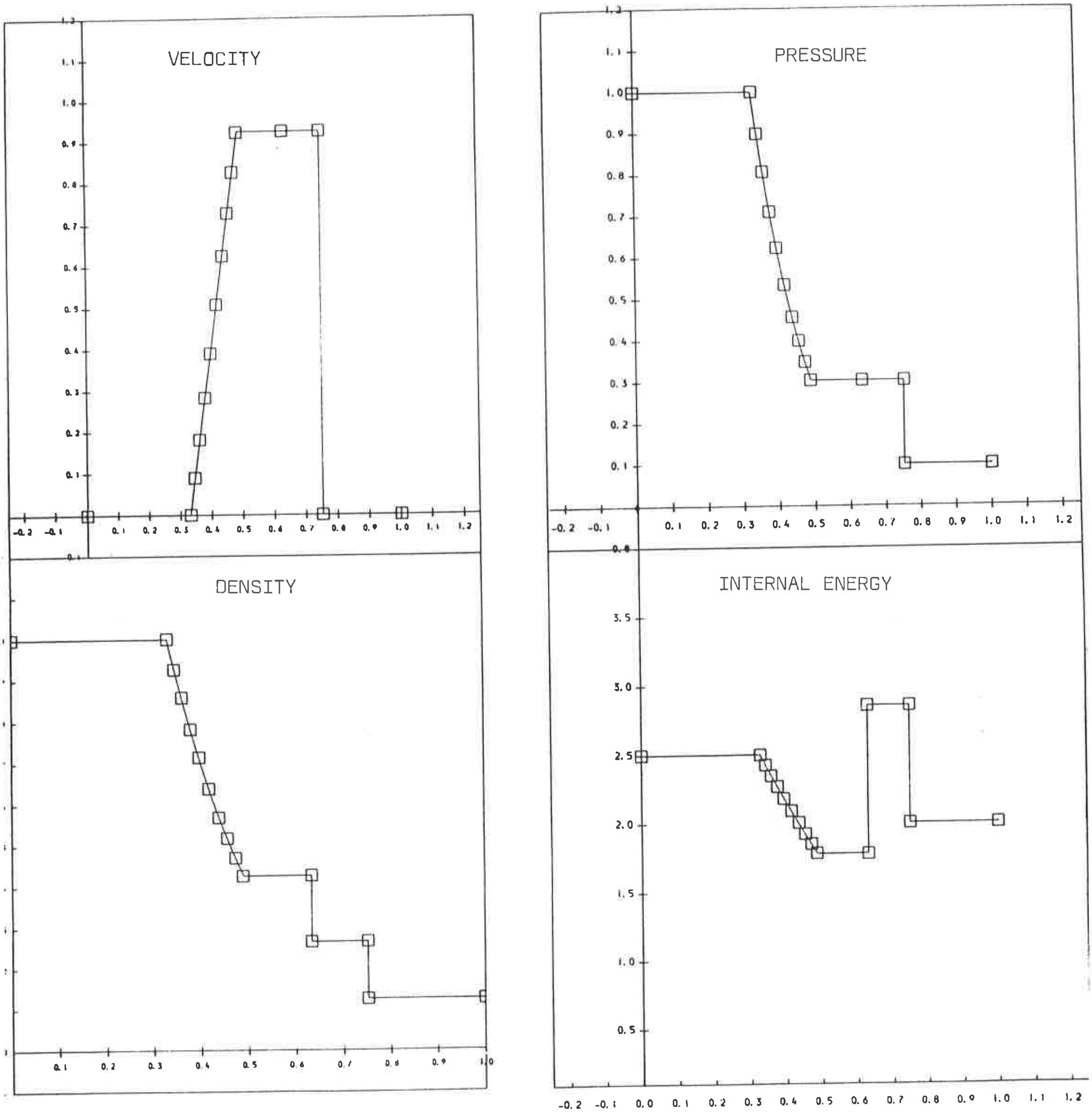


FIG. 9 MEM SINGLE GRID

Exact solution

Computed

Initial time = 0.1
Output time = 0.144
