Output Feedback via State Feedback and Least Squares

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Abstract

We examine a procedure for achieving partial pole placement via output feedback by combining state feedback and least squares techniques and analyse the errors incurred.

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1. Introduction

Automatic control of time continuous systems is an important criterion in many practical areas. The stability of such systems is linked, as will be shown, to the eigenvalues of their state matrices. These matrices, if unsatisfactory from this point of view, may be changed or their eigenvalues reassigned so that they adopt more suitable values. This process is known as pole placement and may be attained by means of a feedback which uses knowledge of the states of, or outputs from, the system to manipulate the inputs in order to change its future performance.

Pole placement by output feedback is often a complex control problem, limited by matrix dimensional conditions and yet is of more practical use than state feedback, which will assign the same characteristics with less constraints but requires a great deal more information to put into effect. However, there exists a link between the equation describing the state matrix of the system obtained by means of state feedback and that obtained by output feedback so that, given a computed state feedback state matrix, we may employ a least squares method to compute the best approximation to the corresponding output feedback matrix.

In section 2 we give the background to both aspects of this problem and carry out an error analysis in section 3. Section 4 contains some numerical examples, while conclusions are given in section 5.

2. Background to the problem

In the first part of this section we explore the motivation for choosing specific eigenvalues of the system state matrix for reassignment.

2.1 Motivation

A general time-continuous linear system can be described by the differential problem

$$\dot{\underline{x}} = A\underline{x} , \qquad \underline{x}(0) = \underline{x}_0$$

where A is a constant $n \times n$ matrix, known as the state matrix, which has the solution

$$\underline{x}(t) = \exp(At)\underline{x}_0 . \tag{2.1}$$

Expanding the exponential term and taking norms, we have

$$\|\exp(At)\| \leq \sum_{k=1}^{q} \sum_{l=1}^{\alpha_k} t^{l-1}\exp(\mathrm{Re}(\lambda_k)t)\|Z_{kl}\| \tag{2.2}$$

where $\mathrm{Re}(\lambda_k)$ denotes the real part of the eigenvalues of A,

q is the number of distinct eigenvalues,

α_k is the order of the largest Jordan block associated with λ_k and

Z_{kl} are constant matrices determined entirely by A.

From (2.2), $\|\exp(At)\| \longrightarrow 0$ as $t \longrightarrow \infty$, provided that $\mathrm{Re}(\lambda_k) < 0 \ \forall k$ since (2.2) is a sum of finite terms which each tend to zero as $t \longrightarrow \infty$. Since, from (2.1),

$$\|\underline{x}(t)\| \leq \|\exp(At)\|\|\underline{x}_0\| .$$

the system is asymptotically stable.

We see, then, that for stability it is sufficient that the eigenvalues of the system state matrix have negative real parts and we aim to design a feedback which is able to alter the state matrix so that this is the case. This may be done in two ways, depending on the information available, and we describe them below.

2.2 Introduction to State Feedback

In this section, we give a brief outline of the characteristics of state feedback and the basic method for achieving stable state feedback. For greater detail, see Kautsky et al (1984).

We consider the time-invariant linear multivariable system whose state equation is given by:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{2.3}$$

where $\underline{x}$ is an $n \times 1$ state vector

$\underline{u}$ is an $m \times 1$ input vector

and A, B are matrices of appropriate dimensions, and B is of full rank.

We assume that we have full knowledge of the states of the system at any time so that we may employ a feedback matrix F which alters the system inputs by acting on the whole state vector. That is, the 'new' input to the system becomes:

$$\underline{u} = F\underline{x} + \underline{v} \tag{2.4}$$

where $\underline{v}$ is an $m \times 1$ input vector unaffected by feedback.

