ON THE CONTROLLABILITY OF DESCRIPTOR SYSTEMS

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Abstract:

Different concepts of controllability for descriptor systems $E\dot{x} = Ax + Bu$ have been proposed and investigated. In this paper, their relationship, their interpretation in terms of the Kronecker canonical form of (E,A), and the implications in terms of pole assignment, regularity and robustness of the closed-loop system, are discussed.

1. Introduction.

Consider a time - invariant, linear, multivariable, descriptor system in $\mathbb{R}^n$ with linear state feedback, described by

$$\begin{cases} E\dot{x} = Ax + Bu , & (1) \\ u = Fx & (2) \end{cases}$$

where x and u are n- and m-dimensional vectors, the matrix B is assumed to be full ranked and E can be singular.

The generalized eigenvalue problem (GEVP) of the matrix-pencil $A_\lambda = (A - \lambda E)$ has been studied in detail by Gantmacher [1974], Van Dooren [1981] and Wilkinson [1978], and the references therein. (See also the related perturbation analysis in Stewart [1978] and Chu [1985].) The corresponding differential equations, of the type (1), have been studied by Wilkinson [1978] and Campbell [1980,1982]. The pole assignment problem (PAP) has been investigated by Cobb [1981, 1984], Lewis and Ozcaldiran [1984], Ozcaldiran and Lewis [1984], Armentano [1984], Fletcher [1982], Chu and Nichols [1983], Chu [1986b] and the references therein. The PAP is a difficult problem and a lot more work, especially numerical, has to be done.

Apart from the usual complexity arising from the GEVP of the matrix-pencil A_λ , one also has to cope with the following problems:

(i) There are different concepts of "controllability", depending on the allowable initial conditions and whether one is interested in the infinite eigenvalues or not.

(ii) Depending on how "controllable" the system (1) is, one may not know how many eigenvalues one can assign.

(iii) The closed-loop matrix pencil $\tilde{A}_\lambda$, defined as $(A + BF - \lambda E)$, may be singular for some feedback matrix F , in the sense that

$$\det \ (\tilde{A}_\lambda) = 0 \tag{3}$$

independent of λ . (c.f. Gantmacher [1974], Golub and Van Loan [1983].)

(iv) Given the eigenvector matrices X and Y such that $Y^H \tilde{A}_\lambda X$ is in the Kronecker canonical form, it is not clear whether X and Y are well-conditioned or "robust" in any sense. Here $(.)^H$ denotes the Hermitian.

In this paper, different concepts of controllability and their mutual relationship are discussed, in the hope that a better understanding of the above problems in (i)-(iv) can be achieved, using the Kronecker canonical form and the related Yip-Sincovec decomposition (Gantmacher [1972], Yip and Sincovec [1981]). Implications on the PAP and its robustness problem are also considered.

2. Controllability.

First one can write the system (1) in the Yip-Sincovec decomposition (Yip and Sincovec [1981]):

$$\begin{cases} \dot{x}_1 = E_1 x_1 + B_1 u & \text{(4a)} \\ E_2 \dot{x}_2 = x_2 + B_2 u \ , & \text{(4b)} \end{cases}$$

with x_i being n_i - dimensional vectors, and the matrix E_2 being nilpotent.

Equation (3) is essentially the result of the transformation of the matrix-pencil A_λ to Kronecker canonical form (Gantmacher [1972]), with the E_i's not restricted to be in Jordan canonical forms. Note that the decomposition in (4) is not unique.

Different concepts of controllability can then be defined as follows:-

(a) R-controllability (RC):- (Van Dooran [1981], Yip and Sincovec [1981], Wonham [1979].)

The system (1) is RC if and only if

$$\text{rank } [\ s E - A, B] = n \tag{5}$$

for all finite complex number s .

(b) C-controlability (CC) : (Yip and Sincovec [1981])

The system (1) is CC if and only if it is RC and

$$\text{rank } (E \ , B) = n \ . \tag{6}$$

(c) S-controllability (SC) : (Yip and Sincovec [1981]. Verghese et al [1981].)

