Comparison of explicit and implicit TVD schemes for the Shallow Water Equations on a Sphere

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Abstract

A comparison is made between earlier results obtained from solving the shallow water equations on the sphere using an explicit TVD scheme. and those from using a corresponding implicit method. The equations considered result from a transformation of the flux form of the shallow water equations into a set of conservation laws with source terms due to the earth's rotation and topography. The technique of operator splitting is used to allow the methods to be applied to the two-dimensional model which is solved on a regular latitude/longitute grid. Solutions are presented for a meteorological test case which demonstrate that the implicit scheme considered is highly diffusive and not suitable for the particular application of interest.

Contents

1 Introduction

In this report the application of a class of schemes known as Total Variational Diminishing (TVD) to solve the Shallow Water Equations (SWE) on a sphere is considered. The SWE are a test case for numerical schemes proposed to solve atmospheric flow problems, and describe the height and velocity profiles of a fluid. The equations are derived by assuming that the fluid is inviscid and that the vertical component of acceleration is negligible. Integration of the Navier Stokes equations over the fluid depth then gives the SWE (see [4] for details) which are a 2-dimensional non-linear system of equations. The particular form of the equations of interest here is the flux form

$$\frac{\partial h^*}{\partial t} + \frac{1}{a\cos\theta}\left[\frac{\partial(h^*u)}{\partial\phi} + \frac{\partial(h^*v\cos\theta)}{\partial\theta}\right] = 0$$

$$\frac{\partial(h^*u)}{\partial t} + \frac{1}{a\cos\theta}\left[\frac{\partial(h^*u^2)}{\partial\phi} + \frac{\partial(h^*uv\cos\theta)}{\partial\theta}\right] + \frac{gh^*}{a\cos\theta}\frac{\partial h}{\partial\phi} = \left(f + \frac{u}{a}\tan\theta\right)h^*v$$

$$\frac{\partial(h^*v)}{\partial t} + \frac{1}{a\cos\theta}\left[\frac{\partial(h^*uv)}{\partial\phi} + \frac{\partial(h^*v^2\cos\theta)}{\partial\theta}\right] + \frac{gh^*}{a}\frac{\partial h}{\partial\theta} = -\left(f + \frac{u}{a}\tan\theta\right)h^*u, \quad (1)$$

where h^* is the fluid depth and h is the height of the free surface above sea level such that $h = h^* + h_s$ (if h_s represents the orographic height). The velocity components u and v are defined in the longitudinal ϕ and latitudinal θ direction respectively and the Coriolis parameter f is given by $f = 2\Omega\sin\theta$ where Ω is the earth's rate of rotation, and a is its radius.

In previous work [1] an explicit TVD method, Roe's scheme, was used to solve a set of conservation laws derived from the above equations, namely

$$\frac{\partial h'}{\partial t} + \frac{\partial}{\partial\theta}\left(\frac{h'u}{a\cos\theta}\right) + \frac{\partial}{\partial\theta}\left(\frac{h'v}{a}\right) = 0$$

$$\frac{\partial}{\partial t}(h'u) + \frac{\partial}{\partial\phi}\left(\frac{h'u^2}{a\cos\theta} + \frac{h'^2g}{2a\cos^2\theta}\right) + \frac{\partial}{\partial\theta}\left(\frac{h'uv}{a}\right) = \left(f + \frac{u\tan\theta}{a}\right)h'v - \frac{gh'}{a\cos\theta}\frac{\partial h_s}{\partial\phi} \quad (2)$$

$$\frac{\partial}{\partial t}(h'v) + \frac{\partial}{\partial\phi}\left(\frac{h'uv}{a\cos\theta}\right) + \frac{\partial}{\partial\theta}\left(\frac{h'v^2}{a} + \frac{h'^2g}{2a\cos\theta}\right) = -\left(f + \frac{u\tan\theta}{a}\right)h'u - \frac{gh'}{a}\frac{\partial h_s}{\partial\theta} - \frac{h'^2g\tan\theta}{2a\cos\theta}.$$

where a [illegible] stable [illegible].

It was found that when flow occured in the polar regions a very small time step was needed for the CFL condition to be satisfied and the solution to remain stable. The main conclusion from the study was that an implicit scheme was needed if the severe time step restriction imposed by the stability criterion was to be overcome. To that end this project was instigated to identify suitable implicit schemes which might be of benefit to the meteorological community. Any such scheme would have to be conservative, TVD, at least second order accurate and perform better than the schemes used in current operational models.

In the preceding sections the original explicit method is described, followed by details of the implicit scheme used. Results are shown for a particular test case, and finally the conclusions of the project are presented.

2 The explicit scheme

Roe [5] proposed a method to obtain an approximate solution to a set of conservation laws of the form

$$\mathbf{w}_t + \mathbf{F}_x = 0 \tag{3}$$

based on regarding the data as piecewise constant and solving a set of Riemann problems. A Riemann problem is one where the initial data is constant either side of a discontinuity. If the discontinuity lies at the point $x = x'$ then the initial values are

$$\mathbf{w}(0, x) = \begin{cases} \mathbf{w}_L & \text{if } x < x' \\ \mathbf{w}_R & \text{if } x > x' \end{cases}$$

where $\mathbf{w}_L$ and $\mathbf{w}_R$ denote the left and right states and x' is the interface between them.

