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DYNAMIC–ALGEBRAIC EQUATIONS AND CONTROL SYSTEM DESIGN

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Abstract Techniques for the robust design of automatic feedback controllers and state estimators for systems governed by implicit linear dynamic–algebraic equations are investigated. Two computational procedures for achieving robust designs are presented: singular value and eigenstructure assignment. The procedures are based on stable decompositions of the system matrices using unitary transformations.

1. INTRODUCTION

Many control systems that arise in practice can be described by implicit dynamic–algebraic equations of the form

$$E \frac{dx}{dt} = A\, x(t) + B\, u(t) \tag{1}$$

$$y(t) = C\, x(t) \tag{2}$$

or, in the discrete–time case, of the form

$$E\, x(k+1) = A\, x(k) + B\, u(k) \tag{3}$$

$$y(k+1) = C\, x(k+1)\,. \tag{4}$$

Here the vector $x \in \mathscr{R}^n$ defines the state of the system at time t (or t_k), $u \in \mathscr{R}^m$ is the control or input to the system, and $y \in \mathscr{R}^p$ is the measured output from the system, where $m, p \leq n$. It is assumed that the matrices B, C are of full rank. The matrix E may be *singular*. Examples of such systems, called *descriptor* or *generalized state–space* systems, occur in a variety of contexts, including aircraft guidance, chemical processing, mechanical body motion, power generation, network fluid flow and many others.

The behaviour of a descriptor system is governed by the generalized eigenstructure of the matrix pencil

$$\alpha E - \beta A\,, \qquad (\alpha, \beta) \in \mathscr{C} \times \mathscr{C}. \tag{5}$$

The response of the system can have a complicated structure and may even contain impulsive modes. Numerical methods are needed to assess the properties of these systems and to aid system design. The main computational technique used for both system analysis and design synthesis is the reduction of the system by unitary transformation to a condensed matrix form. Such forms reveal both the structure of the system and the degrees of freedom available for the design. The synthesis problems are generally under–determined, and it is desirable to select the free parameters to give *robust* designs that are insensitive to plant disturbances and model uncertainties.

In this paper we examine two computational tools for achieving robust designs – eigenstructure assignment and singular value assignment. These methods are used in solving two basic synthesis problems: the design of automatic controllers by proportional–plus–derivative feedback and the design of observers (or state–estimators) for estimating the states of the system from measured data. In the next section we define the basic design synthesis problems. In Section 3 we examine properties of the system. In Sections 4 and 5 we develop robust design procedures based on singular value and eigenstructure assignment, respectively. Conclusions are given in Section 6.

Throughout the paper we consider continuous–time systems of form (1)–(2) only. All the results presented here also hold, however, with minor modifications, for discrete–time systems (3)–(4).

2. CONTROL SYSTEM DESIGN

The aim of a feedback controller is to ensure that the system responds automatically in a required manner to any given reference input. This is achieved by altering the sytem dynamics by 'feeding back', through the control input, information on the current state of the system, thus creating a new 'closed–loop' system.

If all the states x of the system (1)–(2) and their derivatives $\dot{x}$ can be measured (i.e. $C = I$), then a full proportional–plus–derivative state feedback can be used. The input to the system is taken to be

$$u = Fx - G\dot{x} + r , \tag{6}$$

where $F,G \in \mathcal{R}^{m \times n}$ are the feedback matrices to be selected and $r \in \mathcal{R}^{m}$ is the reference input vector. Substituting for u in (1) and rearranging gives the closed loop system equations

$$(E + BG)\dot{x} = (A + BF)x + Br . \tag{7}$$

The matrices F and G must be chosen to ensure that the new closed loop matrix pencil

$$\alpha(E + BG) - \beta(A + BF) \tag{8}$$

has the desired properties.

In practice all of the states of a system cannot generally be measured. In this case an auxilliary dynamical system, known as an observer, or state–estimator can be constructed to provide estimates $\hat{x}$ for all the states x of the system (1)–(2) from the measured data y :

$$E\dot{\hat{x}} = A\hat{x} + Bu + F(C\hat{x}-y) - G(C\dot{\hat{x}}-\dot{y}) . \tag{9}$$

The observer is driven by the differences between the measured system outputs and their derivatives y and $\dot{y}$ and the estimated values $C\hat{x}$ and $C\dot{\hat{x}}$. Rearranging (9) gives the system equations for the observer

$$(E + GC)\dot{\hat{x}} = (A + FC)\hat{x} + Bu - Fy + G\dot{y} \tag{10}$$

with the system pencil

$$\alpha(E + GC) - \beta(A + FC) . \tag{11}$$

The matrices $F, G \in \mathcal{R}^{n \times p}$ must now be selected to ensure that the response of the observer $\hat{x}(t)$ converges to the system state $x(t)$ for any arbitrary starting conditions, that is, system (10) must be asymptotically *stable*. The convergence should be rapid and the converged estimate $\hat{x}$ should then track the true state x closely.