Time step = 0.044

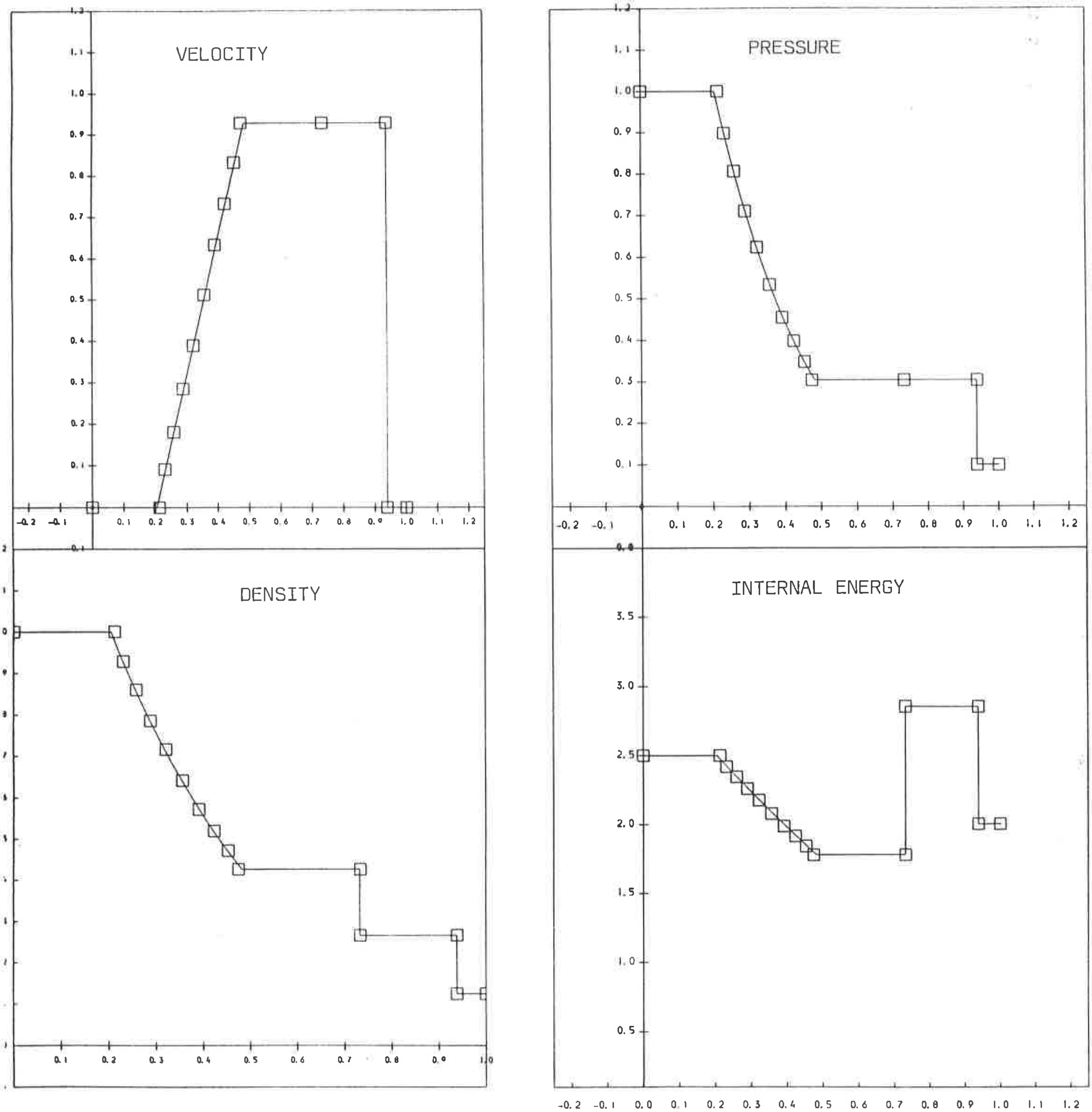


FIG. 10 MEM SINGLE GRID

———— Exact solution

□ Computed

Initial time = 0.1