The solution to equation (3), $\mathbf{w}_j^n$, is regarded as an approximation to the average state between two interfaces, where the interfaces are placed at the mid points of the cells, i.e.

$$\mathbf{w}_j^n = \frac{1}{\Delta x} \int_{(i-\frac{1}{2})dx}^{(i+\frac{1}{2})\Delta x} \mathbf{w}(x, n\Delta t)\, dx$$

where Δx is the grid spacing on a regular grid, and Δt is the time step.

If the problem (3) is approximated

$$\mathbf{w}_t + \tilde{A}\mathbf{w}_x = 0 \tag{4}$$

where $\tilde{A}$ is a constant matrix, then an approximate solution to the exact problem (3) can be taken to be the exact solution to the approximate problem (4).

The matrix $\tilde{A}$, which depends on $\mathbf{w}_L$ and $\mathbf{w}_R$, can be picked in many ways but in Roe's scheme is chosen to satisfy the following properties:

(i) $\tilde{A}$ constitutes a linear mapping from the vector space $\mathbf{w}$ to the vector space $\mathbf{F}$.

(ii) As $\mathbf{w}_L \rightarrow \mathbf{w}_R \rightarrow \mathbf{w}$, $\tilde{A}(\mathbf{w}_L, \mathbf{w}_R) \rightarrow A(\mathbf{w})$, where $A = \dfrac{\partial \mathbf{F}}{\partial \mathbf{w}}$.

(iii) For any $\mathbf{w}_L$, $\mathbf{w}_R$, $\tilde{A}(\mathbf{w}_L, \mathbf{w}_R).(\mathbf{w}_L - \mathbf{w}_R) = \mathbf{F}_L - \mathbf{F}_R$.

(iv) The eigenvectors of $\tilde{A}$ are linearly independent.

The above set of conditions, termed 'Property U' by Roe [5] ensures that the Riemann solver has the desirable properties that the solution is consistent and conservative and therefore gives the correct shock speeds across a shock.

For any two states, $\mathbf{w}_L$ and $\mathbf{w}_R$, the flux difference across the interface can be expressed as [6]

$$\mathbf{F}_R - \mathbf{F}_L = \sum_k \tilde{\alpha}_k \tilde{\lambda}_k \tilde{\mathbf{e}}_k \tag{5}$$

where $\tilde{\mathbf{e}}_k$ are the right eigenvectors of $\tilde{A}$, $\tilde{\lambda}_k$ are the eigenvalues or wave speeds and $\tilde{\alpha}_k$ are coefficients known as the wave strengths. This results in the flux at the interface being [6]

$$\mathbf{F}_{i+\frac{1}{2}}(\mathbf{w}_L, \mathbf{w}_R) = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\sum_k \tilde{\alpha}_k \left|\tilde{\lambda}_k\right| \tilde{\mathbf{e}}_k.$$

A local linearisation can be introduced by choosing an $\tilde{A}$ with property U which implies (as a result of (iii) above) that its eigenvalues and eigenvectors not only satisfy equation (5) but also

$$\mathbf{w}_R - \mathbf{w}_L = \sum_k \tilde{\alpha}_k \tilde{\mathbf{e}}_k.$$

Each $\tilde{\alpha}$ satisfies a scalar scheme and the method of updating is to add

$$-\frac{\Delta t}{\Delta x}\tilde{\lambda}_k \tilde{\alpha}_k \tilde{\mathbf{e}}_k \text{ to } \mathbf{w}_R \text{ if } \tilde{\lambda}_k > 0$$

and

$$-\frac{\Delta t}{\Delta x}\tilde{\lambda}_k\tilde{\alpha}_k\tilde{\mathbf{e}}_k \text{ to } \mathbf{w}_L \text{ if } \tilde{\lambda}_k < 0$$

where $\tilde{\lambda}_k$, $\tilde{\alpha}_k$ and $\tilde{\mathbf{e}}_k$ are determined in the calculation using the values of $\mathbf{w}$ and $\mathbf{F}$ from the current time step, and so the method is explicit.

To allow the method to be applied to a problem with source terms, denoted by $\mathbf{f}$, the source terms may be expanded in terms of the eigenvectors of $\tilde{A}$ i.e.

$$\tilde{\mathbf{f}}(\mathbf{w}^n) = -\frac{1}{\Delta x}\sum_{k=1}^{3}\tilde{\beta}_k\tilde{\mathbf{e}}_k,$$

which enables the difference equation to be written as

$$\mathbf{w}_P^{n+1} = \mathbf{w}_P^n + \frac{\Delta t}{\Delta x}\sum_{k=1}^{3}\tilde{\lambda}_k\tilde{\gamma}_k\tilde{\mathbf{e}}_k$$

where $\tilde{\gamma}_k = \tilde{\alpha}_k + \tilde{\beta}_k/\tilde{\lambda}_k$ and P corresponds to either the left or the right state. The resulting scheme is then to add

$$-\frac{\Delta t}{\Delta x}\tilde{\lambda}_k\tilde{\gamma}_k\tilde{\mathbf{e}}_k \text{ to } \mathbf{w}_R \text{ if } \tilde{\lambda}_k > 0$$

or

$$-\frac{\Delta t}{\Delta x}\tilde{\lambda}_k\tilde{\gamma}_k\tilde{\mathbf{e}}_k \text{ to } \mathbf{w}_L \text{ if } \tilde{\lambda}_k < 0.$$

Details of how to calculate $\tilde{\lambda}_k$, $\tilde{\gamma}_k$ and $\tilde{\mathbf{e}}_k$ for the SWE maybe found in [1].

