MATHEMATICS DEPARTMENT

The Implementation of Boundary Conditions in the

Application of Roe's Scheme to Aeronautical Flows

on Cartesian Grids

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Numerical Analysis Report 18/88

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Introduction

In applying Roe's scheme, a time-accurate characteristic based method, Roe (1981) to the flow around an aerofoil, a projectile or a re-entrant vehicle for example, there are three options concerning the grid.

Firstly the physical space can be transformed so that the body becomes a rectangular block in the transformed space. A Cartesian grid can then be used in the transformed space and Roe's scheme can then be used to solve the transformed equations, see Glaister 1988. This approach has problems, though, in that the transformation can be very expensive to calculate for non-regular three-dimensional shapes and the equations themselves become more complicated to solve.

Secondly, using a body fitted mesh for example, we can work in physical space but on a non-Cartesian mesh. There is still the problem of how to generate the mesh, but one of the advantages of this approach is that an adaptive mesh refinement/enlargement could be incorporated. Again, though, more work is needed at each point to rotate the Euler equations in the directions normal and tangential to local cell interfaces. It is also not immediately clear how to achieve second order accuracy for this scheme. See Chakravarthy & Osher (1985), and Barley (1989) concerning this approach.

Thirdly, a Cartesian mesh can be used in physical space. Clearly this has advantages in that no grid information needs to be stored, schemes revert to their simplest form and the passing of information between regions of fine and coarse mesh is straightforward. Also the schemes generally become easily vectorizable. The difficulty is in modelling the rigid wall. In recent years Cartesian meshes have regained some favour and not just for characteristic based

methods. (See Clarke et al (1986), Leveque (1988), Moretti & Dadone (1988) for some recent work) and it is this approach that will be considered here.

In the next section we will describe how the rigid wall boundary conditions are applied and the advantages & disadvantages over the approach of Leveque (1988) are discussed.

In Section 3 results will be presented for the two-dimensional double ellipse problem that show the method to be a very accurate and robust proposition. Difficulties, when using Roe's scheme, caused by slow moving shocks as they near steady-state, see Roberts (1988), were found to occur and a crude, but effective, solution was implemented. Methods for improving convergence rates will also be discussed briefly.

Finally a summary of the work is given.

2. Rigid Walls in Three Dimensions

The application of reflecting boundary conditions becomes a much simpler problem if the rigid wall is a flat plane. Indeed, this is what is done here: the (curved) surface is replaced by a set of flat planes.

In an obvious notation consider the point (i,j,k) at the centre of a block of 27 points $\{(i-1, i, i+1 ; j-1, j, j+1 ; k-1, k, k+1)\}$. We shall assume that we have certain information available to us concerning the body surface. The functions in the procedure are as follows:

1) A boolean function INSIDE (x,y,z) that given a point in space returns a value of .TRUE. or .FALSE. according to whether the point is within the body or not.

2) A function $z(x,y)$ that, given a point (x,y), returns the value of z that lies on the surface.

3) Functions $\dfrac{\partial z(x,y)}{\partial x}$ and $\dfrac{\partial z(x,y)}{\partial y}$ that return derivatives of the above function.

4) Similarly defined functions $x(y,z)$, $y(x,z)$, $\dfrac{\partial x}{\partial y}$, $\dfrac{\partial x}{\partial z}$, $\dfrac{\partial y}{\partial x}$, $\dfrac{\partial y}{\partial z}$.

In the case of the double ellipse problem for which results will be presented later, there are analytic formulae for these functions. In a more general case where the body is defined by a set of points on the boundary they can de derived from a cubic spline fit of these points for example.

The procedure is now as follows.

(a) If the point, (i,j,k), does not lie inside the body, then get the next point, else go to (b).

(b) Given that the point (i,j,k) lies inside the boundary test to see if the point is a boundary point. This is done by looking at the three pairs of points $\{x_{i-1}, x_{i+1}\}$, $\{y_{j-1}, y_{j+1}\}$ and $\{z_{k-1}, z_{k+1}\}$. If there exists a pair for which one point is inside and the other outside, then the point (i,j,k) is a boundary point and we proceed to (c). If, for all 3 pairs, both points are either inside or outside, then the point is not on the boundary.