Output time = 0.25
Time step = 0.15

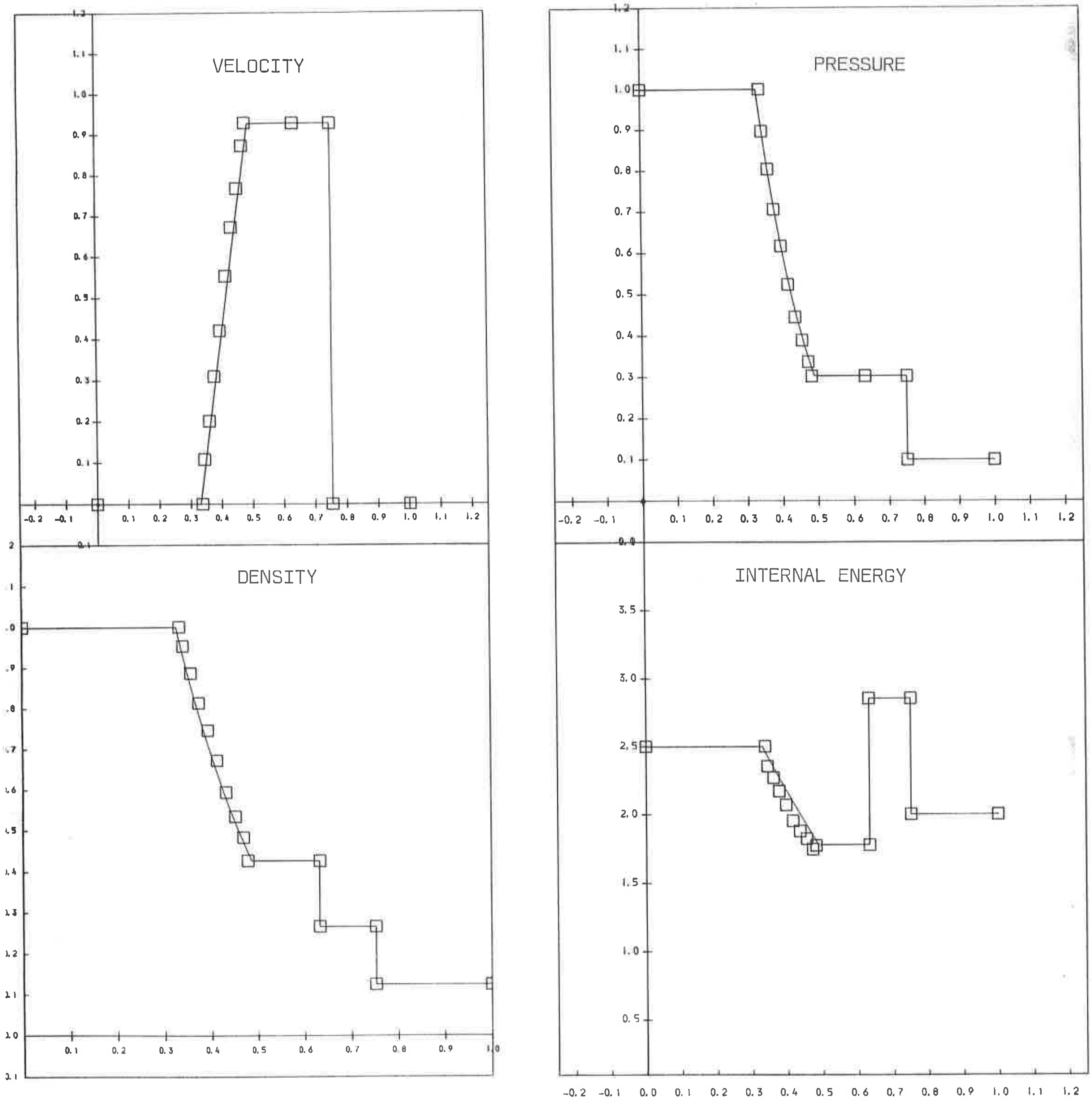


FIG. 11 MEM SINGLE GRID

Initial time = 0.0072
Output time = 0.144