If (2) is written in the form

$$\mathbf{w}_t + \mathbf{F}_\phi + \mathbf{G}_\theta = \mathbf{f} + \mathbf{g} + \mathbf{s} \qquad (6)$$

where

$$\mathbf{w} = (h', h'u, h'v)^T$$

$$\mathbf{F} = \left(\frac{h'u}{a\cos\theta}, \frac{h'u^2}{a\cos\theta} + \frac{h'^2 g}{2a\cos^2\theta}, \frac{h'uv}{a\cos\theta}\right)^T$$

$$\mathbf{G} = \left(\frac{h'v}{a}, \frac{h'uv}{a}, \frac{h'v^2}{a} + \frac{h'^2 g}{2a\cos\theta}\right)^T$$

$$\mathbf{f} = \left(0, -\frac{gh'}{a\cos\theta}\frac{\partial h_s}{\partial\phi}, 0\right)^T$$

$$\mathbf{g} = \left(0, 0, -\frac{gh'}{a}\frac{\partial h_s}{\partial\theta}\right)^T$$

$$\mathbf{s} = \left(0, h'v\left(f + \frac{u\tan\theta}{a}\right), -h'u\left(f + \frac{u\tan\theta}{a}\right) - \frac{h'^2 g\tan\theta}{2a\cos\theta}\right)^T.$$

Then using the [illegible] the problem is equivalent to solving

$$\frac{1}{2}\mathbf{w}_t - \mathbf{F}_\lambda = \mathbf{f} + \frac{1}{2}\mathbf{s} \tag{7}$$

and

$$\frac{1}{2}\mathbf{w}_t - \mathbf{G}_\theta = \mathbf{g} + \frac{1}{2}\mathbf{s}. \tag{8}$$

Only the terms from $\mathbf{f}$ and $\mathbf{g}$ are incorporated within the decomposition (5). The terms in $\mathbf{s}$ which result from the earths rotation are added pointwise at the end of the update.

3 The implicit scheme

The implicit scheme chosen for investigation was one used in previous work [2] to solve flow problems in channels and pipes. In [2] the method was applied to a 1-dimensional form of the SWE known as the St. Venant equations. A detailed analysis of the scheme may be found in [8] and [9].

If the method is applied to a scalar conservation law, in the form of (3), then the flux function is one coming from the TVD version of a centred scheme without the term responsible for the space–time combined discretisation and may be written as

$$f^{*n}_{i+\frac{1}{2}} = \frac{1}{2}[f^n_{i+1} + f^n_i - (1 - \phi(r^n_{i+\frac{1}{2}}))|\tilde{\lambda}^n_{i+\frac{1}{2}}|\delta u^n_{i+\frac{1}{2}}],$$

where $\delta u^n_{i+\frac{1}{2}} = u^n_{i+1} - u^n_i$, $\tilde{\lambda}^n_{i+\frac{1}{2}}$ is the approximate advection speed and $\phi(r)$ represents a limiting function (see [7] for details). The scheme may be extended to a system of equations by discretising the equations in the following manner

$$\frac{\mathbf{w}^{n+1}_i - \mathbf{w}^n_i}{\Delta t} + \frac{1}{\Delta x}[\theta\delta^-(\mathbf{F}^{*n+1}_{i+\frac{1}{2}}) + (1-\theta)\delta^-(\mathbf{F}^{*n}_{i+\frac{1}{2}})] = \theta\mathbf{r}^{n+1}_i + (1-\theta)\mathbf{r}^n_i$$

where $\delta^-(\mathbf{F}^*_{i+\frac{1}{2}}) = \mathbf{F}^*_{i+\frac{1}{2}} - \mathbf{F}^*_{i-\frac{1}{2}}$ and $\mathbf{r} = \mathbf{f} + \mathbf{g} + \mathbf{s}$ are the source terms. The flux function is formed using an approximate Jacobian matrix $\tilde{A}_{i+\frac{1}{2}}$ and can be written as

$$\mathbf{F}^*_{i+\frac{1}{2}} = \frac{1}{2}[\mathbf{F}_{i+1} + \mathbf{F}_i - \sum_{k=1,2} \psi^k_{i+\frac{1}{2}}(1 - \phi(r^k_{i+\frac{1}{2}}))\alpha^k_{i+\frac{1}{2}}\tilde{\mathbf{e}}^k_{i+\frac{1}{2}}] \tag{9}$$

