AN APPROXIMATE RIEMANN SOLVER FOR THREE-DIMENSIONAL MULTIFLUID FLOWS USING OPERATOR SPLITTING

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ABSTRACT

An approximate (linearised) Riemann solver is presented for the solution of the Euler equations of gas dynamics in three dimensions for multifluid (ideal) flows. The solver incorporates operator splitting for two and three dimensional problems. The scheme, without operator splitting, is applied to the one dimensional problem of shock refraction at an interface of two ideal gases.

1. INTRODUCTION

The linearised approximate Riemann solver of Roe [1] was proposed in 1981 for the solution of the Euler equations of gas dynamics for a single polytropic fluid. This scheme has proved successful when applied to one and two dimensional problems (using operator splitting), (see Glaister [2], [3], [4]). Roe [5] has also proposed an approximate Riemann solver for one dimensional flows containing more than one polytropic fluid: in particular, the scheme does not directly use interface tracking. In the present report this scheme is extended to the three-dimensional Euler equations for multifluid flows, incorporating the technique of operator splitting. In particular, we show that there is only one consistent choice for the cell averages required in this type of scheme. We also find that, in the test problem considered, it is possible to maintain a sharp interface between fluids by choice of a suitable mechanism for obtaining second order accuracy.

In §2 we consider the Jacobian matrix of the modified flux functions for the Euler equations for multifluid flows, and in §3 derive an approximate Riemann solver for the solution of these equations. In §4 we describe the mechanism used to obtain second order accuracy together with a sharp interface, and in §5 we describe a one-dimensional test problem, and derive the exact solution. Finally, in §6 we display the numerical results achieved for this test problem using the scheme of §3 and §4.

2. EQUATIONS OF FLOW

In this section we state the equations of motion for an inviscid compressible ideal fluid in three dimensions, and add a further equation that enables us to deal with multifluid flows. We then derive the eigenvalues and eigenvectors of the Jacobian of the corresponding flux functions.

2.1 Equations

The Euler equations governing the flow of an inviscid, compressible fluid in three dimensions can be written in conservation form as

$$\rho_t + \operatorname{div}(\rho \underline{u}) = 0 \tag{2.1}$$

$$(\rho \underline{u})_t + \operatorname{div}(\rho \underline{u}\,\underline{u}) = -\operatorname{grad} p \tag{2.2}$$

$$e_t + \operatorname{div}\left(\underline{u}(e + p)\right) = 0 , \tag{2.3}$$

together with

$$e = \frac{p}{\gamma - 1} + \tfrac{1}{2}\rho \underline{u}.\underline{u} \tag{2.4}$$

where $\rho = \rho(\underline{x},t)$, $\underline{u} = \underline{u}(\underline{x},t) = \left(u_1(\underline{x},t),\ u_2(\underline{x},t),\ u_3(\underline{x},t)\right)^T$, $p = p(\underline{x},t)$, $e(\underline{x},t)$ and γ represent the density, velocity (in the three co-ordinate directions), pressure, total energy and the ratio of the specific heat capacities of the ideal fluid, respectively, at a general position $\underline{x} = (x_1,x_2,x_3)^T$ and at time t .

Equations (2.1)-(2.3) represent conservation of mass, momentum and energy, respectively. If the flow contains more than one ideal fluid we can add a further equation by realising that the value of γ for each fluid particle remains constant. This can be expressed mathematically as

$$\frac{D\gamma}{Dt} = 0 \tag{2.5}$$

where D/Dt represents the material derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\underline{u}.\underline{\nabla}) \; , \tag{2.6}$$

i.e. the additional differential equation for the flow variable $\gamma = \gamma(\underline{x},t)$ is

$$\gamma_t + (\underline{u}.\underline{\nabla})\gamma = 0 \; . \tag{2.7}$$

Now from equations (2.1) and (2.7) we have

$$(\rho\gamma)_t + \rho(\underline{u}.\underline{\nabla})\gamma + \gamma \; \mathrm{div}(\rho\underline{u}) = 0 \; , \tag{2.8}$$

and it can be shown (see Appendix) that

$$\mathrm{div}(\rho\gamma\underline{u}) = \rho(\underline{u}.\underline{\nabla})\gamma + \gamma \; \mathrm{div}(\rho\underline{u}) \; . \tag{2.9}$$