We remark that the design of an observer is equivalent to the design of an automatic controller for the *dual* system

$$E\dot{e} = Ae + v$$
$$w = Ce ,$$

using a feedback of the form

$$v = Fw - G\dot{w} + r .$$

The corresponding closed loop system is

$$(E + GC)\dot{e} = (A + FC)e + r$$

with system pencil given by (11), which is equivalent to the observer (10) with an appropriate choice for the reference r.

An automatic controller for the state system (1)–(2) can be obtained in the case $C \neq I$ by combining the system (1) with the observer system (10) and feeding back the estimated states $\hat{x}$ and derivatives $\dot{\hat{x}}$. A closed loop system of twice the dimension of the original system is derived. The over–all response of the closed loop system is controlled by selecting the free matrices in both the observer and the controller appropriately.

Alternatively a controller for the system (1)–(2) where $C \neq I$ can be obtained by 'feeding back' the measured output data directly. The input is taken to be

$$u = Fy - G\dot{y} + r \equiv FCx - GC\dot{x} + r , \tag{12}$$

giving the closed loop system

$$(E + BGC)\dot{x} = (A + BFC)x + Br . \tag{13}$$

The response of the closed loop system is determined by the properties of the matrix pencil

$$\alpha(E + BGC) - \beta(A + BFC) . \tag{14}$$

The results that may be achieved by selecting the feedback matrices $F, G \in \mathcal{R}^{m \times p}$ are now restricted, however, in comparison with those that may be achieved with full state feedback.

In summary, the objective of these system design problems is to choose matrices F and G such that the matrix pencil (14) has desired properties, which ensure the appropriate response of the system. The state feedback control problem, where $C = I$, and the state estimator problem, where $B = I$, are special cases. For *robust* designs it is necessary to ensure also that the properties of the pencil (14) are insensitive to perturbations in the system matrices E, A, B and C.

In the next section we examine the properties of descriptor systems and formulate design objectives.

3. PROPERTIES OF DESCRIPTOR SYSTEMS

The descriptor system (1)–(2) and its corresponding matrix pencil (5) are said to be *regular* if

$$\det(\alpha E - \beta A) \neq 0 \quad \text{for some} \quad (\alpha,\beta) \in \mathscr{C} \times \mathscr{C} \setminus \{(0,0)\} . \tag{15}$$

Regularity of the system guarantees the existence and uniqueness of classical solutions to (1) [6] [16].

For a regular system, the solutions to (1) can be characterized in terms of the eigenstructure of the pencil. The generalized eigenvalues are defined by the pairs $(\alpha_j, \beta_j) \in \mathscr{C} \times \mathscr{C} \setminus \{(0,0)\}$ such that

$$\det(\alpha_j E - \beta_j A) = 0 , \quad j = 1, 2, 3 \ldots . \tag{16}$$

If $\beta_j \neq 0$, then $\lambda_j = \alpha_j/\beta_j$ is a *finite* eigenvalue, and if $\beta_j = 0$, then $\lambda_j \sim \infty$ is an infinite eigenvalue of the system. The right and left generalized eigenvectors and principal vectors are given by the columns of the non–singular matrices $X = [X_1, X_2]$ and $Y = [Y_1, Y_2]$ (respectively) that transform the pencil into the *Kronecker canonical form* (KCF)

$$Y^T E X = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} , \quad Y^T A X = \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} , \tag{17}$$

where J is the $r \times r$ Jordan matrix associated with the $r \leq \text{rank}(E) \equiv q$ finite eigenvalues of the pencil and N is the nilpotent Jordan matrix corresponding to the $n-r$ infinite eigenvalues [8]. The *index* of the system is defined to be equal to the degree of nilpotency of the matrix N; that is, the index is equal to k, the smallest non–negative integer such that $N^k = 0$.

For a regular system, the solution to (1) is given explicitly in terms of the KCF by

$$x(t) = X_1 z_1(t) + X_2 z_2(t) , \tag{18}$$

where

$$z_1(t) = e^{tJ} z_1(0) + \int_0^t e^{(t-s)} Y_1^T Bu(s)ds , \quad z_1(0) \in \mathscr{R}^r ,$$

$$z_2(t) = -\sum_{i=0}^{k-1} N^i Y_2^T Bu^{(i)}(t) .$$

It is easy to see that for the solution to be continuous, the input function u must be such that $d^i\left[N^i Y_2^T Bu(t)\right]/dt^i$ exists and is continuous for all $i = 1, 2, \dots k-1$, where k is the index of the system.

The index of a regular system is $k = 0$, by convention, if and only if rank (E) = n. The index k is less than or equal to one if and only if rank $[E, AS_\infty] = n$ or, equivalently, rank $[E^T, A^T T_\infty] = n$, where the columns of S_∞ and T_∞ span the null spaces of E and E^T, respectively. In this case the pencil has precisely $q \equiv \text{rank}\,(E)$ finite eigenvalues and $n - q$ *non–defective* infinite eigenvalues [7], [10].