where [illegible] represents a correction factor for the eigenvalues [illegible] with corresponding eigenvectors $\mathbf{e}^k_{i+\frac{1}{2}}$. The definition of $\alpha^k_{i+\frac{1}{2}}$ is the same as that in Roe's scheme. Here the argument of the limiter is taken to be

$$r^k_{i+\frac{1}{2}} = \frac{\tilde{\lambda}^k_{i+\frac{1}{2}-s}\alpha^k_{i+\frac{1}{2}-s}}{\tilde{\lambda}^k_{i+\frac{1}{2}}\alpha^k_{i+\frac{1}{2}}}$$

where $s = sign(\tilde{\lambda}^k_{i+\frac{1}{2}})$. Inspite of coming from a centred scheme this choice of s causes the scheme to be upwinded.

The Jacobian is written in diagonal form using

$$\tilde{A}_{i+\frac{1}{2}} = \tilde{P}_{i+\frac{1}{2}} diag(\tilde{\lambda}^k_{i+\frac{1}{2}})\tilde{P}^{-1}_{i-\frac{1}{2}}$$

where $\tilde{P}_{i+\frac{1}{2}}$ is the matrix of column eigenvectors. If the matrix $B_{i+\frac{1}{2}}$ is defined as

$$B_{i+\frac{1}{2}} = \tilde{P}_{i+\frac{1}{2}}\Gamma_{i+\frac{1}{2}}\tilde{P}^{-1}_{i-\frac{1}{2}}$$

with $\Gamma_{i+\frac{1}{2}} = diag[\psi^k_{i+\frac{1}{2}}(1 - \varphi^k_{i+\frac{1}{2}})]$, then the flux can then be expressed as

$$\mathbf{F}^*_{i+\frac{1}{2}} = \frac{1}{2}[\mathbf{F}_{i+1} + \mathbf{F}_i - B_{i+\frac{1}{2}}\delta\mathbf{U}_{i+\frac{1}{2}}].$$

Both the flux vectors and the source terms can be approximated at the implicit level by a Taylor series expansion of the form

$$\mathbf{F}^{n+1}_i = \mathbf{F}^n_i + A^n_i\Delta\mathbf{w}_i + O(\Delta t^2)$$

$$\mathbf{R}^{n+1}_i = \mathbf{R}^n_i + G^n_i\Delta\mathbf{w}_i + O(\Delta t^2)$$

where $\Delta\mathbf{w}_i$ is the time increment to $\mathbf{w}_i$ and A^n_i and G^n_i are the exact Jacobians of the flux and source term vectors evaluated at the explicit time level. By using this linearisation and approximating $B^{n+1}_{i+\frac{1}{2}}$ by

$$B^{n+1}_{i+\frac{1}{2}} = B^n_{i+\frac{1}{2}}$$

a block tridiagonal matrix system is obtained, with off diagonal terms corresponding to the periodic boundary conditions of the problem, i.e.

$$AA_i\Delta\mathbf{U}_{i-1} + BB_i\Delta\mathbf{U}_i + CC_i\Delta\mathbf{U}_{i+1} = \mathbf{D}_i. \qquad (10)$$

The coefficients of equation (10) are 3×3 matrices of the form

$$AA_i = -\frac{\Delta\theta}{2}[A_{i-1} + B_{i-\frac{1}{2}}]^n$$

$$B_i = I - \lambda \Delta G_i \text{ [illegible]}$$

$$C_i = \frac{\lambda}{2}(A_{i+1} - B_{i+\frac{1}{2}})$$

$$\mathbf{D}_i = -\lambda(\mathbf{F}^*_{i+\frac{1}{2}} - \mathbf{F}^*_{i-\frac{1}{2}})^n + \Delta t \mathbf{R}_i^n.$$

Again using the technique of operator splitting, two equations of the form of (7) and (8) will need to be solved for the separate coordinate directions. Working through the algebra it is found that the corresponding sets of matrices for the two problems are, for the longitudinal (ϕ) direction,

$$\breve{A} = \frac{1}{a\cos\theta}\begin{pmatrix} 0 & 1 & 0 \\ -\tilde{u}^2 + \tilde{\psi}^{*2} & 2\tilde{u} & 0 \\ -\tilde{u}\tilde{v} & \tilde{v} & \tilde{u} \end{pmatrix}$$

$$B = -\frac{1}{2\tilde{\psi}^*}\begin{pmatrix} d^1(\tilde{u} - \tilde{\psi}^*) - d^2(\tilde{u} + \tilde{\psi}^*) & -d^1 + d^2 & 0 \\ (d^1 - d^2)(\tilde{u} + \tilde{\psi}^*)(\tilde{u} - \tilde{\psi}^*) & -d^1(\tilde{u} + \tilde{\psi}^*) + d^2(\tilde{u} - \tilde{\psi}^*) & 0 \\ d^1\tilde{v}(\tilde{u} - \tilde{\psi}^*) - d^2\tilde{v}(\tilde{u} + \tilde{\psi}^*) + 2d^3\tilde{v}\tilde{\psi}^* & -d^1\tilde{v} + d^2\tilde{v} & -2d^3\tilde{\psi}^* \end{pmatrix}$$