Time step = 0 .1368

______ Exact solution

□ Computed

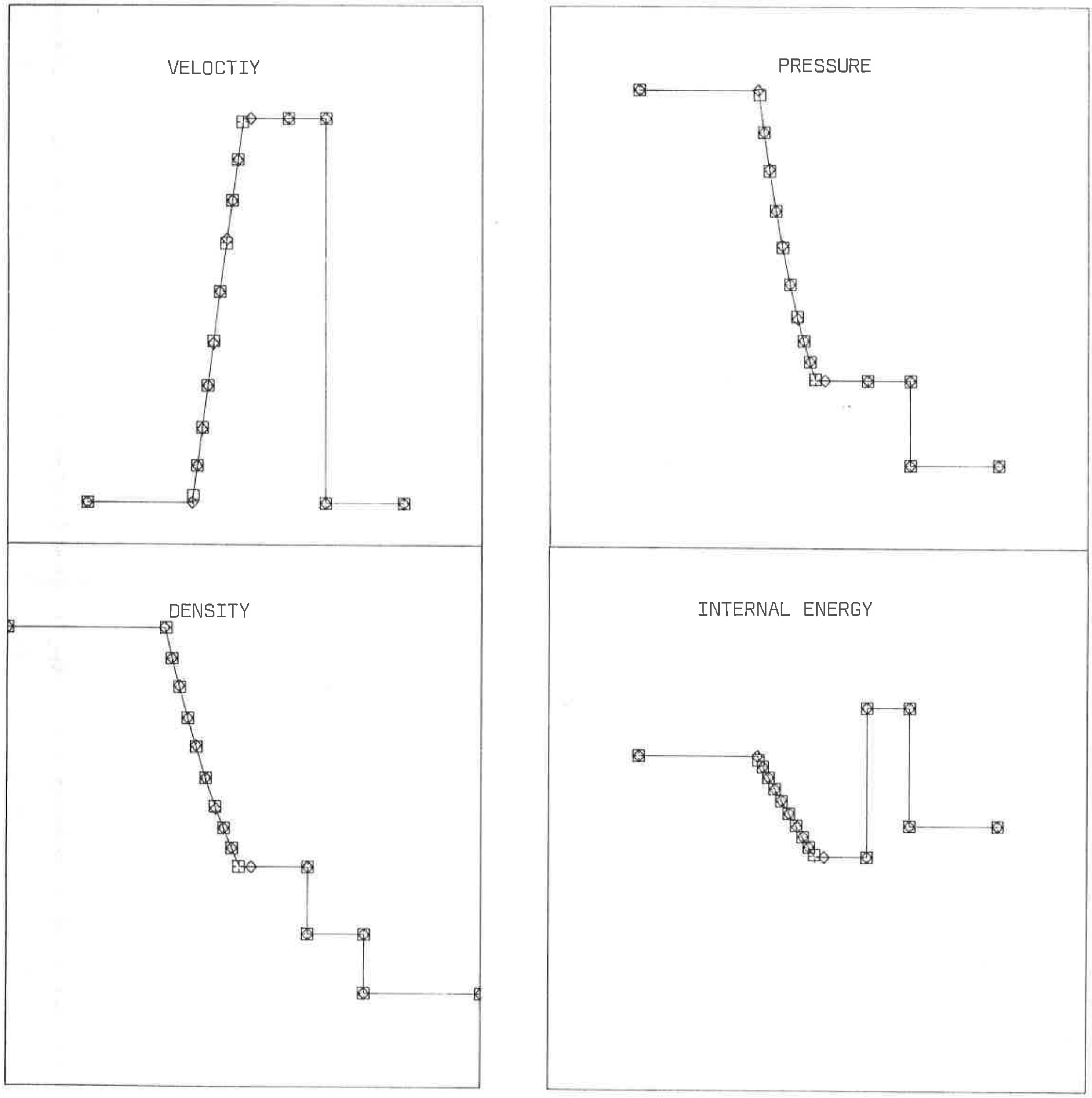
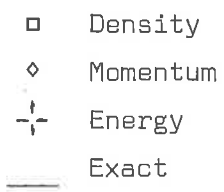