An example of a regular system of index one is given by the semi–implicit equations

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \,, \tag{19}$$

where E_{11} and A_{22} are non–singular. The first block row of equations describes the dynamical behaviour of the system, while the second block row gives algebraic constraints on the states. For systems of this type, the algebraic conditions can be eliminated to give a purely dynamical *explicit* linear system. Since A_{22} is of full rank, we may write

$$x_2 = -A_{22}^{-1}(A_{21}x_1 + B_2 u) \,.$$

Since E_{11} is also of full rank, the system (19) then reduces to the explicit system

$$\dot{x}_1 = E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + E_{11}^{-1}(B_1 - A_{12}A_{22}^{-1}B_2)u \,. \tag{20}$$

We remark that the reduction to explicit form is *not* numerically reliable if E_{11}, A_{22} are ill–conditioned with respect to inversion.

Any regular system that has index at most one can, in fact, always be unitarily transformed and separated into a purely dynamical and a purely algebraic part, and the algebraic variables can be eliminated to give an explicit system of (possibly) reduced order. Higher index descriptor systems cannot be reduced to explicit systems in this way, and impulses can arise in the response if the control is not sufficiently smooth. The system can even lose causality [15] [4]. The eigenstructure of higher index systems is also necessarily less robust with respect to perturbations than systems of index at most one, since higher index systems always have *defective* multiple eigenvalues at infinity [10].

It is desirable, therefore, to design systems that are regular and of index at most one. In practice, there exist physical systems that do *not* have these properties. Such systems can, however, often be made regular and of index at most one by appropriate choice of feedback designs. Systems that *are* regular and of index at most one can, on the other hand, *lose* these properties under linear feedback. It is important, therefore, to establish conditions that ensure regularity/index ≤ 1 under feedback, and to develop numerically reliable techniques for constructing regular systems of index at most one. In the next section we give algebraic conditions that enable the regularization of a system by feedback, and describe a singular value assignment technique for designing *robust* systems that are regular and of index at most one.

Regularity and index ≤ 1 are not sufficient properties to define a satisfactory design, however. In general, we also require the system to be asymptotically stable; that is, we require the response of the system to a constant reference input to converge asymptotically to a constant state of equilibrium from any intitial state. This property holds if the finite eigenvalues of the system lie in the left–half complex plane (see [5]). In order to "shape" the response more explicitly we may wish to assign a *specific* set of (stable) eigenvalues to the system, thus guaranteeing a given modal behaviour. In Section 5 we describe methods for achieving *robust* eigenstructure assignment by feedback. The algebraic "regularizability" conditions given in Section 4 are sufficient to enable the system to be made stable as well as regular and index ≤ 1, and to permit arbitrary assignment of the finite eigenvalues of the system.

For full flexibility a combination of the singular value and eigenstructure assignment techniques can be used in practice to obtain closed loop system designs.

4. REGULARIZATION BY SINGULAR VALUE ASSIGNMENT

The aim of the system design problem is now:

Given real system matrices E, A, B, C, *select real matrices* F *and* G *such that the closed loop pencil*

$$\alpha(E + BGC) - \beta(A + BFC) \tag{21}$$

is regular and has index less than or equal to one.

The pencil (21) has the required properties if and only if

$$\operatorname{rank} \begin{bmatrix} E + BGC \\ T_\infty^T(A+BFC) \end{bmatrix} = n , \tag{22}$$

where the columns of T_∞ span the null space of $(E + BGC)^T$ [10]. For a *robust* solution to the problem we want the closed loop system to retain these properties under reasonable perturbations. We aim, therefore, to select matrices F and G to ensure that the matrix in (22) is as far from losing rank as possible under perturbations that preserve the range space of T_∞. Such perturbations preserve the space of admissible controls (see [5]).

It is well–known that for a matrix with full rank, the distance to the nearest matrix of lower rank is equal to its minimum singular value [9]. Hence, for robustness, we select F and G such that the pencil (21) is unitarily equivalent to a pencil of the form

$$\alpha \begin{bmatrix} \Sigma_R & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \Sigma_L \end{bmatrix}, \tag{23}$$

where the condition numers $cond(\Sigma_R)$ and $cond(\Sigma_L)$ are minimal. This choice maximises a lower bound on

$$\sigma_{\min}\left\{\begin{bmatrix} \Sigma_R & 0 \\ A_{21} & \Sigma_L \end{bmatrix}\right\} \equiv \sigma_{\min}\left\{\begin{bmatrix} E + BGC \\ T_\infty^T(A+BFC) \end{bmatrix}\right\}, \tag{24}$$

whilst retaining an upper bound on the magnitude of the gains F and G. This choice also ensures that the reduction of the closed loop descriptor system to an explicit (reduced order) system, as described in the previous section, is as well–conditioned as possible. In practice such robust systems also have improved performance characteristics (see [12] [14]).

The existence of solutions to the design problem can be established under simple conditions. The proof is based on a unitary transformation of the system to a condensed form that reveals both the structure of the system and the degrees of freedom available in the design. In the next subsection the conditions for regularizability are given, together with the main result. In the following subsections, the condensed system form is presented and a technique for selecting a robust solution to the design problem is described.