Substituting (2.4) into (2.3) gives a new state matrix equation

$$\dot{\underline{x}} = (A+BF)\underline{x} + B\underline{v} \tag{2.5}$$

and we seek to choose F so that $A + BF$ has eigenvalues with negative real parts. More specifically, we seek an F which will assign predetermined eigenvalues to the system. In other words, state feedback requires us to find an F and an X s.t.

$$(A+BF)X = X\Lambda \quad \text{for some given} \quad \Lambda = \text{diag}(\lambda_i) \tag{2.6}$$

where X is a non-singular matrix having the eigenvectors $\underline{x}_i \in \mathbb{R}^n$, $i = 1,\ldots,n$ of $A + BF$, corresponding to each λ_i, as its columns, and the eigenvalues λ_i are self-conjugate complex.

There are many choices of eigenvector for each eigenvalue corresponding to different choices of F and we may select each so that the resulting system has certain properties.

From (2.6) we have that

$$BF = X\Lambda X^{-1} - A \, . \tag{2.7}$$

Taking the QR decomposition of B gives

$$B = UZ_B \qquad \text{where } U \text{ is orthogonal}$$

and partitioning in such a way that Z_B consists of a non-singular, upper triangular block and a zero block corresponding to the partitioned blocks of U, we obtain

$$B = [U_0, U_1] \begin{bmatrix} Z \\ 0 \end{bmatrix} .$$

Then, pre-multiplying (2.7) by U^T gives:

$$ZF = U_0^T(X\Lambda X^{-1} - A) \tag{2.8}$$

and $$0 = U_1^T(X\Lambda X^{-1} - A) \ .$$

The first of these equations gives us a solution matrix F ; the second gives the condition to be satisfied for such an F to exist, i.e. a solution exists iff

$$U_1^T(X\Lambda X^{-1} - A) = 0$$

i.e. iff $$U_1^T(X\Lambda - AX) = 0 \ . \tag{2.9}$$

In other words, we require each column $\underline{x}_i$ of X , corresponding to each eigenvalue λ_i , to belong to the null space

$$\mathcal{S}_i = \mathcal{N}\{U_1^T(A - \lambda_i I)\} \ .$$

To form the null space we take the QR decomposition of $[U_1^T(A - \lambda_i I)]^T$ for each i and partition,

i.e. $$[U_1^T(A - \lambda_i I)]^T = [\hat{S}_i, \ S_i]\begin{bmatrix} R_i \\ 0 \end{bmatrix} \ . \tag{2.10}$$

Then S_i is an orthonormal basis for $\mathcal{S}_i$ since

$$S_i^T[\hat{S}_i, \ S_i]\begin{bmatrix} R_i \\ 0 \end{bmatrix} = [0, \ I]\begin{bmatrix} R_i \\ 0 \end{bmatrix} = 0$$

and

$$S_i^T[U_1^T(A - \lambda_i I)]^T = [U_1^T(A - \lambda_i I)S_i]^T \ .$$

So, for each i , we choose a vector $\underline{x}_i$ from S_i to form $X = [\underline{x}_1, \ \underline{x}_2, \ldots, \underline{x}_n]$ such that X lends the required characteristics to the system.

We may now employ the computed X in (2.8) to find the feedback matrix

$$F = Z^{-1}U_0^T(X\Lambda X^{-1} - A) . \qquad (2.11)$$

2.3 Introduction to output feedback

Output feedback may be described in a similar way to state feedback by means of the equations (2.3)

and $$\underline{y} = C\underline{x} \qquad (2.12)$$

where A, B, $\underline{x}$, $\underline{u}$ are as definition in §2.2 and $\underline{y}$ is a $p \times 1$ output vector

and C is a $p \times n$ matrix of full rank.

In this instance, information is not available for all states of the system. Here we may only observe the outputs of the system. For example, we may think of a chemical reaction which as it occurs has many states, e.g. quantities of each compound as they are formed, temperature, volumes of gases, liquids and solids etc. Perhaps only some of these may be measurable and, moreover, may only be measured indirectly. In this way system outputs are related less comprehensively to system states - they are able to give limited information about the states and are not directly related to them. Any feedback which is desired to alter the state matrix may depend only on output information to adjust system input so that the new input is given by

$$\underline{u} = F_0\underline{y} + \underline{v} \qquad (2.13)$$

where $\underline{u}$ and $\underline{v}$ are $m \times 1$ input vectors.