FIG. 12 MEM SEPARATE GRIDS

Initial Time 0.1

Output Time 0.144

Time step 0.044



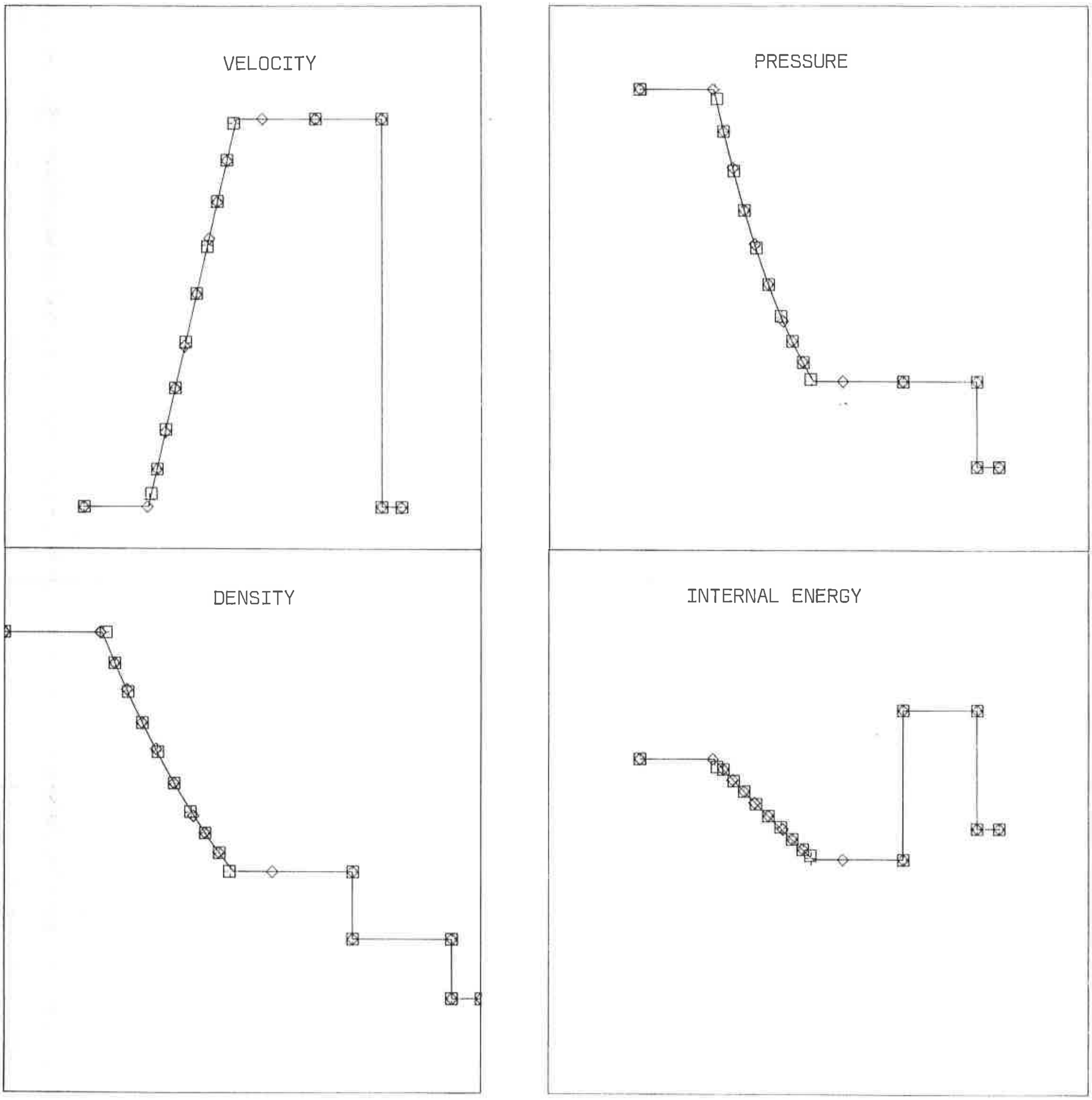


FIG. 13 MEM SEPARATE GRIDS

Initial Time 0.1

Output Time 0.25