(c) Find the plane that locally approximates the surface. Let us assume that it was the $\{z_{k-1}, z_{k+1}\}$ pair that satisfied the criterion of (b). The general equation of a plane is

$$ax + by + cz + d = 0$$

and we can immediately put $c = -1$ (so that $z = z(x,y)$. The parameters a and b can be found straight away from the functions $\partial z/\partial x$ and $\partial z/\partial y$. The value of d can also be calculated although it is not actually needed.

(d) From the normal to the plane, given by (a,b,c), find a value, s, such that the point T, say, with coordinates

$$(x_i, y_j, z_k) + s(a,b,c)$$

lies on the surface of the cube centred on the point (i,j,k). This gives us the place from which the values

of (ρ, u, v, w, p) are interpolated in order to apply the reflection conditions.

Then, using a corresponding suffix T,

$$\rho_{i,j,k} = \rho_T \tag{2.1a}$$

$$p_{i,j,k} = p_T \; . \tag{2.1b}$$

For the velocities only the normal component is reflected, the tangential components remaining unaltered. To resolve into normal and tangential components at the point T we need to solve

$$
\left[\begin{array}{ccc}
a & 1 & 1 \\
b & -a/b & b/a \\
c & 0 & -(\dfrac{a}{c} + \dfrac{b^2}{ac})
\end{array}\right]
\left[\begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array}\right]_T
=
\left[\begin{array}{c}
u \\
v \\
w
\end{array}\right]_T
\tag{2.2}
$$

Denoting the matrix in (2.2) by Λ, the normal and tangential components at T are then given by

$$\underline{n}_T = \Lambda^{-1} \underline{u}_T \; . \tag{2.3}$$

To reflect the normal vector and hence obtain the normal and tangential components at (i,j,k) we multiply equation (2.3) by a matrix D to get

$$\underline{n}_{i,j,k} = D\Lambda^{-1}\underline{u}_T \tag{2.4}$$

where $\quad D = \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] .$

To return to velocities in x,y,z space we now just need to
multiply (2.4) by Λ, so that the final equation for the velocities
at (i,j,k) is

$$\underline{u}_{i,j,k} = \Lambda \, D \, \Lambda^{-1} \underline{u}_T \ .$$
(2.5)

Evaluating $\Lambda \, D \, \Lambda^{-1}$ in (2.5) explicitly gives

$$
\begin{bmatrix} u \\ v \\ w \end{bmatrix}_{i,j,k}
=
\begin{bmatrix}
1 - \dfrac{2a^2}{R} & -\dfrac{2ab}{R} & -\dfrac{2ac}{R} \\[2mm]
-\dfrac{2ab}{R} & 1 - \dfrac{2b^2}{R} & -\dfrac{2bc}{R} \\[2mm]
-\dfrac{2ac}{R} & -\dfrac{2ac}{R} & 1 - \dfrac{2c^2}{R}
\end{bmatrix}
\begin{bmatrix} u \\ v \\ w \end{bmatrix}_T
$$
(2.6)

where $R = a^2 + b^2 + c^2$.

The main difference between the method described here and that of
Leveque (1988) is that of ease of application versus flexibility of
application. This procedure is certainly very simple. Only the
points T have to be found, which is quite straightforward, and then it
is a trivial matter to update points on the boundary from formulae (2.1)
and (2.6). This means that we can perform Roe's scheme in the same way
at all points.

Leveque's procedure does not use 'imaginary' points and literally
reflects waves hitting the rigid wall, spreading the flux associated
with the wave over the area swept out by the wave. This is certainly a
more complicated procedure and involves much logical testing to see if a
wave is going to impinge upon the boundary. The significant advantage
of Leveque's (1987) approach is when it is required to work with larger

CFL numbers, as we might well want to consider doing if iterating towards steady-state. Leveque's approach can cope with this situation at no extra cost whereas when imaginary points are used, as in the present proposal, more imaginary points need to be introduced and some may need to be doubly defined. Whilst there is no reason why this cannot be done it is certainly not such a neat and tidy approach.