Substituting (2.12) and (2.13) into (2.3) gives

$$\dot{\underline{x}} = (A + BF_0C)\underline{x} + B\underline{v} \quad . \tag{2.14}$$

Again, for stability, we desire the eigenvalues of $M = A + BF_0C$ to have negative real parts and we seek an F_0 to ensure this.

Output feedback introduces dimensional requirements unlike state feedback which requires only that there are at least as many states as inputs. Kimura (1975) has shown it is sufficient that $m + p > n$ for arbitrary assignment. We shall continue by showing how, in certain circumstances, we may soften this restriction.

2.4 Partial pole placement

We may find that we need not change all of the eigenvalues of A, a process which can prove to be expensive. So we construct an output feedback matrix F_0 which changes those eigenvalues which are unsuitable and leaves the others in their original positions.

Taking the Schur decomposition of A^T, we have:

$$A^T = QRQ^T \tag{2.15}$$

where Q is orthogonal and R is a (block) upper triangular matrix which has the eigenvalues of $A, \mu_1, \ldots, \mu_n$, along its diagonal.

Pre- and post-multiplying M by Q^T and Q respectively gives

$$Q^T(A + BF_0C)Q = Q^TAQ + Q^TBF_0CQ \quad . \tag{2.16}$$

Suppose that we wish to assign only k new eigenvalues, $\lambda_1, \ldots, \lambda_k$, and leave the remaining $n - k$, $\mu_{k+1}, \ldots, \mu_n$, in their original positions. We reorder the rows and columns of Q and R so that the

eigenvalues to be reassigned lie in the first k positions along the diagonal of R, those to be left along the final $n - k$ positions of the diagonal and so that the upper triangular structure of R is retained, and we partition the decomposition thus:

$$A^T = [Q_1, Q_2]\begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix}\begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$$

so that R_1 has the k eigenvalues to be overwritten along its diagonal and R_3 contains the $n - k$ eigenvalues to be left.

Then,

$$A = [Q_1, Q_2]\begin{bmatrix} R_1^T & 0 \\ R_2^T & R_3^T \end{bmatrix}\begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}. \qquad (2.17)$$

Define $\tilde{B}_i = Q_i^T B$ $i = 1,2$ so that $\tilde{B}_1$ is $k \times m$, $\tilde{B}_2$ is $(n-k)xm$

and $\tilde{C}_i = CQ_i$ $\tilde{C}_1$ is $p \times k$, $\tilde{C}_2$ is $px(n-k)$. (2.18)

Then
$$Q^T(A + BF_0C)Q = \begin{bmatrix} R_1^T & 0 \\ R_2^T & R_3^T \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}F_0\begin{bmatrix} \tilde{C}_1, & \tilde{C}_2 \end{bmatrix}$$

$$= \begin{bmatrix} R_1^T + \tilde{B}_1F_0\tilde{C}_1 & \tilde{B}_1F_0\tilde{C}_2 \\ R_2^T + \tilde{B}_2F_0\tilde{C}_1 & R_3^T + \tilde{B}_2F_0\tilde{C}_2 \end{bmatrix}. \qquad (2.19)$$

We aim to assign eigenvalues to $R_1^T + \tilde{B}_1F_0\tilde{C}_1$ as for the state feedback. Here the dimensional condition reduces to $k \geq m$ since there are now k states in the partial system.

So we assign k new eigenvalues to $R_1^T + \tilde{B}_1\tilde{F}_1$ and solve the equation $\tilde{F}_1 = F_0\tilde{C}_1$ with the constraint that $F_0\tilde{C}_2 = 0$, so that (2.19) reduces to

$$Q^T(A + BF_0C)Q = \begin{bmatrix} R_1^T + \tilde{B}_1\tilde{F}_1 & 0 \\ R_2^T + \tilde{B}_2\tilde{F}_1 & R_3^T \end{bmatrix} \qquad (2.20)$$

and the system then has eigenvalues $\lambda_1,\ldots,\lambda_k$ which are assigned to $R_1^T + \tilde{B}_1\tilde{F}_1$ by $\tilde{F}_1$ and $\mu_{k+1},\ldots,\mu_n$ which remain in their original positions.