$$G = \frac{\partial \mathbf{R}}{\partial \mathbf{w}} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{uv\tan\theta}{2a} - \frac{g}{a\cos\theta}\frac{\partial h_s}{\partial\phi} & \frac{v\tan\theta}{2a} & \frac{1}{2}(f + \frac{u\tan\theta}{a}) \\ \frac{u^2\tan\theta}{2a} - \frac{h'g\tan\theta}{2a\cos\theta} & -\frac{1}{2}(f + \frac{2u\tan\theta}{a}) & 0 \end{pmatrix},$$

and for the latitudinal (θ) direction,

$$\breve{A} = \frac{1}{a}\begin{pmatrix} 0 & 0 & 1 \\ -\tilde{u}\tilde{v} & \tilde{v} & \tilde{u} \\ -\tilde{v}^2 + \tilde{\psi}^{*2} & 0 & 2\tilde{v} \end{pmatrix}$$

$$B = \frac{1}{2\tilde{\psi}^*}\begin{pmatrix} d^2(\tilde{v} + \tilde{\psi}^*) - d^1(\tilde{v} - \tilde{\psi}^*) & 0 & d^1 - d^2 \\ \tilde{u}d^2(\tilde{v} + \tilde{\psi}^*) - \tilde{u}d^1(\tilde{v} - \tilde{\psi}^*) - 2\tilde{u}\tilde{\psi}^*d^3 & 2\tilde{\psi}^*d^3 & \tilde{u}(d^1 - d^2) \\ (\tilde{v} + \tilde{\psi}^*)(\tilde{v} - \tilde{\psi}^*)(d^2 - d^1) & 0 & d^1(\tilde{v} + \tilde{\psi}^*) - d^2(\tilde{v} - \tilde{\psi}^*) \end{pmatrix}$$

$$G = \frac{\partial \mathbf{R}}{\partial \mathbf{w}} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{uv\tan\theta}{2a} & \frac{v\tan\theta}{2a} & \frac{1}{2}(f + \frac{u\tan\theta}{a}) \\ \frac{u^2\tan\theta}{2a} - \frac{h'g\tan\theta}{2a\cos\theta}\frac{g}{a}\frac{\partial h_s}{\partial\theta} & -\frac{1}{2}(f + \frac{2u\tan\theta}{a}) & 0 \end{pmatrix}$$

where $\tilde{\psi}^{*2} = \tilde{h}'g/\cos\theta$ and a tilde above a quantity denotes the approximate value at the midpoint of the cell.

4 Application to a test case

The scheme was applied to the first of a series of test cases proposed by Williamson et al [11] wherein a cosine bell is translated around the globe. The advecting wind is specified by

$$u = u_0 (\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

$$v = -u_0 \sin\phi\sin\alpha.$$

where α is the angle between the axis of solid body rotation and the polar axis of the spherical coordinate system.

The initial cosine bell is defined by

$$h^*(\phi,\theta) = \begin{cases} \frac{h_0^*}{2}\left(1+\cos\left(\frac{\pi r}{R}\right)\right) & \text{if } r < R \\ 0 & \text{if } r \geq R \end{cases}$$

where $h_0^* = 1000m$, $R = a/3$ and r is the radius of the great circle distance between (ϕ,θ) and the centre of the bell which is initially at $(\phi_c,\theta_c) = (3\pi/2, 0)$. The radius r can be calculated from

$$r = a\cos^{-1}\left[\sin\theta_c\sin\theta + \cos\theta_c\cos\theta\cos(\phi-\phi_c)\right],$$

the parameter values are taken to be

$$a = 6.37122 \times 10^6 m$$

$$\Omega = 7.292 \times 10^5 s^{-1}$$

$$g = 9.80616 ms^{-2}$$

and u_0 is to be set as $2\pi a/(12 \text{ days})$ which is equivalent to about $40ms^{-1}$. There are no mountains in this problem, corresponding to h_s being zero everywhere.

If the program is run for 12 days then the initial profile should return to its starting point, without any change of shape.

The chosen grid has points equally distributed in the longitudinal and latitudinal directions, with grid spacings $\Delta\phi$ and $\Delta\theta$ respectively. In this instance the angle θ is not the standard polar coordinate but is instead measured from the equator so that θ lies in the interval $[-\pi/2, \pi/2]$ where $-\pi/2$ is the South pole and $\pi/2$ is the North pole.

There are no nodes at the poles, nodes which lie directly opposite one another closest the poles being $\frac{1}{2}\Delta\theta$ from the pole. To avoid nodes at the pole points, it is necessary to take an even number of intervals in the longitudinal direction.

5 Results

For the initial testing and debugging of the program, 6 day runs were performed with $\alpha = 0$. This choice should result in the profile being translated halfway round the globe along the equator. In each instance the solution is found using a time step of 1800 seconds ($\frac{1}{2}$ hour). The initial profile is shown in figure 1. Comparing this with the results in figure 2 using the scheme with $\theta = 0$ and $\phi = 0$ (corresponding to Roe's explicit scheme) we see that a highly diffusive solution is obtained, as seen previously [1]. If the Superbee limiter is then applied the results are much improved (see figure 3). Setting $\theta = 1$ gives figure 4. The solution is again diffusive, but this time the introduction of a limiter has little effect (see figure 5). At this stage it is apparent that, when the scheme is used explicitly, a limiter must be used to achieve satisfactory results, while if an implicit scheme is used the results are always too diffusive.