Thus the additional differential equation can be written in conservation form as

$$(\rho\gamma)_t + \mathrm{div}(\rho\gamma\underline{u}) = 0 \; . \tag{2.10}$$

We are interested in the solution of equations (2.1)-(2.4) and (2.10) in cartesian geometry. If we write $\underline{x} = (x,y,z)^T$, $\underline{u} = (u,v,w)^T$, equations (2.1)-(2.4) and (2.10) give rise to the following system of hyperbolic equations

$$\underline{w}_t + \underline{F}_x + \underline{G}_y + \underline{H}_z = \underline{0} \; , \tag{2.11}$$

where

$$\underline{w} = (\rho,\ \rho u,\ \rho v,\ \rho w,\ e,\ \rho\gamma)^T \tag{2.12a}$$

$$\underline{F}(\underline{w}) = \left(\rho u,\ p+\rho u^2,\ \rho uv,\ \rho uw,\ u(e+p),\ \rho u\gamma\right)^T \tag{2.12b}$$

$$\underline{G}(\underline{w}) = \left(\rho v,\ \rho uv,\ p+\rho v^2,\ \rho vw,\ v(e+p),\ \rho v\gamma\right)^T \tag{2.12c}$$

$$\underline{H}(\underline{w}) = \left(\rho w,\ \rho uw,\ \rho vw,\ p+\rho w^2,\ w(e+p),\ \rho w\gamma\right)^T \tag{2.12d}$$

with

$$e = \frac{p}{\gamma - 1} + \tfrac{1}{2}\rho(u^2 + v^2 + w^2) \ . \tag{2.12e}$$

2.2 Jacobian

We now construct the Jacobian A of the flux function $\underline{F}(\underline{w})$, given by

$$A = \frac{\partial \underline{F}}{\partial \underline{w}} \ , \tag{2.13}$$

and find its eigenvalues and (right) eigenvectors, since this information, together with a similar analysis for the Jacobians of $\underline{G}$ and $\underline{H}$ will form the basis for the approximate Riemann solver.

Defining the momentum $\underline{m} = (\mu,\nu,\sigma)^T$ as $\underline{m} = \rho\underline{u} = (\rho u,\rho v,\rho w)^T$ and the quantity $\Gamma = \rho\gamma$, equations (2.12a-b) and (2.12e) can be written in the form,

$$\underline{w} = (\rho,\mu,\nu,\sigma,e,\Gamma) \tag{2.14a}$$

$$\underline{F}(\underline{w}) = \left(\mu,\ p+\frac{\mu^2}{\rho},\ \frac{\mu\nu}{\rho},\ \frac{\mu\sigma}{\rho},\ \mu(e+p),\ \Gamma u\right)^T \tag{2.14b}$$

where

$$p = p(\rho,\mu,\nu,\sigma,e,\Gamma) = \left(\frac{\Gamma}{\rho} - 1\right)\left(e - \frac{1}{2}\frac{\mu^2}{\rho} - \frac{1}{2}\frac{\nu^2}{\rho} - \frac{1}{2}\frac{\sigma^2}{\rho}\right) \ . \tag{2.14c}$$