Note that state feedback requires there to be more than, or an equal number of states than inputs, so that, in this partial case, the condition is imposed that $k > m$.

2.5 Least squares method

Assume that we have assigned the eigenvalues λ_i, $i = 1,\ldots,k$ to the matrix $R_1^T + \tilde{B}_1\tilde{F}_1$ and have computed $\tilde{F}_1$. Then we wish to solve

$$\tilde{F}_1 = F_0\tilde{C}_1 \quad \text{subject to} \quad F_0\tilde{C}_2 = 0 \; . \qquad (2.21)$$

We may cast this problem in the least squares form:

$$\min_{F_0^T} \left\| \tilde{C}_1^T F_0^T - \tilde{F}_1^T \right\| \quad \text{subject to} \quad \tilde{C}_2^T F_0^T = 0 \qquad (2.22)$$

where $\|\cdot\|$ is any Holder norm.

i.e. we minimise the residual of the solution to (2.21).

The most common method for solution is the direct QR method for the least squares problem with equality constraints (LSE problems). In the following section we examine this method and comment on its suitability.

2.5.1 Direct method for LSE

We seek to solve the problem (2.22). First, to ensure satisfaction of the constraint, we take the QR decomposition of $\tilde{C}_2$:

$$\tilde{C}_2 = \tilde{Q}\begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1, \tilde{Q}_2 \end{bmatrix}\begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} = \tilde{Q}_1\tilde{R} . \qquad (2.23)$$

Here we require that $p \geq n-k$ or there would be more constraints than variables and we would not, in general, be able to satisfy all of the constraints, i.e. we would have an over-determined set of equations in $\tilde{C}_2^T F_0^T = 0$.

Define

$$\tilde{Q}^T F_0^T = Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \qquad (2.24)$$

so that $$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1^T \\ \tilde{Q}_2^T \end{bmatrix} F_0^T = \begin{bmatrix} \tilde{R}^{-T}\tilde{C}_2^T F_0^T \\ \tilde{Q}_2^T F_0^T \end{bmatrix}$$ from (2.23) .

Then, since $\tilde{C}_2^T\tilde{Q}_2 = 0$ and $\tilde{Q}_2$ is the full orthonormal basis for the null space of $\tilde{C}^T$, the constraint is satisfied iff

$$F_0^T = \tilde{Q}_2 Y_2 .$$

Hence,

$$\| \tilde{C}_1^T F_0^T - \tilde{F}_1^T \| = \| \tilde{C}_1^T\tilde{Q}_2 Y_2 - \tilde{F}_1^T \| .$$

Therefore, $Y_2 = \left[\tilde{C}_1^T\tilde{Q}_2 \right]^+ \tilde{F}_1^T$ where $^+$ denotes the Moore-Penrose pseudo-inverse

and $$F_0^T = \tilde{Q}_2 [\tilde{C}_1^T\tilde{Q}_2]^+ \tilde{F}_1^T . \qquad (2.25)$$

Note that this method yields three dimensional requirements as a result of QR decompositions and the ensuing partitioning, that $n \geq k$, $n \geq p$ and $p \geq n-k$. The first two conditions are trivial and should be satisfied for any system. The final condition, however, gives some cause for concern, and an alternative method is sought for the cases for which it is not satisfied. We explore such a method in the following section.

2.5.2 Weighted least squares method

Here we alter the problem slightly and relax the constraint by bringing it into the minimisation so that we have

$$\min_{F_0^T} \left\| \begin{bmatrix} \tilde{C}_1^T \\ \tilde{C}_2^T \end{bmatrix} F_0^T - \begin{bmatrix} \tilde{F}_1^T \\ 0 \end{bmatrix} \right\| . \tag{2.26}$$

We can place more emphasis on the constraint by introducing a weighting parameter (Golub and van Loan (1986)), ϕ , which in theory may be large, but in practice is restricted in size by the potential rounding errors it may introduce. Barlow (1986) has shown that an optimal value of ϕ is $\mu^{-1/3}$, where μ is the machine unit round-off.