The results so far seem reasonable, though a little disapointing. When the code is run for 12 days with $\theta = 0$ the results shown in figures 6-7 are obtained. When θ is set to 1, figures 8-9 are produced. A number of subsequent tests have been performed, results for which are not shown here, which demonstrate a similar behaviour when the implicitness parameter is set to 1. There is no evidence to suggest that the phenomenon which occurs in figures 8-9 is present in any of the 6 day plots, and the behaviour is unexpected. It is believed that the cause is a fault in the code, rather than a consequence of the scheme.

6 Conclusion

A scheme has been presented which had previously been found suitable for modelling flows in channels and pipes. It was hoped that the scheme would also prove to be suitable to solve the SWE on a sphere. However, as can be seen from the results obtained. [illegible] an implicit scheme results in the solution being highly diffusive. It is not clear at this stage whether the computer code is functioning completely correctly, and this may account for the poor results. We have experimented with other implicit schemes of higher order but without resolving the difficulties. Unfortunately due to time restrictions, work on this problem is unable to continue, though it is believed that a suitable scheme could still be found.

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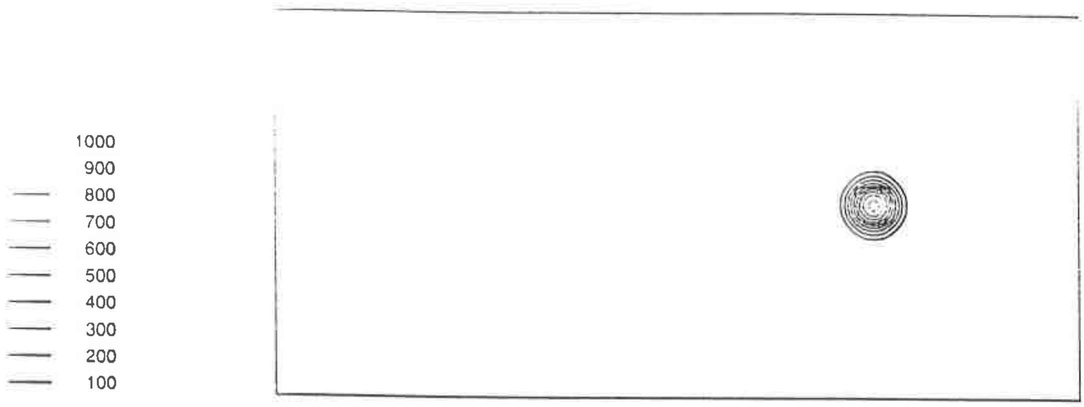