4.1 Conditions for Regularizability

Algebraic conditions that ensure regularizability of the descriptor system (1)–(2) are given by the following:

C1: $\operatorname{rank}\,[\lambda E - A,\ B] = n$ for all $\lambda \in \mathscr{C}$;

C2: $\operatorname{rank}\,[E,\ AS_\infty,\ B] = n$, where the columns of S_∞ span the null space of E;

01: $\operatorname{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathscr{C}$;

02: $\operatorname{rank}\begin{bmatrix} E \\ T_\infty^T A \\ C \end{bmatrix} = n$, where the columns of T_∞ span the null space of E^T.

For systems that are regular, these conditions characterize the controllability and observability of the system. The conditions **C1/C2** and **01/02** guarantee that a regular system is *strongly controllable* and *strongly observable*, respectively. The conditions **C1** and **01**, together with the stronger conditions

$$\operatorname{rank}\,[E,\ B] = n\,, \qquad \operatorname{rank}\,[E^T,\ C^T] = n \tag{25}$$

guarantee that a regular system is *completely* controllable and *completely* observable (see [16] [1] [2]). The conditions **C2** and **02** ensure 'controllability and observability at infinity' (see [15]). A regular system that has index at most one *always* satisfues **C2** and **02**.

The conditions **C1, C2, 01** and **02** are all preserved under certain transformations of the system. Specifically, these conditions are all preserved under non–singular 'equivalence' transformations of the pencil and under proportional feedback. The conditions **C1** and **01** are also preserved under derivative feedback [1] [4].

The key result is given by the following:

THEOREM 1 *Given the real system matrices* E, A, B, C, *then* **C2** *and* **02** *hold if and only if there exist real matrices* F *and* G *such that*

$$\alpha(E + BGC) - \beta(A + BFC)$$

is regular with index at most one and

$$\text{rank}\,(E + BGC) = r\,, \tag{26}$$

where $\text{rank}(E) \le r \le s \equiv \text{rank}(E) + t_2$. *(Here* t_2 *is an integer determined by the decomposition of the system given in Theorem* 2*).*

If **C1** *and* **01** *also hold, then* F *and* G *can be selected to ensure, in addition, that the corresponding closed loop system is strongly controllable and strongly observable.*

Proof. The proof follows by construction from the condensed form given in Theorem 2 of Section 4.2 [2]. □

We remark that the value of s in Theorem 1 is equal to n, the system dimension, if and only if the stronger conditions (25) hold [2]. In the special cases where $C = I$ (the state feedback control problem) and $B = I$ (the state estimator problem), the value of s is given by $s = \text{rank}\,[E, B]$ and $s = \text{rank}\,[E^T, C^T]$, respectively [1].

The value $r = s$ in Theorem 1 is attained in all cases by the feedback G alone, with $F = 0$. The value $r = \text{rank } E$ is attained by feedback F alone, with $G = 0$. If **C2** and **02** hold, solutions such that $r < \text{rank}(E)$ may also exist, but the converse of the theorem does not necessarily hold [2].

4.2 A Condensed Form

The main result in Section 4.1 depends on the following.

THEOREM 2 *Given real system matrices* E, A, B, C, *there exist unitary matrices* U, V, W, Z *such that*

$$U^H E V = \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix}, \qquad U^H B W = \begin{bmatrix} B_{11} & B_{12} \\ \hat{B}_{21} & 0 \end{bmatrix}$$

$$Z^H C V = \begin{bmatrix} C_{11} & \hat{C}_{12} \\ C_{21} & 0 \end{bmatrix}, \qquad U^H A V = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \hat{A}_{22} \end{bmatrix} \tag{27}$$

where

$$\hat{A}_{22} = \begin{bmatrix} A_{22} & A_{23} & A_{24} & 0 & 0 \\ A_{32} & A_{33} & A_{34} & \Sigma_{35} & 0 \\ A_{42} & A_{43} & \Sigma_{44} & 0 & 0 \\ 0 & \Sigma_{53} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \hat{B}_{21} = \begin{bmatrix} B_{21} \\ B_{31} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{C}_{12} = [\,C_{12} \;\; C_{13} \;\; 0 \;\; 0 \;\; 0\,] \tag{28}$$

and Σ_E, Σ_{35}, Σ_{44}, Σ_{53} *are non–singular, square diagonal matrices of dimensions* t_1, t_3, t_4 *and* t_5, *respectively.* B_{12} *and* C_{21} *are of full rank, and*

$$\begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix} \quad \text{and} \quad [\, C_{12}, C_{13} \,]$$

are non–singular, square matrices of dimensions $t_2 + t_3$ *and* $s_2 + t_5$, *respectively. (All partitionings are compatible.)*

Proof. The proof is by construction via Algorithm 1 given in [2]. The construction uses a sequence of singular value (SVD) decompositions [9]. □

From the condensed form (27)–(28), it follows that the system pencil $\alpha E - \beta A$ is regular and of index at most one if and only if the matrix $\hat{A}_{22}$ is non–singular. Necessary and sufficient conditions for this to hold are that the last zero block rows and columns of $\hat{A}_{22}$ (and corresponding blocks of A_{21}, A_{22}) are empty (so that $t_2 = s_2$) and the square matrix A_{22} is non–singular.