We operate then, on the system

$$\min_{F_0^T} \left\| \begin{bmatrix} \tilde{C}_1^T \\ \phi\, \tilde{C}_2^T \end{bmatrix} F_0^T - \begin{bmatrix} \tilde{F}_1^T \\ 0 \end{bmatrix} \right\| \tag{2.27}$$

by using the QR technique on the matrix

$$\begin{bmatrix} \tilde{C}_1^T \\ \phi\, \tilde{C}_2^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1, \tilde{Q}_2 \end{bmatrix} \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix}$$

so that (2.27) which is equivalent to

$$\min_{F_0^T} \left\| \tilde{Q}^T \left[\begin{bmatrix} \tilde{C}_1^T \\ \phi \; \tilde{C}_2^T \end{bmatrix} F_0^T - \tilde{Q}^T \begin{bmatrix} \tilde{F}_1^T \\ 0 \end{bmatrix} \right] \right\|$$

$$= \min_{F_0^T} \left\| \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} F_0^T - \begin{bmatrix} \tilde{Q}_1^T \\ \tilde{Q}_2^T \end{bmatrix} \begin{bmatrix} \tilde{F}_1^T \\ 0 \end{bmatrix} \right\|$$

$$= \min_{F_0^T} \left\| \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} F_0^T - \begin{bmatrix} \tilde{Q}_1^T & \begin{bmatrix} \tilde{F}_1^T \\ 0 \end{bmatrix} \\ \tilde{Q}_2^T & \begin{bmatrix} \tilde{F}_1^T \\ 0 \end{bmatrix} \end{bmatrix} \right\| \qquad (2.28)$$

which is minimised when $F_0^T = \tilde{R}^{-1} \tilde{Q}_1^T \begin{bmatrix} \tilde{F}_1^T \\ 0 \end{bmatrix}$

with residual $\left\| \tilde{Q}_2^T \begin{bmatrix} \tilde{F}_1^T \\ 0 \end{bmatrix} \right\|$.

Again, this method yields dimensional conditions, but in this case they are limited to the previous trivial conditions, i.e. $n \geq k$ and $n \geq p$.

The disadvantage of this method is that neither the original minimisation nor the constraint may be fully satisfied. Relaxing the constraint in this way ultimately leads to some change in the eigenvalues which had been designated as fixed, particularly because the structure (2.21) is not achieved; neither does it improve the solution of $\min_{F_0^T} \left\| \tilde{C}_1^T F_0^T - F_1^T \right\|$.

3. Error analysis

Having described two least squares methods which, in theory, should cover all cases for the purposes of partial pole placement provided that $k \geq m$, we examine in this section the errors incurred by these methods and the effects that any such errors might have on the eigenvalues of the resulting output feedback matrix.

3.1 Eigenvalue perturbations

Let the eigenvalues of $M = R_1^T + \tilde{B}_1\tilde{F}_1$ be $\lambda_1,\ldots,\lambda_k$, where $X^{-1}MX = \text{diag}(\lambda_i)$, $i = 1,\ldots,k$, and suppose that M is approximated by $\tilde{M} = R_1^T + \tilde{B}_1F_0\tilde{C}_1$. Then the error matrix, E, is given by

$$\begin{aligned} E &= M - \tilde{M} \\ &= \tilde{B}_1(\tilde{F}_1 - F_0\tilde{C}_1) . \end{aligned} \tag{3.1}$$

Let the eigenvalues of $\tilde{M}$ be $\mu_1,\ldots,\mu_k$, then by the Bauer-Fike theorem (1960), a weak bound is given by

$$\min_{\lambda\in\lambda(M)} |\lambda-\mu| \leq \kappa(X)\ \|E\|$$

where $\kappa(X)$ is the condition number of X.