Figure 1: Initial profile

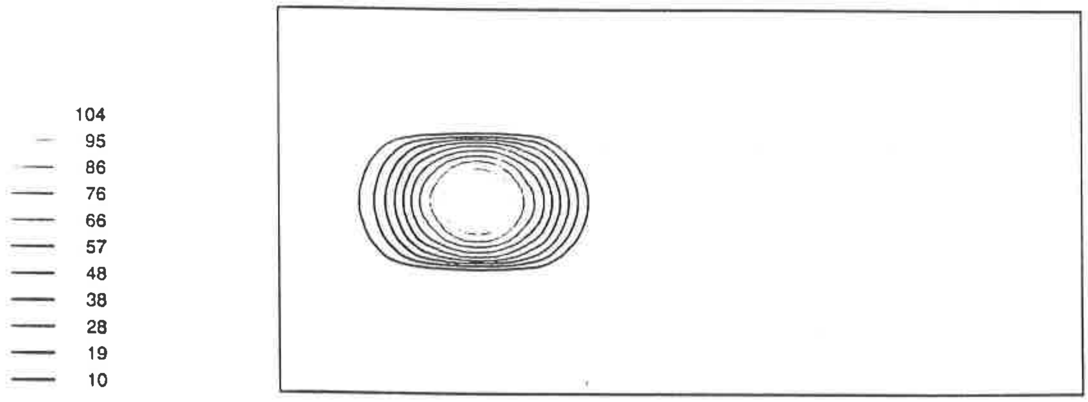

Figure 2: Explicit scheme with $\phi = 0$. 6 day run

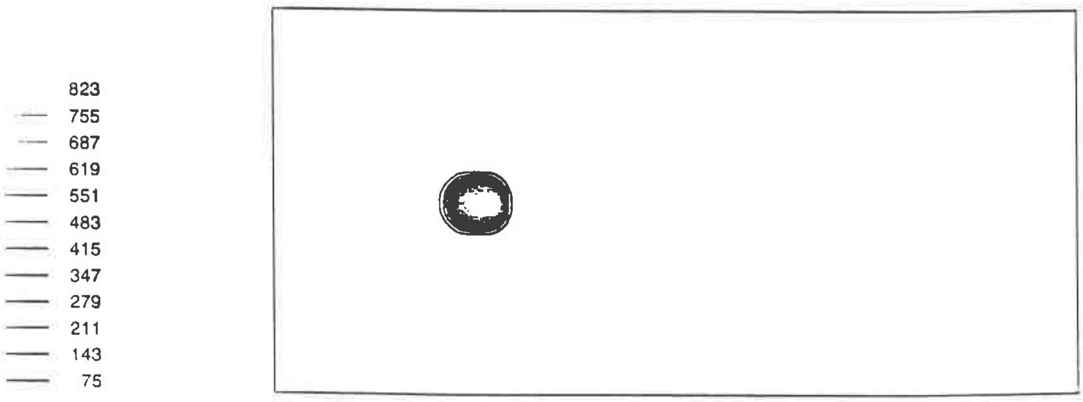

Figure 3: Explicit scheme with Superbee. 6 day run

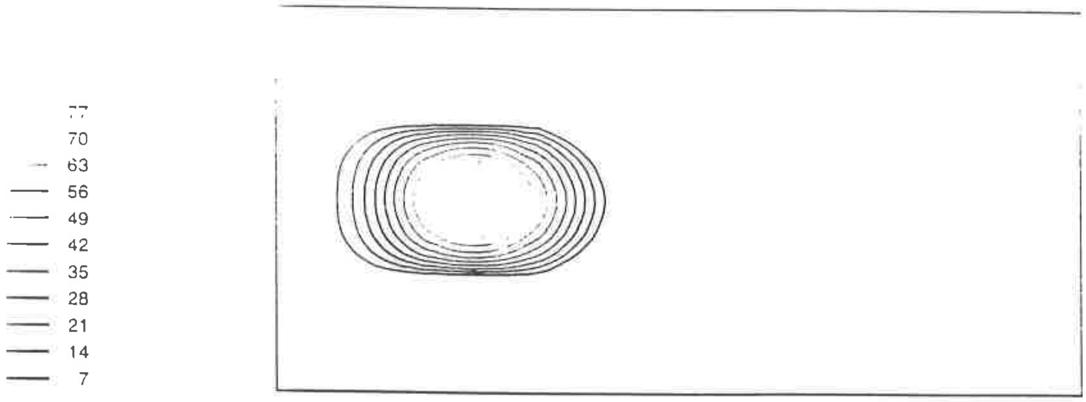

Figure 4: Implicit scheme with $\phi = 0$, 6 day run

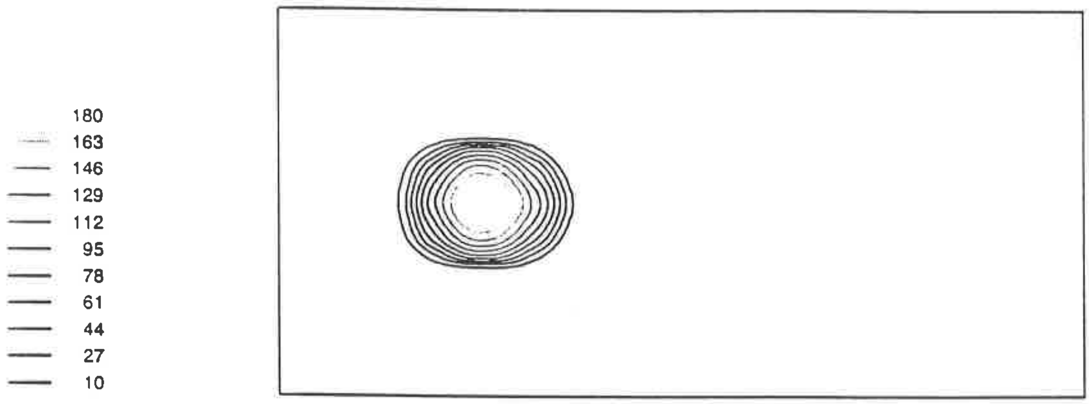

Figure 5: Implicit scheme with Superbee, 6 day run