From Theorem 2 it also follows that conditions **C2** and **02**, respectively, hold if and only if the last zero block rows and last zero block columns of $\hat{A}_{22}$ and $\hat{B}_{21}$, $\hat{C}_{12}$, respectively, are empty (together with the corresponding blocks of A_{21}, A_{22}). Furthermore, if these conditions hold, then it can be seen from the condensed form (27)–(28) that feedback matrices F and G can be constructed such that the closed loop pencil (21) is unitarily equivalent to a pencil of form (23), where Σ_R, Σ_L are square and non–singular and Σ_R is $r \times r$ with $\text{rank}\, \Sigma_E \leq r \leq s = t_1 + t_2$ (for details see [2]). These results, together with the fact that the conditions **C1, C2, 01** and **02** are preserved under equivalence transformations, effectively establishes Theorem 1.

In the next subsection we examine how the feedback matrices F, G can be constructed explicitly to give a *robust* solution to the regularization problem.

4.3 Robust Singular Value Assignment

To obtain a system pencil that is regular and of index at most one, the matrices F and G can be selected such that the pencil (21) is unitarily equivalent to a pencil of form (23) provided certain algebraic conditions hold, as described in the previous subsections. For a *robust* system design, we choose F and G to assign the singular values of the subsystems Σ_R, Σ_L in the equivalent system (23) to ensure that

$$cond\,(\Sigma_R) \equiv \sigma_{\max}\{\Sigma_R\}/\sigma_{\min}\{\Sigma_R\}\,, \quad cond\,(\Sigma_L) \equiv \sigma_{\max}\{\Sigma_L\}/\sigma_{\min}\{\Sigma_L\}$$

are as small as possible, and also to ensure reasonable gaps between singular values.

In general, not all singular values can be assigned arbitrarily. In the case of the state feedback controller and its dual, the state estimator problem, where $C = I$ and $B = I$, respectively, the complete singular value structure can be identified and an optimal solution can be found to the robust regularization problem. To establish the form of the optimal solution in the case $C = I$, it is convenient to reduce the form (27)–(28) still further. In this case, assuming **C2** holds so that $t_6 = 0 = s_6$ and $t_2 = s_2$, we find that $t_3 = 0 = t_4$ and, therefore, the third, fourth, and sixth block rows and fourth, fifth and sixth block columns of the form (27)–(28) are empty. We may then apply further unitary row and column operations to the first and third block rows and columns of E, A and B. It follows that there exist unitary matrices $\tilde{U}, \tilde{V}, \tilde{W}$ such that

$$\tilde{U}^H E \tilde{V} = \begin{bmatrix} \Sigma_E & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{U}^H B \tilde{W} = \begin{bmatrix} 0 & 0 \\ B_{21} & B_{22} \\ \Sigma_B & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{U}^H A \tilde{V} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & 0 & \Sigma_A \end{bmatrix}, \tag{29}$$

where Σ_E, Σ_A and Σ_B are square, non–singular diagonal matrices of dimensions ℓ, $n - m - \ell$ and $m - t$ respectively, and E_{22}, B_{22} are square, non–singular matrices of dimension t. Here $\text{rank}(E) = \ell + t$ and $\text{rank}[E,B] = \ell + m$. (The form (29) can also be derived directly from the results of [1].)

From the decomposition (29) it can be seen that the matrices Σ_E and Σ_A cannot be altered by feedback. We can, however, select matrices F and G such that the system pencil $\alpha\tilde{U}^H(E + BG)\tilde{V} - \beta\tilde{U}^H(A + BF)\tilde{V}$ is equal to the pencil (23), where

$$\Sigma_R = \begin{bmatrix} \Sigma_E & 0 & 0 \\ 0 & \Sigma_1 & 0 \\ 0 & 0 & \Sigma_2 \end{bmatrix}, \qquad \Sigma_L = \begin{bmatrix} \Sigma_3 & 0 \\ 0 & \Sigma_A \end{bmatrix},$$

$\text{rank}(\Sigma_R) = r$, with $\text{rank}(E) \le r \le \text{rank}[E,B]$, and $\Sigma_1, \Sigma_2, \Sigma_3$ are *arbitrary*, positive diagonal matrices. Appropriate feedback matrices F and G are given by

$$\tilde{W}^H G = \begin{bmatrix} 0 & 0 & G_{13} & 0 \\ G_{21} & G_{22} & G_{23} & 0 \end{bmatrix}, \tag{30}$$

where

$$G_{13} = \Sigma_B^{-1} \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad G_{22} = B_{22}^{-1}(\Sigma_1 - E_{22}),$$

$$G_{21} = -B_{22}^{-1} E_{21}, \qquad G_{23} = -B_{22}^{-1} B_{21} G_{13},$$

and

$$\tilde{W}^H F = \begin{bmatrix} 0 & 0 & F_{13} & F_{14} \\ 0 & 0 & F_{23} & F_{24} \end{bmatrix}, \tag{31}$$

where

$$F_{13} = \Sigma_B^{-1} \left[\begin{bmatrix} 0 & 0 \\ 0 & \Sigma_3 \end{bmatrix} - A_{33} \right], \qquad F_{14} = -\Sigma_B^{-1} A_{34}$$

$$F_{23} = -B_{22}^{-1} B_{21} F_{13}, \qquad F_{24} = -B_{22}^{-1} B_{21} F_{14}.$$

Some freedom in the solution remains, which has not been exploited here.