The system (1) is SC if and only if it is RC and

$$\text{span} \langle E_2/B_2 \rangle = \text{span } [B_2 \ , E_2 B_2, \ldots ,E_2^{n_2-1} B_2] \supset \text{span } (E_2). \tag{7}$$

(d) Complete assignability (CA): (Armentano [1984], Chu [1986 b] Chu and Nichols [1983],)

The system (1) is CA if and only if it is RC and

$$\text{rank } [\ AS_\infty \ , E \ , B\ ] = n, \tag{8}$$

where $\ker(E) = \text{span}\ (S_\infty)$ and the matrix $(S_E\ , S_\infty)$ is orthogonal.

It is obvious that RC corresponds to the controllability of finite eigenvalues. It can be proved (Armentano [1984]) that condition (8) corresponds to the controllability of the infinite eigenvalues in the sense that no more than n-rank (E) so many infinite eigenvalues are assigned. It can be easily shown to be the case, using a different but interestingly simple argument:

The feedback matrix F can only assign less than rank (E) = q finite eigenvalues if and only if the closed-loop matrix-pencil $\tilde{A}_\lambda$ has more than (n-q) inifinite eigenvalues. As the matrix E is of rank q and thus only (n-q) linearly independent null-vectors, there must exist non-linear elementary divisors for the infinite eigenvalues. As a result, the feedback matrix F will assign exactly q eigenvalues if and only if (i) the system is RA, and (ii) there exists no principal vectors or non-linear elementary divisors for the zero eigenvalues of E , i.e.

$$\nexists\ x_0\ ,\ x_1 \neq 0\ ,\ x_0 \in \ker(E) \text{ such that}$$

$$(A + BF)\, x_0 = Ex_1 \ ;$$

$$\Leftrightarrow \quad E(S_E\ , S_\infty) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (A + BF)\,(S_\infty p_3) \ ;$$

does not have non-trivial solutions p_1 and p_3 ;

$$\Leftrightarrow \quad [ES_E\ , (A + BF)\, S_\infty] \begin{bmatrix} p_1 \\ p_3 \end{bmatrix} \neq 0 \ , \quad \forall p_1\ , p_3 \neq 0 \ ;$$

$$\Leftrightarrow \quad (8)$$

3. Controllability in Terms of the Kronecker Canonical Form

The following observations can easily be made from the above definitions (a)-(d), by considering the Kronecker canonical form or Yip-Sincovec decomposition in (4), or using other standard techniques:

Lemma 1.

(i) The following conditions are equivalent:

(a) RC ;

(b) rank $[sI_n - E , B] = n$, $\forall\ s \in \mathbb{C}$

(c) rank $[sI_{n_1} - E_1 , B_1] = n_1$, $\forall\ s \in \mathbb{C}$;

(d) rank $\langle E_1 \mid B_1 \rangle = n_1$;

(e) rank $[B , (sE - A) S_E , AS_\infty] = n$, $\forall\ s \in \mathbb{C}$.

(ii) The following conditions are equivalent:

(a) CC ;

(b) RC and rank $(E , B) = n$;

(c) RC and rank $(E_2 , B_2) = n_2$;

(d) RC and span $(B) \supset \ker (E^T)$;

(e) RC and span $(B_2) \supset \ker (E_2^T)$.

(iii) The following conditions are equivalent:

(a) SC ;

(b) RC and $\langle E_2 \mid B_2 \rangle \supset$ span (E_2) ;

(c) RC and rank $[\langle E_2 \mid B_2 \rangle , \ker (E_2^T)] = n_2$,

(iv) The following conditions are equivalent:

(a) CA ;

(b) RC and rank $(AS_\infty , E , B) = n$;

(c) RC and rank $(\ker (E_2) , E_2 , B_2) = n_2$;

(d) RC and span $(E_2, B_2) \supset$ span (E_2^T) .

Proof:- Only (iv) requires some explanations.

(b) is the definition of (a) in (8).

(b) $\Leftrightarrow$ (c) : consider the canonical form in (4), one has

rank $(AS_\infty, E, B) = n$

$$\Leftrightarrow \text{rank} \begin{bmatrix} 0 & I & 0 & B_1 \\ \ker(E_2) & 0 & E_2 & B_2 \end{bmatrix} = n$$

and thus (b) $\Leftrightarrow$ (c) .