The first of the two methods employed in the previous section may be analysed in a straightforward manner and we label the error obtained from its residual E_1. The error from the second we denote by E_2. Both share an interesting characteristic which can be used directly to affect the choices of X in (2.9) to form the state feedback matrix.

3.2 The LSE method

In using this method we know that the constraint is enforced and that any error reflects only how well $F_0\tilde{C}_1$ approximates $\tilde{F}_1$. We may consider then the residual:

$$\begin{aligned}\tilde{C}_1^T F_0^T - \tilde{F}_1^T &= \tilde{C}_1^T\tilde{Q}_2(\tilde{C}_1^T Q_2)^+\tilde{F}_1^T - \tilde{F}_1^T \\ &= (\tilde{C}_1^T\tilde{Q}_2(\tilde{C}_1^T Q_2)^+ - I)\tilde{F}_1^T \\ &= P\tilde{F}_1^T \quad \text{say,}\end{aligned}$$

so that the actual residual $F_0\tilde{C}_1 - \tilde{F}_1$ is given by $\tilde{F}_1 P^T$ and the error E_1 is given by

$$E_1 = \tilde{B}_1\tilde{F}_1 P^T . \tag{3.2}$$

3.3 The LS method

From section 2.5.2 we know that

$$\begin{bmatrix}\tilde{C}_1 \\ \phi\tilde{C}_2\end{bmatrix}F_0^T - \tilde{F}_1^T = \tilde{Q}\left[\begin{bmatrix}\tilde{R} \\ 0\end{bmatrix}F_0^T - \begin{bmatrix}\tilde{Q}_1^T & \begin{bmatrix}\tilde{F}_1^T \\ 0^1\end{bmatrix} \\ \tilde{Q}_2^T & \begin{bmatrix}\tilde{F}_1^T \\ 0^1\end{bmatrix}\end{bmatrix}\right] = \tilde{Q}\begin{bmatrix}0 \\ \tilde{Q}_2^T\begin{bmatrix}\tilde{F}_1^T \\ 0\end{bmatrix}\end{bmatrix}$$

at the optimum value of F_0 ,

i.e. the residual is $\tilde{Q}_2\tilde{Q}_2^T\begin{bmatrix}\tilde{F}_1^T \\ 0\end{bmatrix}$. (3.3)

However, (3.3) represents the error for the whole problem and we wish to separate it into the residual of $\tilde{C}_1^T F_0^T - \tilde{F}_1^T$ and that of the

constraint. By partitioning $\tilde{Q}_2\tilde{Q}_2^T$ so that

$$\tilde{Q}_2\tilde{Q}_2^T\begin{bmatrix}\tilde{F}_1^T\\ 0\end{bmatrix} = \begin{bmatrix}Q_3\\ Q_4\end{bmatrix} \quad \text{where } Q_3 \text{ is } k \times m .$$

Then $$\tilde{C}_1^T F_0^T - \tilde{F}_1^T = Q_3$$

and $$\tilde{C}_2^T F_0^T = \frac{1}{\phi} Q_4 . \tag{3.4}$$

Here, $$M = \begin{bmatrix} R_1^T + \tilde{B}_1\tilde{F}_1 & 0 \\ R_2^T + \tilde{B}_2\tilde{F}_1 & R_3^T \end{bmatrix}$$

and $$\tilde{M} = \begin{bmatrix} R_1^T + \tilde{B}_1\tilde{F}_0\tilde{C}_1 & \tilde{B}_1F_0\tilde{C}_2 \\ R_2^T + \tilde{B}_2\tilde{F}_0\tilde{C}_1 & R_3^T + \tilde{B}_2F_0\tilde{C}_2 \end{bmatrix}$$

so that $$E_2 = M - \tilde{M} = \begin{bmatrix} \tilde{B}_1(\tilde{F}_1 - F_0\tilde{C}_1) - \tilde{B}_1F_0\tilde{C}_2 \\ \tilde{B}_2(\tilde{F}_1 - F_0\tilde{C}_1) - \tilde{B}_2F_0\tilde{C}_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\tilde{B}_1Q_3^T - \frac{1}{\phi}\tilde{B}_1Q_4^T \\ -\hat{B}_2Q_3^T - \frac{1}{\phi}\tilde{B}_2Q_4^T \end{bmatrix} . \tag{3.5}$$