We remark that if $r = \text{rank}[E,B]$, then the dynamical part of the closed–loop system is of maximum dimension (equal to the dimension of the reachable space of the original system), and the optimal solution is obtained by feedback G alone, with $F = 0$.

The case of the state estimator problem, where $B = I$, is just the dual of the state feedback controller problem. If **02** holds, the optimal observer design can, therefore, be obtained by replacing the triple (E, A, B) by (E^T, A^T, C^T) and F by F^T in (29), (30) and (31).

For the more general output feedback problem, where $B \neq I$, $C \neq I$, the singular value structure that can be attained is more complicated. The problem of optimizing robustness can, in this case, be reduced to the problem:

Given matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ *find* Δ *such that* $\text{cond}\left[\begin{bmatrix} M_{11}+\Delta & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \right]$ *is minimal.*

This is an **open problem.** In practice an upper bound on the condition number can be minimized, using the structure in Theorem 2. Details of the procedure are described in [2] and [3] and numerical examples are also presented in [3].

In this section we have described a method for obtaining a regular closed loop system pencil of index at most one using derivative and proportional feedback. The feedback is selected to ensure that the properties of the pencil are insensitive to perturbations, using singular value assignment. It is desirable for the system design to have other additional properties, however; in particular, to be stable and, possibly, to have specified finite eigenvalues. One strategy for achieving an overall design is to use derivative feedback alone to obtain a robust, regular, index one system of maximal dimension, and then to use proportional feedback to assign the required eigenvalues to the system. In the next section we describe methods for robust eigenstructure assignment in descriptor systems using proportional feedback.

5. ROBUST EIGENSTRUCTURE ASSIGNMENT

The aim of the design problem is now to select the feedback matrices to assign specified eigenvalues to the closed loop system pencil. We consider here only the state feedback controller problem and the state estimator problem, where $C = I$ and $B = I$, respectively. The full output feedback design problem is much more difficult and is beyond the scope of this paper. We assume that if derivative feedback is available, then it is used to "pre–condition" the system pencil by singular value assignment, as described in Section 4. We therefore consider here only *proportional state* feedback designs.

The design problem is:

Given real system matrices E, A, B *and a set of* $q \equiv \text{rank}(E)$ *self–conjugate complex numbers* $\mathscr{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$, *select real matrix* F *such that the closed–loop pencil*

$$\alpha E - \beta(A + BF)$$

is regular and of index at most one and has the prescribed finite eigenvalues $\lambda_j \in \mathscr{L}$, $j = 1, 2, \ldots, q$.

For a *robust* solution to the problem we want the assigned finite eigenvalues to be non–defective and we want some measure of the sensitivity of the eigenvalues (both finite and infinite) to be minimal.

If we let (α_j, β_j) denote a generalized, *simple* eigenvalue of the pencil $\alpha E - \beta A$ with right and left eigenvectors x_j and y_j satisfying

$$\alpha_j E x_j = \beta_j A x_j\ , \qquad \alpha_j y_j^T E = \beta_j y_j^T A\ ,$$

then a measure of the sensitivity of the eigenvalue is given by the *condition number*

$$c_j = \frac{\|y_j\|_2 \ \|x_j\|_2}{(|y_j^T E x_j|^2 + |y_j^T A x_j|^2)^{\frac{1}{2}}} \tag{33}$$

(see [13]). The condition number is inversely proportional to the angles between the invariant vectors y_j and Ex_j (or y_j and Ax_j, in the case of infinite eigenvalues). The condition number c_j is, moreover, inversely proportional to the quantity $(|y_j^T E x_j|^2 + |y_j^T A x_j|^2)^{\frac{1}{2}}$, which measures how nearly the vector x_j approximates a null vector of both E and A and, hence, how close the pencil is to losing regularity.

If the pencil $\alpha E - \beta A$ is non–defective and a perturbation of order $O(\epsilon)$ is made in E or A, then the corresponding first order perturbation in a simple eigenvalue (α_j, β_j) is of order $0(\epsilon c_j)$, where distance is measured in the chordal metric [13]. If the pencil is defective, then the corresponding perturbation in *some* eigenvalue is at least an order of magnitude worse in ϵ, and, therefore, defective systems are necessarily less robust than those that are non–defective.

The sensitivity of a multiple (non–defective) eigenvalue is proportional to the maximum of the associated condition numbers c_j, taken with respect to an *orthonormal* basis $\{x_j\}$ for the space of right eigenvectors and a corresponding set $\{y_j\}$ of left eigenvectors normalized such that $y_j^T E x_i = 0$, $y_j^T A x_i = 0$, for $i \neq j$. (See [10] for details.)