(c) $\Leftrightarrow$ (d) because span $(E_2^T) \oplus \ker(E_2) = \mathbb{R}^{n_2}$. ■

Note that by attaching the parameter s to the matrix A , instead of E , in (i) (e) and passing the limit $s \to 0$, will produce the condition in (8). Condition (6) and (i) (b) are related in a similar way.

It is also clear from Lemma 1 that SC is a quite different concept from the others.

The following characterizations for various controllability concepts can be proved using the Kronecker canonical form (Gantmacher [1972]) of A_λ : (we cannot prove a similar result for SC.)

Lemma 2 :- (i) RC $\Leftrightarrow B^T Z_1$ is full-ranked, with

span (Z_1) = { left-eigenvectors corresponding to the finite eigenvalues of A_λ.}

(ii) CC $\Leftrightarrow B^T Z_1$ and $B^T Z_2$ are full-ranked, with Z_1 as defined in (i) and

span (Z_2) = { left-eigenvectors corresponding to the infinite eigenvectors of A_λ.}

(iii) $CA \Leftrightarrow B^T Z_1$ and $B^T Z_3$ are full-ranked,

with Z_1 as defined in (i) and

span (Z_3) = { left-eigenvectors corresponding to infinite eigenvalues, with non-linear elementary divisors, of A_λ.}

Proof:- (i) is a trivial generalization of the well-known result for non-descriptor system.

(i), (ii) and (iii) can be proved from the characterizations (i) (b) , (ii) (b) , (iv) (b) in Lemma 1, with A_λ in Kronecker canonical form in (4). ■

The following theorem on the relationship among various concepts of controllability can be stated :

Theorem 3. $CC \Rightarrow CA \Rightarrow RC$, $SC \Rightarrow RC$,

and the converses are not true.

Proof:- $CA \Rightarrow RC$, $SC \Rightarrow RC$ and $CC \Rightarrow CA$ are obvious from the respective definitions, or Lemma 1 or 2. The converses can be disproved bv counter examples, constructed by applying Lemma 1 or 2. ■

One can also use Lemma 2 to obtain the minimum number of linearly independent controls, m , required for the system (1) to be controllable :

Corollary 4 : The minimum value of rank (B) = m required so that the system (1) does not have to be "uncontrollable" in their respective sense, is as follows; (with Z_i 's as defined in Lemma 2)

(i) For RC , $m \geqq m_{RC} = \text{rank}\ (Z_1)$.

with A_λ in Kronecker canonical form, m_{RC} is the number of Jordan blocks in A_λ for the finite eigenvalues.

(ii) For CA , $m \geqq m_{CA} = \max \{m_{RC} , \rho_1\}$

where $\rho_1 = \text{rank } (Z_2)$, the number of non-trivial Jordan blocks corresponding to infinite eigenvalues of A_λ .

(iii) For CC , $m \geqq m_{CC} = \max \{m_{RC} , \rho_2\}$

where $\rho_2 = \text{rank } (Z_3)$, the number of Jordan blocks corresponding to infinite eigenvalues of A_λ .

(iv) $m_{RC} \leqq m_{CA} \leqq m_{CC}$.

Property (iv) in the Corollary shows that requirements on the input matrix B become more and more severe, as one moves from RC to CA , and then to CC . If $m = m_{RC}$, m_{CA} or m_{CC} , the system will be potentially controllable in their respective sense and the components in B can then be chosen with care to satisfy the requirements of Lemma 2.

Corollary 4 provides a simple test of uncontrollability or potential controllability when the Kronecker canonical form, or geometric structure of the eigenvectors, of A_λ is available.

It is unclear how SC is related to other concepts, except SC ⇒ RC . Other properties of the various concepts of controllability can be found in the references in the reference - list, and more work is obviously needed in this area.

We now concentrate on systems which are CA .