Time Step 0.15

□ Density

◇ Momentum

-¦- Energy

— Exact

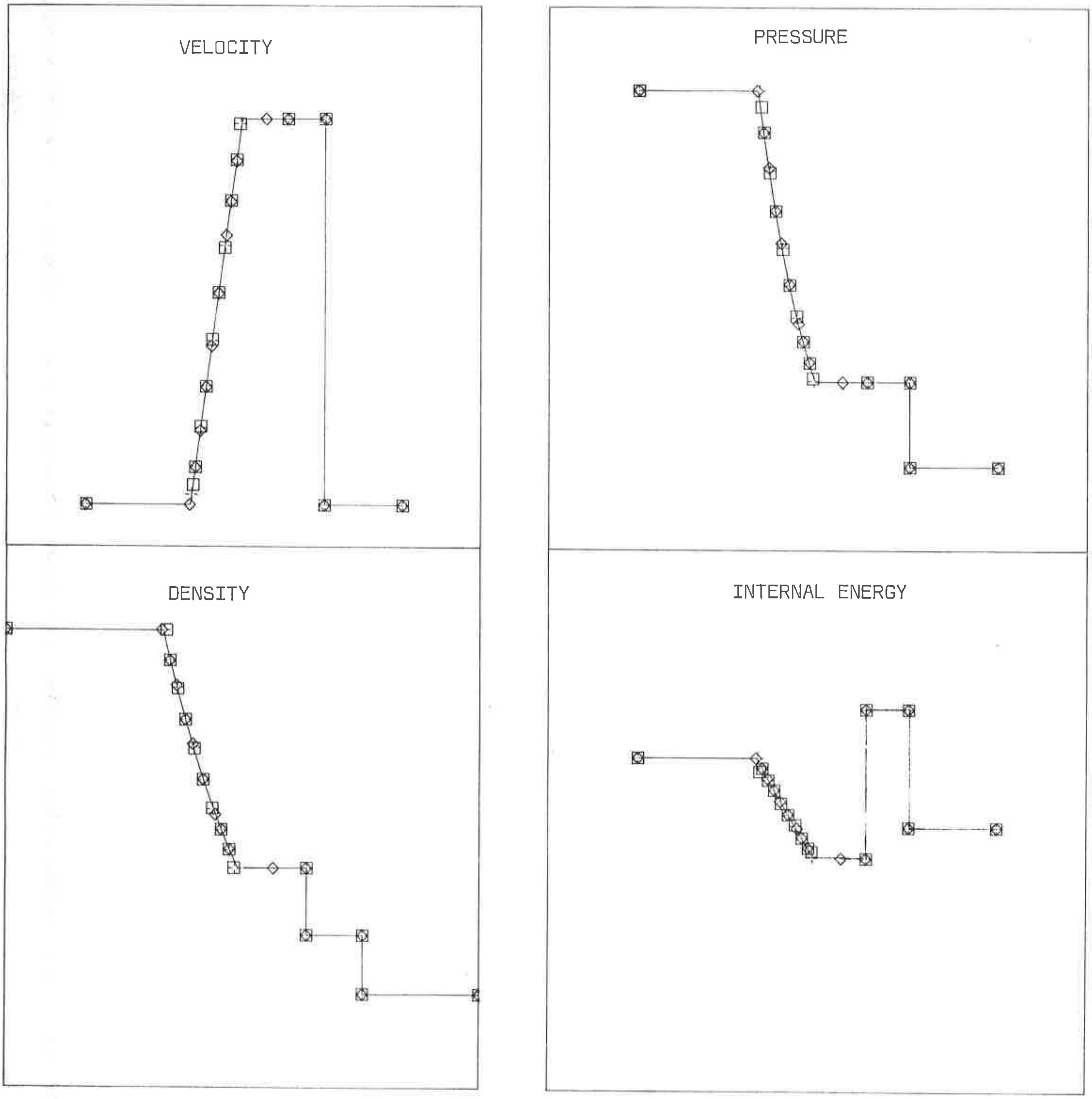


FIG. 14 MEM SEPARATE GRIDS

Initial Time 0.0072

Output Time 0.144

Time Step 0.1368