An overall measure of the sensitivity of the eigenvalues of a regular, non–defective system pencil is given by a weighted sum of all the condition numbers

$$\nu(\omega) = \left[\sum_{j=1}^{u} \omega_j^2 c_j^2 \right]^{\frac{1}{2}}, \tag{34}$$

where $\omega_j > 0$ and $\Sigma\, \omega_j^2 = 1$. A regular non–defective pencil must, by definition, be of index at most one and have precisely $q \equiv \text{rank}\,(E)$ finite eigenvalues. The robustness measure $\nu(\omega)$ can therefore be written [10]

$$\nu(\omega) = \| D_\omega [EX_q, AX_\infty]^{-1} \|_F\ , \tag{35}$$

where D_ω is a diagonal weighting matrix, the columns of X_∞ form an orthonormal basis for the null space of E, and $X_q = [x_1, x_2, \ldots, x_q]$ is the modal matrix of right eigenvectors associated with the finite eigenvalues, normalized such that $\|x_j\| = 1$, $j = 1, 2, \ldots, q$, and such that the vectors associated with each multiple eigenvalue form an orthonormal set. (Here $\|\cdot\|_F$ denotes the Frobenius matrix norm.) A robust feedback design can thus be achieved by selecting the eigenstructure of the closed loop system so as to minimize the measure $\nu(\omega)$ given by (35).

The existence of a solution to the eigenstructure assignment problem can be guaranteed under the algebraic conditions of Section 4.1. In the next subsection the existence results are given, together with a parameterization of the solution. In the following subsection a technique for achieving robust eigenstructure assignment using this parameterization is described.

5.1 Existence of Solutions

Necessary and sufficient conditions for the state feedback eigenvalue assignment problem to have a solution are established by the following.

THEOREM 3 *Given real system matrices* E, A, B *and any arbitrary self–conjugate set* $\mathscr{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ *of* $q \equiv \mathrm{rank}(E)$ *complex numbers, there exists a real matrix* F *such that the matrix pencil* $\alpha E + \beta(A + BF)$ *is regular and of index at most one and has the finite eigenvalues* $\lambda_j \in \mathscr{L}$, $j = 1, 2, \ldots, q$, *if and only if the conditions* **C1** *and* **C2**, *defined in Section* 4.1, *hold.*

Proof: Proofs are given in [7] [10] and [1]. □

We remark that the condition **C2** ensures that the closed loop system pencil (32) can be made regular and of index at most one, and the condition **C1** guarantees that the finite poles can be assigned arbitrarily.

For a *non–defective* closed loop pencil (32) with prescribed eigenvalues $\mathscr{L} = \{\lambda_j, j = 1, 2, \ldots, q\}$, where $q = \mathrm{rank}(E)$, we require that for some matrix $X_q \in \mathscr{C}^{n \times q}$ of full rank, the feedback matrix F satisfies

$$(A + BF)X_q = EX_q \Lambda_q , \qquad \Lambda_q = \mathrm{diag}\{\lambda_j\} , \tag{36}$$

and

$$\mathrm{rank}\,[E, (A + BF)X_\infty] = n , \tag{37}$$

where the columns of X_∞ form an *orthonormal* basis for the null space of E and

$$\mathrm{rank}\,[X_q, X_\infty] = n . \tag{38}$$

The condition (36) ensures that the closed loop pencil has the prescribed finite eigenvalues λ_j with a full set of independent right eigenvectors $x_j = X_q e_j$, $j = 1, 2, 3, \ldots, q$, and condition (37) ensures that the system is regular and of index at most one. The required feedback F can be parameterized in terms of the vectors x_j, $j = 1, 2, 3, \ldots, q$, and a matrix W. We have the following structure theorem.

THEOREM 4 *Given the set* $\mathscr{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ *of* distinct *self–conjugate complex numbers, where* $q = \mathrm{rank}(E)$, *there exist vectors*

$$x_j \in \mathscr{S}_j \equiv \{x \mid (\lambda_j E - A)x \in \text{range}(B)\},\ \lambda_j \in \mathscr{L},\ j = 1, 2, \ldots, q, \tag{39}$$

such that $X_q = [x_1, x_2, \ldots, x_q]$ *satisfies* (38), *and a matrix* W *satisfying*

$$\text{rank}\,[E + AX_\infty X_\infty^T + BWX_\infty^T] = n \tag{40}$$

if and only if conditions C1 *and* C2 *hold. If* (38)–(40) *hold, then the matrix* F *given by*

$$F = [B^+(EX_q \Lambda_q - AX_q),\ W][X_q,\ X_\infty]^{-1} \tag{41}$$

solves the eigenvalue assignment problem, and (36) *and* (37) *are satisfied.* (*Here* B^+ *denotes the Moore–Penrose pseudo–inverse of matrix* B *and* X_∞ *denotes an orthonormal basis for the null space of* E.

Proof: The proof is established in [10]. □

If the prescribed eigenvalues are not distinct, then **C1** and **C2** are necessary but may not be sufficient to guarantee a non–defective solution to the eigenvalue assignment problem.

Theorem 4 gives a parameterization of feedback matrix F in terms of the eigenstructure of the corresponding closed loop system pencil. In the next subsection a technique is described for selecting the free parameters to give a *robust* solution to the system design problem.