4. Numerical results

In this section we examine three numerical examples. The first two employ the standard QR method for LSE and the third, due to the dimensions of the matrices, uses the weighted QR method for LS problems as described in section 2.5.2. All three examples have controllable and observable (Barnett (1975)) systems (A,B,C) and subsystems $(\tilde{R}_1, \tilde{B}_1, \tilde{C}_1)$. All three examples were worked on a Nord computer.

Example 4.1

$$A = \begin{bmatrix} 5.1017 & -0.0808 & -0.4009 & -2.4388 \\ 0.1918 & 1.4069 & 1.1396 & -1.6548 \\ 2.3891 & 0.2025 & 1.5873 & -1.9924 \\ 1.0183 & -0.6796 & -0.2174 & 1.9041 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here $n = 4$, $m = 2$, $p = 3$.

A has eigenvalues $\mu = 1, 2, 3, 4$ and we attempt to reassign $\mu = 2, 3, 4$ to the new values $\lambda = -1, -2, -3$ and leave the remaining eigenvalue as $\mu = 1$.

State feedback produces the state feedback matrix $\tilde{F}$, which assigns eigenvalues $-1, -2, -3$ to the partial system and is given by

$$\tilde{F}_1 = \begin{bmatrix} 28.9274 & -14.6695 & -16.3431 \\ 18.6220 & -2.5108 & -7.9058 \end{bmatrix}.$$

This in turn yields the output feedback matrix F_0, where

$$F_0 = \begin{bmatrix} 29.3759 & 7.6350 & -18.7981 \\ 16.1464 & 6.5981 & -10.3340 \end{bmatrix}$$

and assigns the eigenvalues -2.8984, -0.9001, 2.4645 and 1.000 to the system $A + BF_0C$.

The solution matrix F_0 gives the residual

$$\|F_0\tilde{C}_1 - \tilde{F}_1\| = 7.0021$$

and yields the unscaled Bauer-Fike upper bound:

$$\min|\lambda - \mu| \leq 78.381$$

which validates, albeit pessimistically, the difference between the best matching expected and computed eigenvalues.

Example 4.2

Here

$$A = \begin{bmatrix} 1.3800 & -0.2077 & 6.7150 & -5.6760 \\ -0.5814 & -4.2900 & 0.0000 & 0.6750 \\ 1.0670 & 4.2730 & -6.6540 & 5.8930 \\ -0.0048 & 4.2730 & 1.3430 & -2.1040 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0000 & 0.0000 \\ 5.6790 & 0.0000 \\ 1.1360 & -3.1460 \\ 1.1360 & 0.0000 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

so that $n = 4$, $m = 2$, $p = 3$.

In this case, the eigenvalues of A are given by $\mu = 1.9910$, 0.0635, -8.6659 and -5.0566 and we aim to reassign the first three of these as -0.2, -0.5 and -8.6659 and leave $\mu = -5.0566$ in its original position. Again, the state feedback procedure successfully yields a feedback matrix $\tilde{F}_1$, which assigns the required eigenvalues to the partial system, where

$$\tilde{F}_1 = \begin{bmatrix} 0.6551 & -0.2735 & 0.1049 \\ 0.7971 & -1.2260 & 0.0048 \end{bmatrix}.$$

Employing this matrix in the LSE problem yields:

$$F_0 = \begin{bmatrix} 0.2776 & 0.2152 & 0.2651 \\ 1.1257 & 0.4605 & 0.6467 \end{bmatrix}.$$

The eigenvalues of $A + BF_0C$ are 0.8157, -0.1269, -7.8117 and -5.0566 .