5. Regularity.

In order to find a feedback matrix F so that the closed-loop matrix-pencil $\tilde{A}_\lambda$ is regular, or (3) does not happen, one has the following theorem for systems which are CA :-

Theorem 5. For CA systems, there exists feedback matrix F such that $\tilde{A}_\lambda = [(A + BF) - \lambda E]$ is regular.

Proof:- Let $X = (X_q , S_\infty)$ and $Y = (Y_q , T_\infty)$ be non-singular matrices such that $Y^H \tilde{A}_\lambda X$ is in Kronecker canonical form. The matrix S_∞ and T_∞ can be chosen to be real.

The matrix-pencil is regular if and only if the matrix

$$M = T_\infty^T A S + T_\infty^T B . F X_\infty$$

$$= T_\infty^T A S_\infty + T_\infty^T B . G_\infty \qquad (9)$$

is non-singular, and there are q-finite eigenvalues for $\tilde{A}_\lambda$ (a consequence of CA).

By considering the rows of the following matrix (which is full-ranked because of CA):

$$Y^H (sE - A , B) \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \quad \text{or} \quad Y^H (A S_\infty , E , B)$$

the matrix $(T_\infty^T A S_\infty , T_\infty^T B)$ can be proved to be full-ranked, in turn implies that the matrix M in (9) is non-singular for some matrix G_∞. By selecting $G_q = F X_q$, the feedback matrix can be retrived by solving the matrix equation

$$F X = G = (G_q , G_\infty) , \qquad (10)$$

with the non-singular matrix operator X .

Equation (9) indicates a way of finding G_∞ for a non-singular matrix M , and the PAP for CA descriptor systems now reduces to finding the eigenvector matrix X which assigns the prescribed set of q finite eigenvalues.

Note that if the open-loop matrix-pencil A_λ is already regular one can choose $G_\infty = 0$.

6. CA Controllability Condensed Form.

In Chu [1986b] a descriptor system represented by (E, A, B) can be transformed to a controllability condensed form by orthogonal transformation (P, Q, Z), such that $Q^T(E, A, B) \cdot \operatorname{diag}(Z, Z, P) =$

$$\left[\begin{array}{cc|ccc|c}
\begin{matrix} E_{kk} & & 0 \\ & \ddots & \\ * & & E_{11} \end{matrix} & 0 & \begin{matrix} A_{kk} & A_{k,k+1} & & 0 \\ & \ddots & \ddots & \\ * & & \ddots & A_{11} \end{matrix} & \begin{matrix} 0 \\ A_{12} \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline
* & 0 & * & A_{00} & A_{01} & 0 \\ \hline
* & 0 & * & * & A_{-1,-1} & A_{-1,0}
\end{array}\right]$$

with A_{ii} and E_{ii} being square, and E_{ii} non-singular.

The system will be CA if and only if the matrices $A_{i,i+1}$ are of full-row-rank.

A direct algorithm for the PAP was then proposed based on the above condensed form.

Please refer to Chu [1986a, b] for details, with related work in Miminis and Paige [1982], Paige [1981], Varga [1981], Van Dooren [1985].

7. An Iterative Pole Assignment Algorithm.

For CA descriptor systems, problems (i)-(iii) in section 1 are solved, based on the discussion in the previous sections. The PAP will then be solved if one selects the eigenvectors x_j in the columns of X_q carefully to ensure that

(i) The q finite eigenvalues $\{\lambda_1, \ldots, \lambda_q\}$ are assigned.

(ii) The closed-loop matrix-pencil $\tilde{A}_\lambda$ is regular, based on the selection of G_∞ as discussed in Section 5.

and (iii) The matrix $X = (x_q , S_\infty)$ in (10) is non-singular.

It is well-known, from Chu and Nichols [1983], Kautsky et al [1985] and Wonham [1979], that (i) is satisfied with

$$\text{span} \begin{pmatrix} S_j \\ G_j \end{pmatrix} = \ker (\lambda_j E - A , B) , \tag{11}$$

$x_j = S_j u_j$ and columns of G_q in (10) , q_j , chosen to be $G_j u_j$.

It is obvious from (11) that

$$\mathcal{S}_j = \text{span} (S_j) = \ker \{(I - B B^+) . (\lambda_j E - A)\} \tag{12}$$

with $(.)^+$ denoting the (1,2,3,4) - or Penrose-speudo - inverse (Golub and Van Loan [1983]).