5.2 A Numerical Algorithm

To construct a robust solution to the eigenvalue assignment problem, we use the parameterization of the feedback F given by (41) in Theorem 4 and select the freedom in X_q and W so as to minimize the measure of robustness $\nu(\omega)$ given by (35). We aim also to ensure that the ranks of the matrices in (38) and (40) are insensitive to perturbations; that is, we want these matrices to be well–conditioned.

In practice it is sufficient to minimize the condition numbers

$$\kappa_1 = \text{cond}_F([X_q, X_\infty])\ , \quad \kappa_2 = \text{cond}_2\,(E + AX_\infty X_\infty^T + BWX_\infty^T)\ ,$$

subject to $\|E + AX_\infty X_\infty^T + BWX_\infty^T\|_2$ remaining finite. It can be shown [10] that the measure $\nu(\omega)$ is bounded in terms of the product $\kappa_1 \kappa_2$. We have

$$\gamma_1(\kappa_1 \kappa_2)^{\frac{1}{2}} \leq \nu(\omega)\ \|E + AX_\infty X_\infty^T + BWX_\infty^T\|_2 \leq \gamma_2\, \kappa_1 \kappa_2,$$

where γ_1 and γ_2 are fixed constants. Provided $\|E + AX_\infty X_\infty^{T^+} + BWX_\infty\|_2$ remains bounded, minimizing κ_1 and κ_2 thus minimizes an equivalent measure of the sensitivity of the assigned eigenvalues, as well as ensuring that the matrices in (38) and (40) are well–conditioned. Since the free parameters appear independently in κ_1 and κ_2, these measures can be minimized separately. Optimizing the condition numbers κ_1 and κ_2 also leads to other desirable properties of the closed loop system. In particular the transient response and the magnitude of the gains can be bounded in terms of κ_1 (see [10]), and a lower bound on the distance of the closed loop pencil to instability can be given in terms of κ_1^{-1}(see [5]).

The computational procedure for solving the robust eigenstructure assignment problem consists of four basic steps:

Step 1: Compute orthonormal bases X_∞ for kernel(E) and S_j for the subspaces $\mathcal{S}_j$, defined in (39), for $j = 1, 2, \ldots, q$.

Step 2: Select W to minimize $\sigma_{min}(E + AX_\infty X_\infty^T + BWX_\infty^T)$ subject to $\sigma_{max}(E + AX_\infty X_\infty^T + BWX_\infty^T) \leq \tau$, where τ is a given tolerance.

Step 3: Select vectors $x_j = S_j v_j \in \mathcal{S}_j$ with $\|x_j\|_2 = 1, j = 1, 2, \ldots, q$, to minimize κ_1.

Step 4: Determine F from equation (41).

Reliable library software with procedures for computing QR, SVD and LU decompositions is used to accomplish these steps. Iterative techniques for selecting vectors from given subspaces to minimize κ_1 in *Step 3* are described in [11]. The computation of F in *Step 4* is accurate as long as κ_1 is reasonably small (relative to machine precision). A detailed description of the algorithm is given in [10]. It is not necessary that the prescribed eigenvalues be distinct. Provided that a non–defective solution to the eigenvalue assignment problem exists for the given set $\mathcal{L}$, the algorithm determines a feedback that assigns the prescribed eigenvalues.

In this section we have described a procedure for designing a robust state feedback controller with prescribed eigenvalues. The dual state estimator design problem can be solved using the same technique by replacing the system triple (E, A, B) and the feedback F in the algorithm by the triple (E^T, A^T, C^T) and the feedback F^T.

6. CONCLUSIONS

Two techniques for designing automatic feedback controllers and observers for implicit linear differential–algebraic systems are described here: singular value assignment and eigenstructure assignment. The techniques are based on stable and reliable numerical procedures for factorizing and reducing matrices to condensed forms, using unitary transformations. Measures of sensitivity for the system designs are derived and conditions for the existence of robust solutions to the synthesis problems are established. The degrees of freedom in the design are identified and computational procedures for selecting the free parameters to minimize the sensitivity measures are presented. These results all, with minor modifications, apply also to discrete–time implicit linear systems governed by dynamic–algebraic equations.

In practice a combination of the robust singular value and eigenstructure assignment procedures can be used to synthesize full state feedback controllers and observers. Applications of these combined techniques to the design of state–estimators for flow in gas networks are examined in [12] and [14]. The system dynamics are modelled by implicit, discrete–time, finite difference equations. The importance of robustness in the observer design is demonstrated by the investigations. Effects of model uncertainty and measurement noise are minimized, as far as possible, by the robust design procedures.

The synthesis of output feedback designs is more difficult and complicated than state feedback or state–estimator synthesis, and many aspects of the output design problem are still open. For systems that can be made regular and of index at most one by feedback, the condensed form presented here identifies the system structure and the available freedom for design synthesis. To exploit this freedom fully, further techniques are still needed, and extensions are required to treat higher index systems that cannot be reduced to systems of index less than or equal to one. Work on these developments is in progress.

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