These lead to the following expression for the Jacobian:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\gamma q^2}{2} - H - u^2 & (3-\gamma)u & -(\gamma-1)v & -(\gamma-1)w & \gamma-1 & \frac{a^2}{\gamma(\gamma-1)} \\ -uv & v & u & 0 & 0 & 0 \\ -uv & w & 0 & u & 0 & 0 \\ \frac{\gamma u q^2}{2} - 2uH & H-(\gamma-1)u^2 & -(\gamma-1)uv & -(\gamma-1)uw & \gamma u & \frac{ua^2}{\gamma(\gamma-1)} \\ -\gamma u & \gamma & 0 & 0 & 0 & u \end{bmatrix} \qquad (2.15)$$

where the fluid speed q is given by

$$q^2 = u^2 + v^2 + w^2 , \qquad (2.16)$$

the enthalpy H is defined by

$$H = \frac{e + p}{\rho} = \frac{\gamma p}{\rho(\gamma - 1)} + \tfrac{1}{2}q^2 \qquad (2.17)$$

and the 'sound speed' a is given by

$$a^2 = \frac{\gamma p}{\rho} . \qquad (2.18)$$

2.3 Eigenvalues and Eigenvectors

The eigenvalues λ_i and corresponding eigenvectors $\underline{e}_i$ of A are then found to be

$$\lambda_1 = u + a , \quad \underline{e}_1 = (1, u+a, v, w, H+ua, \gamma)^T \qquad (2.19a)$$

$$\lambda_2 = u - a , \quad \underline{e}_2 = (1, u-a, v, w, H-ua, \gamma)^T \qquad (2.19b)$$

$$\lambda_3 = u , \quad \underline{e}_3 = (1, u, v, w, \tfrac{1}{2}u^2+\tfrac{1}{2}v^2+\tfrac{1}{2}w^2, \gamma)^T \qquad (2.19c)$$

$$\lambda_4 = u , \quad \underline{e}_4 = (0, 0, v, 0, v^2, 0)^T \qquad (2.19d)$$

$$\lambda_5 = u , \quad \underline{e}_5 = (0, 0, 0, w, w^2, 0)^T \qquad (2.19e)$$

$$\lambda_6 = u , \quad \underline{e}_6 = (0, 0, 0, 0, -\frac{a^2}{(\gamma-1)^2}, \gamma)^T , \qquad (2.19f)$$

where

$$H = \frac{a^2}{\gamma - 1} + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \tfrac{1}{2}w^2 \quad . \qquad (2.19g)$$

A similar analysis can be carried out for the Jacobians $\frac{\partial \underline{G}}{\partial \underline{w}}$, $\frac{\partial \underline{H}}{\partial \underline{w}}$.

In the next section we develop an approximate Riemann solver using the results of this section.

3. AN APPROXIMATE RIEMANN SOLVER

In this section we develop an approximate Riemann solver for the Euler equations in three dimensions for multifluid flows incorporating the technique of operator splitting. We follow a similar reasoning to that of Glaister [4].

3.1 Wavespeeds for nearby states

Consider two adjacent states $\underline{w}_L$, $\underline{w}_R$ (Left and right) close to an average state $\underline{w}$, at points L and R on an x coordinate line. We seek constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$, such that

$$\Delta\underline{w} = \sum_{j=1}^{6} \alpha_j \underline{e}_j \tag{3.1}$$

to within $O(\Delta^2)$, where $\Delta(\bullet) = (\bullet)_R - (\bullet)_L$. Writing equation (3.1) in full we have

$$\Delta\rho = \alpha_1 + \alpha_2 + \alpha_3 \tag{3.2a}$$

$$\Delta(\rho u) = \alpha_1(u+a) + \alpha_2(u-a) + \alpha_3 u \tag{3.2b}$$

$$\Delta(\rho v) = \alpha_1 v + \alpha_2 v + \alpha_3 v + \alpha_4 v \tag{3.2c}$$

$$\Delta(\rho w) = \alpha_1 w + \alpha_2 w + \alpha_3 w + \alpha_5 w \tag{3.2d}$$

$$\begin{aligned}\Delta e = {} & \alpha_1\left(\frac{a^2}{\gamma-1} + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \tfrac{1}{2}w^2 + ua\right) \\ & + \alpha_2\left(\frac{a^2}{\gamma-1} + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \tfrac{1}{2}w^2 - ua\right) \\ & + \alpha_3(\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \tfrac{1}{2}w^2) \\ & + \alpha_4 v^2 + \alpha_5 w^2 - \alpha_6 \frac{a^2}{(\gamma-1)^2}\end{aligned} \tag{3.2e}$$

and

$$\Delta(\rho\gamma) = \alpha_1\gamma + \alpha_2\gamma + \alpha_3\gamma + \alpha_6\gamma \,. \tag{3.2f}$$