3.   Results

The double ellipse problem is a GAMM workshop test case and is defined below.

Lower Surface:

$$y = - 0.015 \qquad\qquad\qquad 0 \leq x \leq 0.016$$

$$y = -0.015 \sqrt{1-(\tfrac{x}{0.06})^2} \qquad -0.06 \leq x \leq 0$$

Upper Surface:

$$y = 0.025 \qquad\qquad\qquad 0 \leq x \leq 0.016$$

$$y = 0.025 \sqrt{1-(\tfrac{x}{0.035})^2} \qquad x^* \leq x \leq 0$$

$$y = 0.015 \sqrt{1-(\tfrac{x}{0.06})^2} \qquad -0.06 \leq x \leq x^*$$

where $x^* = - 0.029890588$.

The problem of using a Cartesian mesh and having sufficient resolution around both the body and important flew features whilst at the same time having a mesh that reaches the farfield boundary as quickly as possible is overcome by using a succession of overlapping Cartesian meshes. For this problem four meshes were used. No claim is made that the best nesting of meshes has been used; a finer mesh would have been useful to resolve the flow near the canopy shock and would

also have been beneficial at the nose in the 0° incidence case, while the coarsening of the meshes would probably be done more rapidly. However, the solutions obtained are perfectly adequate to demonstrate the approach.

The two flow regimes for which results are presented both have freestream Mach numbers of 8.15, the first having an angle of attack of 0° and the second of 30°. Figures 1-4 show density (ρ/ρ_{∞}) and Mach numbers contours for both cases. It should be noted that these calculations have not yet fully reached steady-state.

The problem described by Roberts (1988) of noise generated behind a near stationary shock in Roe's scheme first caused severe distortion of the bow shock near the nose of the body in the 0° case. This has been overcome by the inclusion of a small amount of diffusion which is however not enough to make any noticable changes to the shock thickness. Ideally we would have liked to incorporate the diffusion explicitly, in keeping with the explicit nature of Roe's scheme. However, solving with the diffusion calculated explicitly by the formula

$$U_j^{n+1} = (1 - v\Delta_- + k\delta^2)U_j^n \tag{3.1}$$

is well known to reduce CFL stability limits. Here v is the CFL number, k the coefficient of diffusion, and

$$\Delta_- U_j = U_j - U_{j-1}$$

$$\delta^2 U_j = U_{j+1} - 2U_j + U_{j-1} .$$

Typically this problem is overcome by treating the diffusion implicitly, as in

$$(1 - k\delta^2)U_j^{n+1} = (1 \ v\Delta_-)U_j^n \ . \tag{3.2}$$

This, however, involves a matrix inversion and hence considerably more work. A third approach that requires no more work than (3.1) is to use a predictor-corrector approach and to solve

$$U_j^* = (1 \ v\Delta_-)U_j^n \tag{3.3a}$$

$$U_j^{n+1} = (1 + k\delta^2)U_j^* \ . \tag{3.3b}$$

Although equations (3.3) require no more work than (3.1) they give good stability because they represent a first order approximation to the solution of the implicit equations (3.2).

Figures 1-4 were all calculated from an initial state in a time-accurate fashion. This is not a particularly efficient means of reaching a steady-state solution and Sells (1980), Priestley (1987) have discussed means of increasing the convergence rate of Roe's scheme in these situations based on local and regional time-stepping, which can dramatically increase the convergence rate. Local time-stepping would be of particular use here behind the bow shock where the maximum CFL number is less than 50% that of the global maximum.

4. Summary

In this paper an effective and cheap means of enforcing reflecting

boundary conditions has been presented that facilitates the use of Roe's scheme on a Cartesian grid in a very straightforward manner. Results on a double ellipse problem have been presented to demonstrate that the method produces good solutions. Previous work on time-accelerating techniques suggests that when these techniques are incorporated steady state can be achieved in a very efficient manner.