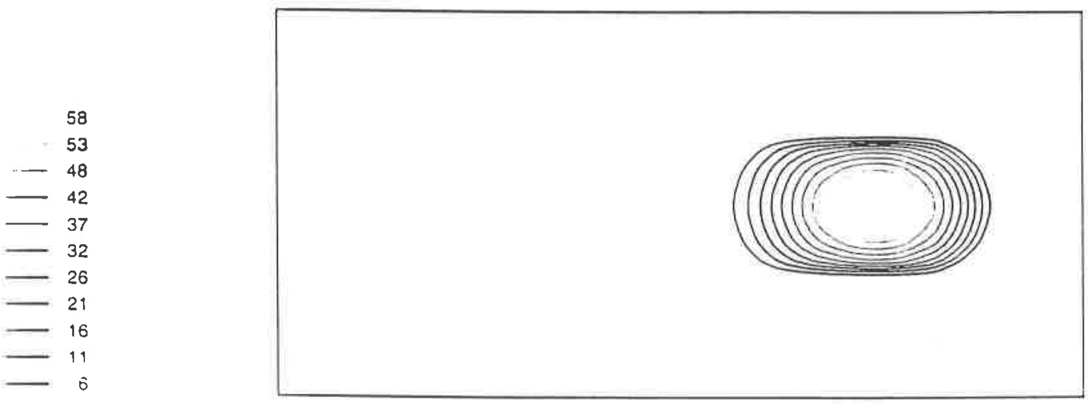

Figure 6: Explicit scheme with $\phi = 0$, 12 day run

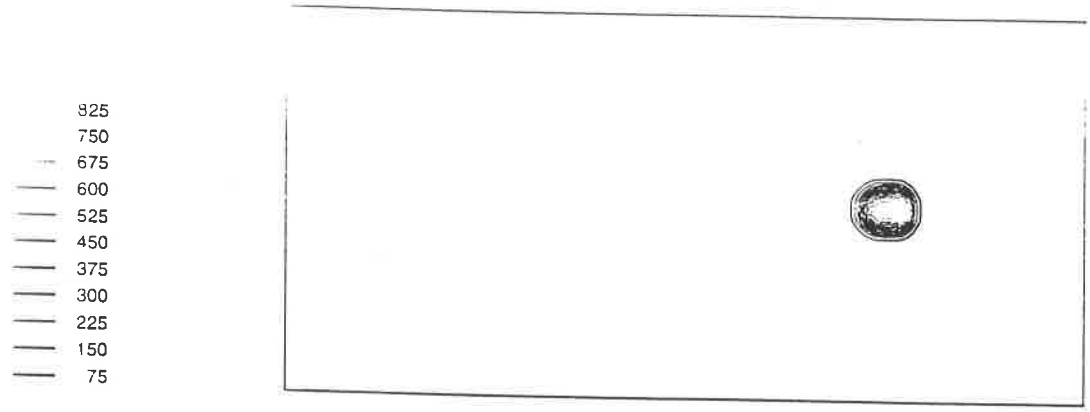

Figure 7: Explicit scheme with Superbee. 12 day run

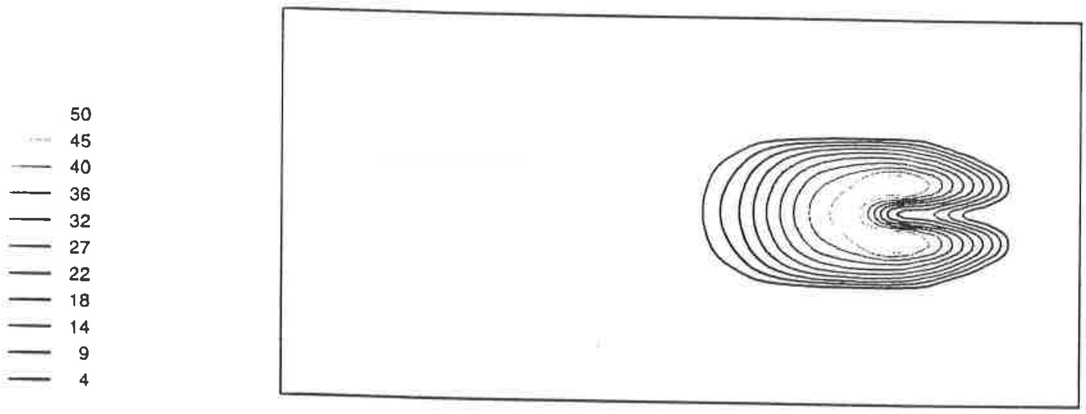

Figure 8: Implicit scheme with $\phi = 0$. 12 day run

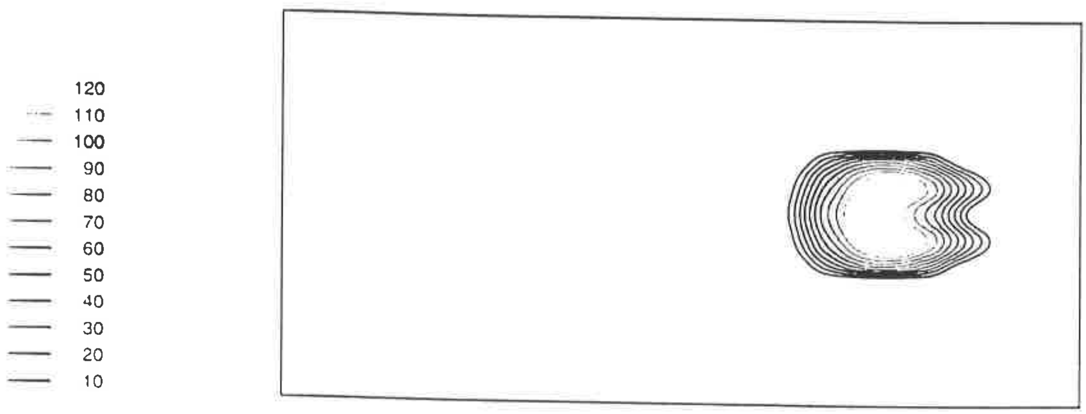

Figure 9: Implicit scheme with Superbee. 12 day run