From equations (3.2a-d) and (3.2f) we have that

$$\Delta(\rho u) - u\Delta\rho = a(\alpha_1 - \alpha_2) \tag{3.3}$$

$$\Delta(\rho v) - v\Delta\rho = \alpha_4 v \tag{3.4}$$

$$\Delta(\rho w) - w\Delta\rho = \alpha_5 w \tag{3.5}$$

$$\Delta(\rho\gamma) - \gamma\Delta\rho = \alpha_6 \gamma \tag{3.6}$$

and from equations (3.2a), (3.2e),

$$\Delta\left(\frac{p}{\gamma-1}\right) + \Delta\left(\frac{\rho u^2}{2}\right) - \tfrac{1}{2}u^2\Delta\rho + \Delta\left(\frac{\rho v^2}{2}\right) - \tfrac{1}{2}v^2\Delta\rho + \Delta\left(\frac{\rho w^2}{2}\right) - \tfrac{1}{2}w^2\Delta\rho$$

$$= \frac{a^2}{\gamma-1}(\alpha_1 + \alpha_2) + ua(\alpha_1 - \alpha_2) + \alpha_4 v^2 + \alpha_5 w^2 - \frac{\alpha_6 a^2}{(\alpha-1)^2} \; . \tag{3.7}$$

Using equations (3.3)-(3.6) together with $\alpha_1 + \alpha_2 = \Delta\rho - \alpha_3$ equation (3.7) yields the following equation for α_3 :

$$\frac{a^2}{\gamma-1}\alpha_3 = \frac{\gamma a^2}{(\gamma-1)^2}\Delta\rho - \frac{a^2}{\gamma(\gamma-1)^2}\Delta(\rho\gamma) - \Delta\left(\frac{p}{\gamma-1}\right)$$

$$- \frac{u^2}{2}\Delta\rho - \Delta\left(\frac{\rho u^2}{2}\right) + u\Delta(\rho u)$$

$$- \frac{v^2}{2}\Delta\rho - \Delta\left(\frac{\rho v^2}{2}\right) + v\Delta(\rho v)$$

$$- \frac{w^2}{2}\Delta\rho - \Delta\left(\frac{\rho w^2}{2}\right) + w\Delta(\rho w) \tag{3.8a}$$

where

$$a^2 = \frac{\gamma p}{\rho} \; . \tag{3.8b}$$

In addition, α_1, α_2, α_4, α_5 and α_6 can now be calculated from equations (3.2a) and (3.3)-(3.6), i.e.

$$\alpha_1 + \alpha_2 = \Delta\rho - \alpha_3 \tag{3.8c}$$

$$\alpha_1 - \alpha_2 = \frac{\Delta(\rho u) - u\Delta\rho}{a} \tag{3.8d}$$

$$\alpha_4 = \frac{\Delta(\rho v) - v\Delta\rho}{v} \tag{3.8e}$$

$$\alpha_5 = \frac{\Delta(\rho w) - w\Delta\rho}{w} \tag{3.8f}$$

$$\alpha_6 = \frac{\Delta(\rho\gamma) - \gamma\Delta\rho}{\gamma} \quad . \tag{3.8g}$$