The residual in solving the minimisation is given by

$$\|F_0\tilde{C}_1 - \tilde{F}_1\| = 0.7511 ,$$

so that the Bauer-Fike bound is:

$$\min_{\lambda} |\lambda - \mu| \leq 2.341 .$$

It should be noted that the weighted method gave almost identical results to those obtained by the LSE method when applied to Examples 4.1 and 4.2.

Example 4.3

In this final example we choose the matrices so that $p < n-k$.

$$A = \begin{bmatrix} -20.2060 & 4.5847 & 12.7400 & 1.4426 & -0.8052 \\ -62.9920 & 15.9710 & 34.9860 & 4.1658 & -0.9161 \\ -13.9080 & 2.7680 & 10.3330 & 0.6951 & 0.9239 \\ -33.2010 & 5.9492 & 18.3740 & 5.5345 & 1.5275 \\ -37.5820 & 6.9960 & 21.4760 & 3.2463 & 3.3680 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Here $n = 5$, $m = 2$, $p = 2$ and the eigenvalues of A are $\mu = 1, 2, 3, 4, 5$. State feedback is employed which reassigns the eigenvalues 1,2 and 5 as -1, -2 and -3 with the feedback matrix

$$\tilde{F}_1 = \begin{bmatrix} -36.319 & 5.5806 & -0.5771 \\ -36.159 & 6.3971 & -1.8434 \end{bmatrix}.$$

The resulting least squares solution, using weighting parameter $\phi = 10^4$, yields an output feedback matrix

$$F_0 = 10^4 \times \begin{bmatrix} -2.0865 & 1.8407 \\ -2.0169 & 1.7776 \end{bmatrix},$$

which leaves all of the original eigenvalues in their original positions.

The residual for the problem are given by:

$$\|F_0\tilde{C}_1 - \tilde{F}_1\| = 51.973 \quad \text{and} \quad \|F_0\tilde{C}_2\| = 1.3 \times 10^{-8}.$$

5. Conclusions

Perhaps the strongest point to emerge from this approach to output feedback is that a small residual in the least squares solution of the equations relating state and output feedback does not guarantee good approximations to the eigenvalues required. Clearly though, reducing the residual can only have beneficial effects and we should consider ways of improving it. We note that both errors resulting from the least squares approach are directly related to the state feedback matrix $\tilde{F}_1$ which assigns eigenvalues λ_i, $i = 1,\ldots k$, to the partial system $R_1^T + \tilde{B}_1\tilde{F}_1$ and that only the construction of $\tilde{F}_1$ may affect the errors E_1 and E_2.

Kautsky et al have shown that the state feedback matrix may be written as:

$$\tilde{F}_1 = \tilde{B}_1^{+}(X\Lambda X^{-1} - R_1^T) \qquad (5.1)$$

where Λ is a diagonal matrix having the eigenvalues of R_1^T along its diagonal

and X is the matrix of eigenvectors corresponding to Λ.

Taking the norms of (5.1), we see that

$$\|\tilde{F}_1\| \leq \|\tilde{B}_1^{+}\|(\|X\|\|X^{-1}\|\max_i|\lambda_i| + \|R_1^T\|)$$

$$= \|\tilde{B}_1^{+}\|(\kappa(X)\max_i|\lambda_i| + \|R_1^T\|) \; . \qquad (5.2)$$

Since we only have control over the choices of $\underline{x}_i$, $i = 1,\ldots,k$, which make up the columns of X, (5.2) suggests that we may minimise an upper bound on $\|\tilde{F}_1\|$ by minimising the condition number of X. The state feedback procedure employed by Kautsky et al (1984) does, in fact, already

incorporate a method for selecting an X which minimises its condition number as this also minimises the sensitivity of the eigenvalues resulting from state feedback.

We might conclude then that this particular approach to output feedback is not the most reliable of methods, although it should be noted that by using the direct least squares method we can at least bypass the dimensional restrictions imposed by other methods. However, this also means that often no eigenvalues are assigned exactly so that the advantages of partial pole placement no longer exist.

Future work is expected to include an investigation into the relationship between the two errors E_1 and E_2, together with more extensive numerical testing and to incorporate some of the early theory developed.

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