□ Density

◇ Momentum

Energy

Exact

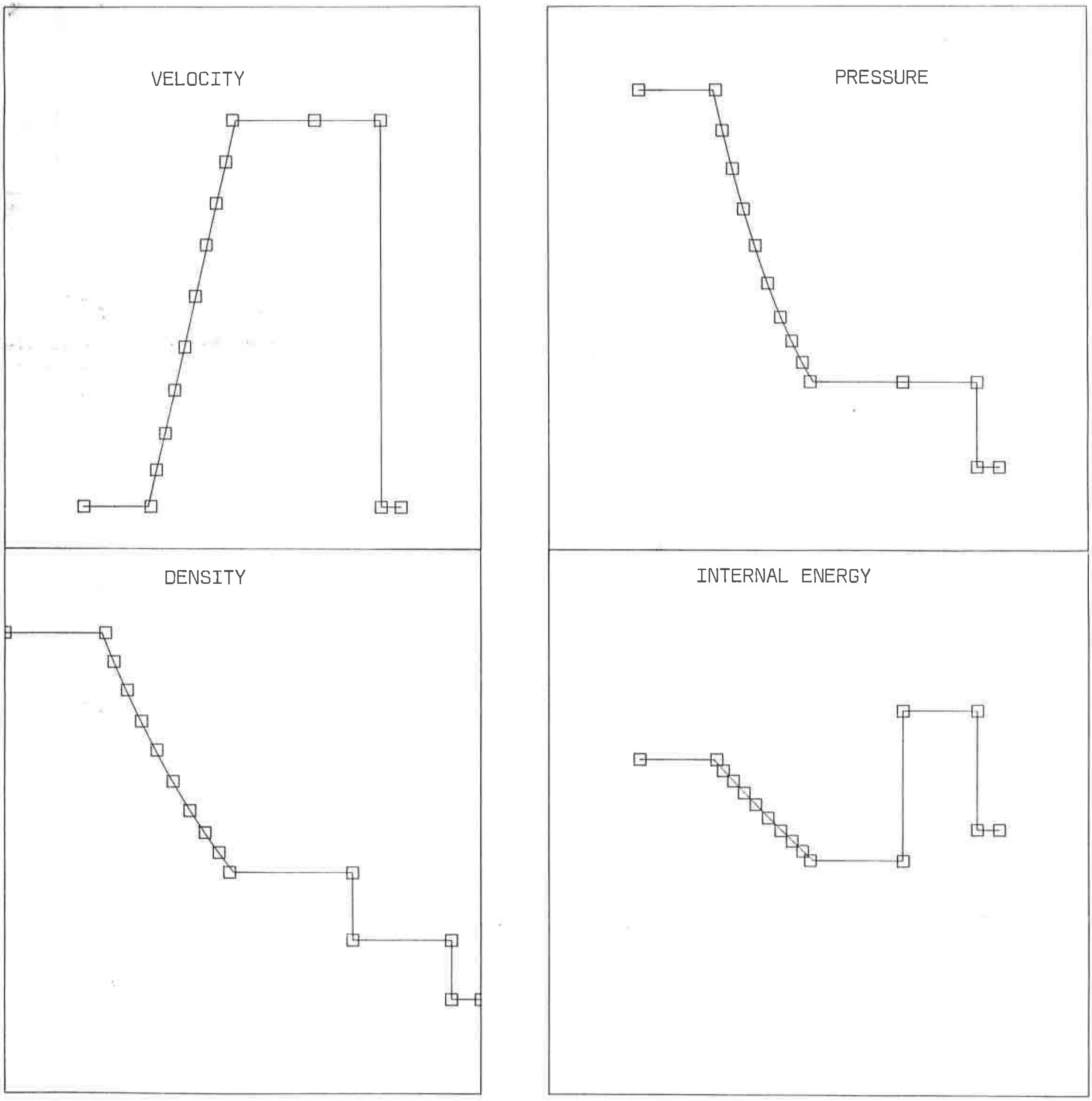


FIG. 15   MEM 2 SINGLE GRID

Initial Time	0.1
Output Time	0.25
Time Step	0.15

□ Computed