We have made the assumption that the left and right states $\underline{w}_L$, $\underline{w}_R$ are close to some average state $\underline{w}$, so that, to within $O(\Delta^2)$,

$$\Delta(\rho U) = U\Delta\rho + \rho\Delta U \; , \qquad U = u,\ v,\ w \ \text{ or } \ \gamma \tag{3.9a-d}$$

$$\Delta(\rho U^2) = U^2\Delta\rho + 2\rho U\Delta U \; , \quad U = u,\ v \ \text{ or } \ w \tag{3.9e-g}$$

and

$$\Delta\left(\frac{p}{\gamma-1}\right) = \frac{\Delta p}{\gamma-1} - \frac{p}{(\gamma-1)^2}\,\Delta\gamma \ . \tag{3.9h}$$

In that case equations (3.8a-b) give

$$\frac{a^2}{\gamma-1}\,\alpha_3 = \frac{a^2}{\gamma-1}\,\Delta\rho - \frac{\Delta p}{\gamma-1} \tag{3.10}$$

i.e.

$$\alpha_3 = \Delta\rho - \frac{\Delta p}{a^2} \quad . \tag{3.11}$$

Finally, equations (3.8c-g) become

$$\alpha_1 + \alpha_2 = \frac{\Delta p}{a^2} \tag{3.12}$$

$$\alpha_1 - \alpha_2 = \rho \frac{\Delta u}{a^2} \tag{3.13}$$

$$\alpha_4 = \frac{\rho}{v} \Delta v \tag{3.14}$$

$$\alpha_5 = \frac{\rho}{w} \Delta w \tag{3.15}$$

$$\alpha_6 = \frac{\rho}{\gamma} \Delta \gamma \, , \tag{3.16}$$

to give the following expressions for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6 ,

$$\alpha_1 = \frac{1}{2a^2}(\Delta p + \rho a \Delta u) \tag{3.17a}$$

$$\alpha_2 = \frac{1}{2a^2}(\Delta p - \rho a \Delta u) \tag{3.17b}$$

$$\alpha_3 = \Delta \rho - \frac{\Delta p}{a^2} \tag{3.17c}$$

$$\alpha_4 = \frac{\rho}{v} \Delta v \tag{3.17d}$$

$$\alpha_5 = \frac{\rho}{w} \Delta w \tag{3.17e}$$

$$\alpha_6 = \frac{\rho}{\gamma} \Delta \gamma \, . \tag{3.17f}$$

We have found $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ such that

$$\Delta \underline{w} = \sum_{j=1}^{6} \alpha_j \underline{e}_j \tag{3.18}$$

to within $O(\Delta^2)$, and a routine calculation verifies that

$$\Delta \underline{F} = \sum_{j=1}^{6} \lambda_j \alpha_j \underline{e}_j \tag{3.19}$$

to within $O(\Delta^2)$. We are now in a position to construct the approximate Riemann solver.

3.2 Decomposition for general $\underline{w}_L$, $\underline{w}_R$

Consider the algebraic problem of finding average eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4, \tilde{\lambda}_5, \tilde{\lambda}_6$ and corresponding average eigenvectors $\tilde{\underline{e}}_1, \tilde{\underline{e}}_2, \tilde{\underline{e}}_3, \tilde{\underline{e}}_4, \tilde{\underline{e}}_5, \tilde{\underline{e}}_6$ such that relations (3.18) and (3.19) hold exactly for arbitrary states $\underline{w}_L$, $\underline{w}_R$, not necessarily close. Specifically, we seek averages $\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{a}, \tilde{\gamma}, \tilde{\delta}, \tilde{\varepsilon}$ and $\tilde{\kappa}$ in terms of two adjacent states $\underline{w}_L$, $\underline{w}_R$ (on an x-coordinate line) such that

$$\Delta \underline{w} = \sum_{j=1}^{6} \tilde{\alpha}_j \tilde{\underline{e}}_j \tag{3.20}$$

and

$$\Delta \underline{F} = \sum_{j=1}^{6} \tilde{\lambda}_j \tilde{\alpha}_j \tilde{\underline{e}}_j , \tag{3.21}$$