A consequence of (11) and (12) is that

$$\dim (\mathcal{S}_j) = m ,$$

and it will be more convenient to assume that the eigenvalues λ_j have no non-linear elementary divisors and the multiplicity of λ_j is less than or equal to m , as in Chu and Nichols [1983].

The eigenvectors x_j are then selected iteratively to ensure that (iii) is satisfied, with any degree of freedom left used to optimize the conditioning of the closed-loop eigensystem. (For more detail, see Chu and Nichols [1983], Kautsky et al [1985], Chu [1985] and Stewart [1978]; see also Section 8.)

An equivalent algorithm was also suggested by Armentano [1984], with the restriction that $(\lambda_j E - A)$ has to be invertible. The restriction can be removed by better management of numerical processes.

8. Robustness.

For algorithms which solve the PAP by the selection of eigenvectors x_j of $\tilde{A}_\lambda$, (e.g. Chu [1986b], Chu and Nichols [1983]; see Sections

6 and 7.), we can prove some useful results concerning the robustness of the closed-loop system, involving the conditioning of the eigenvector matrices X and Y .

(i) From (10) - (12), one has

$$F X = G = (G_q , G_\infty) \quad (13)$$

with $G_q = B^+(X_q \Lambda_q - A X_q)$ (14)

and $\Lambda_q = \text{diag} \{\lambda_1 , \dots ,\lambda_q\}$.

Equation (13) implies that

$$\| F \|_2 \leqq \| X^{-1} \|_2 \cdot \| G \|_2 , \quad (15)$$

and thus the feedback gain matrix F will not be too large if X is not too ill-conditioned and G is reasonably small. In Chu [1986b], the conditioning of X and the size of G are implicitly optimized.

In Chu and Nichols [1983], the conditioning of X is optimized, and using (14), (15) implies that

$$\| F \|_2 \leqq \| X^{-1} \|_2 \cdot \{ \| B^+ \|_2 \cdot \| X_q \Lambda_q - A X_q \| + \| G_\infty \| \} .$$

(ii) From $E\dot{x} = (A + BF) x$ and using the Drazin inverse in Campbell [1980, 1982], one has

$$x(t) = X_q e^{\Lambda_q t} Y_q^H \cdot x_0 , \quad (16)$$

with X_q and Y_q containing the right- and left-eigenvectors for the finite eigenvalues λ_i , and x_0 denoting the initial state in span $[X_q Y_q^H]$. (λ_i are assumed to be non-defective in this case.) Equation (16) implies

$$\| x(t) \|_2 \leqq \tilde{\kappa}_2(X_q) \cdot \| x_0 \|_2 \cdot \max \{ |e^{\lambda_i t}| \} \quad (17)$$

with $\tilde{\kappa}_2 (X_q) = \| X_q \|_2 \cdot \| Y_q \|_2$.

Note that it has been proved in Chu [1985] that $\tilde{\kappa}_2 (X_q)$ is related to a condition number for the finite eigenvalues of the GEVP of $\tilde{A}_\lambda$.

In equality (17) provides us with an upper bound of the state vector $x(t)$, and the bound will be tighter if $\tilde{\kappa}_2(X_q)$ is smaller, or λ_i , $i = 1 , \ldots , q$; better conditioned. Note that $x(t) \to 0$ when all λ_i have negative real parts.

(iii) Similar to Kautsky et al [1984], one can prove the following result for the stability margin of the descriptor system:

Assume that all λ_i are non-defective. Similar to (4), the Kronecker canonical form of $\tilde{A}_\lambda$ will be in the analogous form:

$$Y^H(A + BF) X = \begin{pmatrix} \Lambda_q & 0 \\ 0 & I \end{pmatrix} . \tag{18a}$$

and
$$Y^H E X = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} . \tag{18b}$$

Using a standard argument, any matrix $M + \Delta = M(I + M^{-1} \Delta)$ will be non-singular, assuming that M already is, provided that

$$\| M^{-1} \Delta \|_2 \leq \| M^{-1} \|_2 \cdot \| \Delta \|_2 < 1$$

$$\Leftarrow \quad \| \Delta \|_2 < \| M^{-1} \|_2^{-1} = \sigma_n(M) .$$

Here $\sigma_n(M)$ denotes the smallest singular value of the $n \times n$ matrix M .