____ Exact

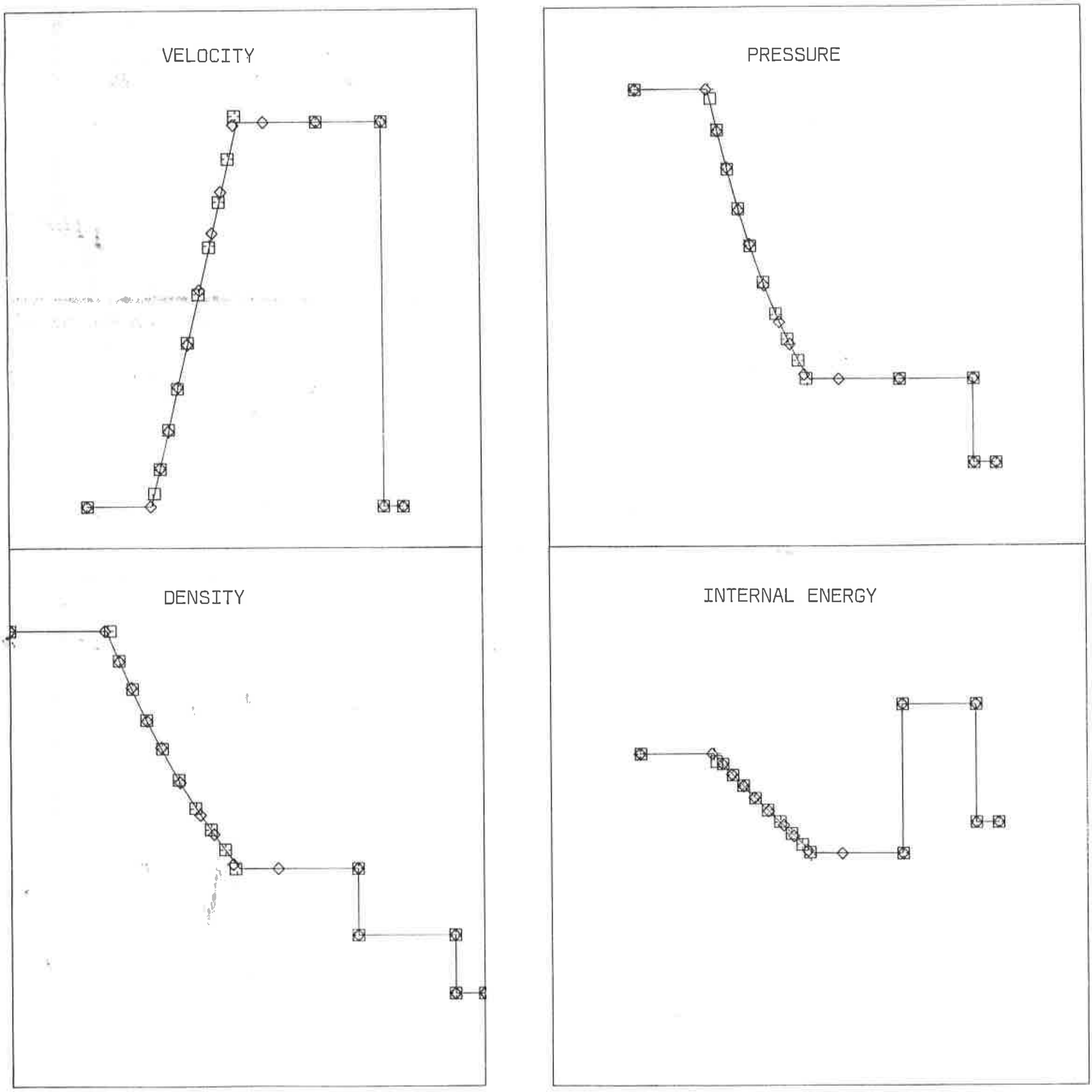


FIG. 16

MEM 2 SEPARATE GRIDS

Initial Time	0.1
Output Time	0.25
Time Step	0.15

Separate Grids

- □ Density
- ◇ Momentum
- -¦- Energy
- —— Exact