5. Acknowledgements

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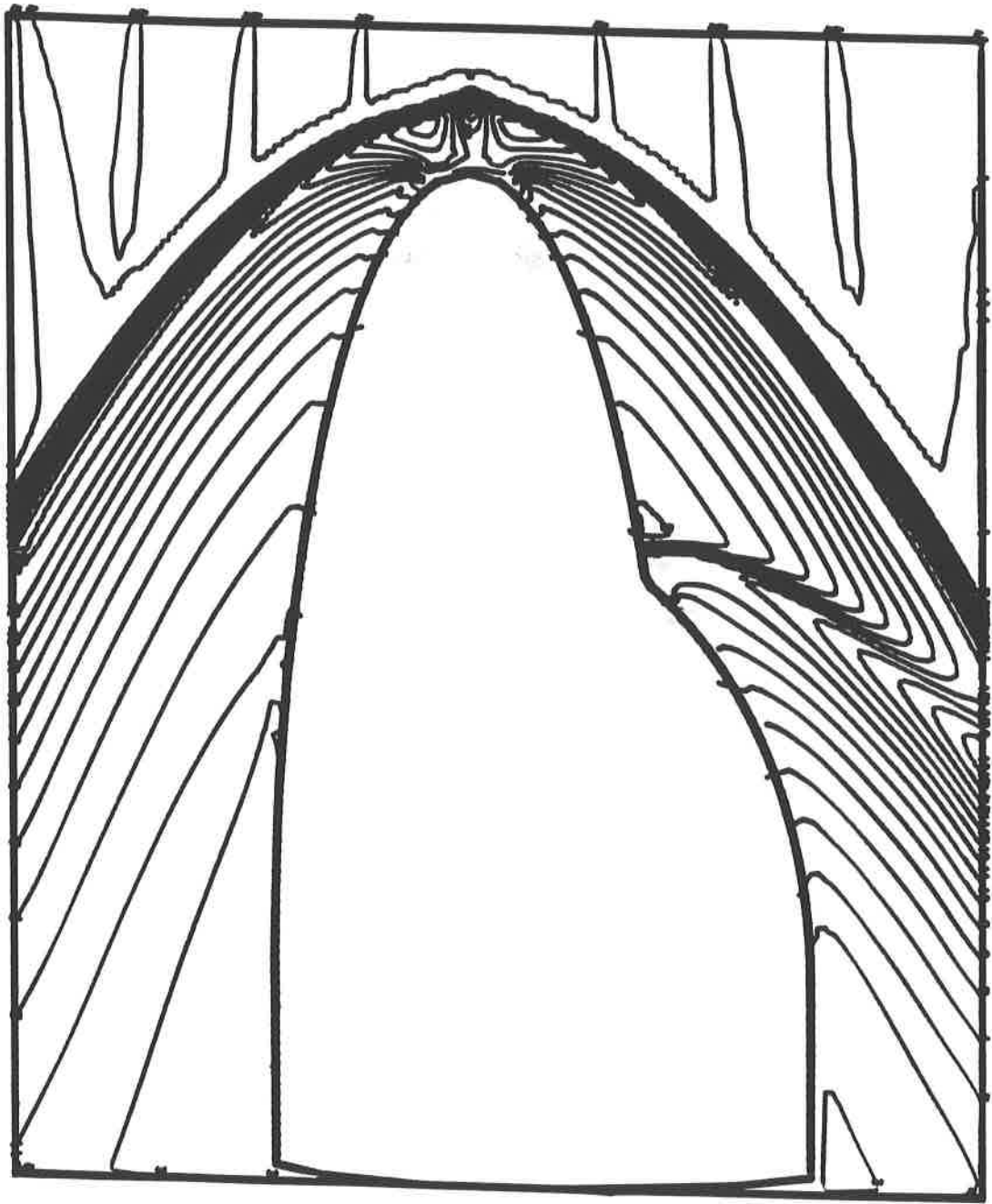
Fig. 1 Density 0° case. En route to steady-state

Fig. 2 Mach number 0⁰ case. En route to steady-state

Fig. 3. Density 30^b case. En route to steady-state

Fig. 4. Mach number 30° case. En route to steady-state