Apply the same argument to the closed-loop system matrix $A + BF$, then the perturbed closed-loop system matrix $A + BF + \Delta$ remains stable for all disturbaces Δ which satisfies

$$\| \Delta \|_2 \leq \min_{s = jw} \sigma_n\{s E - (A + BF)\} \equiv \delta(F) , \tag{19}$$

where $j = \sqrt{-1}$.

From (19), $\delta(F)$ has the lower bound

$$\delta(F) = \min_{s=jw} \sigma_n \left\{ Y^{-H} \begin{bmatrix} sI - \Lambda q & 0 \\ 0 & -I \end{bmatrix} X^{-1} \right\}$$

$$\geq \sigma_n(Y^{-1}) \cdot \sigma_n(X^{-1}) \cdot \min_{s=jw} \sigma_n \left\{ \begin{bmatrix} sI - \Lambda_q & 0 \\ 0 & -I \end{bmatrix} \right\}$$

$$\geq \min \{Re(-\lambda_i) , 1\} \Big/ \|X\|_2 \cdot \|Y\|_2 \quad . \qquad (20)$$

In equality (20) means that if X and Y are ill-conditioned, then the lower bound of $\delta(F)$ will be small, and thus the allowable size of $\|\Delta\|_2$ for the closed-loop system matrix to remain stable may be small. $\|\Lambda_q^{-1}\|_2$ and $\|X\|_2 \cdot \|Y\|_2$ in the RHS of (20) have been proved to be related to a conditon number of the GEVP of $\tilde{A}_\lambda$ (Chu [1985].)

Consider the stability margin $\tilde{\delta}(F)$, where the return difference $I + G(s) + \tilde{\Delta}(s)\ G(s)$ of the disturbed closed-loop system, with $G(s) = -F\ (sI - A)^{-1}\ B$, remains non-singular at $s = jw$ for disturbances $\tilde{\Delta}(s)$ which satisfies $\|\hat{\Delta}(jw)\|_2 < \tilde{\delta}(F)$.

It is easy to show that

$$\det [sI - (A + BF + \Delta)] = \det (sI - A) \,.\, \det [I + (I + \tilde{\Delta}(s))\, G(s)]$$

with $\Delta = B\,\tilde{\Delta}(s)\,F$. Hence $I + G(s) + \tilde{\Delta}(s)\,G(s)$ is non-singular at $s = jw$ provided that

$$\| \Delta \|_2 \leq \| B \|_2 \,.\, \| \tilde{\Delta}(jw) \|_2 \,.\, \| F \|_2 < \delta(F) \,. \qquad (21)$$

A lower bound of the stability margin is thus

$$\delta(F) \geq \delta(F)/(\| B \|_2 \,.\, \| F \|_2) \qquad (22)$$

from (21).

Other lower bounds can be obtained when $\| F \|_2$ in (22) is further bounded by using (15).

From (20), (22) and (15), the stability margin will thus be larger if the closed-loop eigensystem is well-conditioned in the sense that

$\hat{\kappa} = \| X \|_2 \,.\, \| Y \|_2$ in (20), is small.

9. Conclusion

It is shown in this paper that, for CA descriptor systems, q finite eigenvalues can be assigned so that the closed-loop system is regular. Based on the discussions on robustness in Section 8, the problems (i)-(iv) in Section 1 have been countered.

However, it is still unclear even for CA descriptor systems whether it is desirable to assign all q finite eigenvalues, or assign some but leave others to remain infinite. Obviously, other controllability concepts (such as SC) may well be more appropriate in different circumstances.

A few numerical algorithms for CA descriptor systems have been proposed (Chu [1986b], Chu and Nichols [1984], Armentano [1984]) but more work, especially numerical, have to be done, in comparison to the vast amount of literature available for the non-descriptor problem.

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