where

$$\Delta(\bullet) = (\bullet)_R - (\bullet)_L \tag{3.22a}$$

$$\underline{w} = (\rho,\ \rho u,\ \rho v,\ \rho w,\ e,\ \rho\gamma)^T \tag{3.11b}$$

$$F(\underline{w}) = \left(\rho u,\ p+\rho u^2,\ \rho uv,\ \rho uw,\ u(e+p),\ \rho u\gamma\right)^T \tag{3.22c}$$

$$e = \frac{p}{\gamma-1} + \tfrac{1}{2}\rho u^2 + \tfrac{1}{2}\rho v^2 + \tfrac{1}{2}\rho w^2 \tag{3.22d}$$

$$\tilde{\lambda}_{1,2,3,4,5,6} = \tilde{u}+\tilde{a},\ \tilde{u}-\tilde{a},\ \tilde{u},\ \tilde{u},\ \tilde{u},\ \tilde{u} \tag{3.23a}$$

$$\tilde{\underline{e}}_1 = (1,\ \tilde{u}+\tilde{a},\ \tilde{v},\ \tilde{w},\ \tilde{\delta},\ \tilde{\gamma})^T \tag{3.23b}$$

$$\underline{\tilde{e}}_2 = (1,\ \tilde{u}-\tilde{a},\ \tilde{v},\ \tilde{w},\ \tilde{\varepsilon},\ \tilde{\gamma})^T \tag{3.23c}$$

$$\underline{\tilde{e}}_3 = (1,\ \tilde{u},\ \tilde{v},\ \tilde{w},\ \tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\tilde{v}^2 + \tfrac{1}{2}\tilde{w}^2,\ \tilde{\gamma})^T \tag{3.23d}$$

$$\underline{\tilde{e}}_4 = (0,\ 0,\ \tilde{v},\ 0,\ \tilde{v}^2,\ 0)^T \tag{3.23e}$$

$$\underline{\tilde{e}}_5 = (0,\ 0,\ 0,\ \tilde{w},\ \tilde{w}^2,\ 0)^T \tag{3.23f}$$

$$\underline{\tilde{e}}_6 = (0,\ 0,\ 0,\ 0,\ \tilde{\kappa},\ \tilde{\gamma})^T \tag{3.23g}$$

$$\tilde{\alpha}_1 = \frac{1}{2\tilde{a}^2}(\Delta p + \tilde{\rho}\tilde{a}\Delta u) \tag{3.24a}$$

$$\tilde{\alpha}_2 = \frac{1}{2\tilde{a}^2}(\Delta p - \tilde{\rho}\tilde{a}\Delta u) \tag{3.24b}$$

$$\tilde{\alpha}_3 = \Delta\rho - \frac{\Delta p}{\tilde{a}^2} \tag{3.24c}$$

$$\tilde{\alpha}_4 = \frac{\tilde{\rho}}{\tilde{v}}\,\Delta v \tag{3.24d}$$

$$\tilde{\alpha}_5 = \frac{\tilde{\rho}}{\tilde{w}}\,\Delta w \tag{3.24e}$$

$$\tilde{\alpha}_6 = \frac{\tilde{\rho}}{\tilde{\gamma}}\,\Delta\gamma\ . \tag{3.24f}$$

We note that no particular form for the representation of the fifth component of $\underline{\tilde{e}}_1$, $\underline{\tilde{e}}_2$ and $\underline{\tilde{e}}_6$ is assumed.

The problem of finding averages $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{a}$, $\tilde{\gamma}$, $\tilde{\delta}$, $\tilde{\varepsilon}$ and $\tilde{\kappa}$ subject to equations (3.20)-(3.24f) will subsequently be denoted by (*). (N.B. The quantities $\tilde{\delta}$, $\tilde{\varepsilon}$ and $\tilde{\kappa}$ denote approximations to the fifth components of $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_6$, respectively.)

The solution of problem (*) is sought in a similar way to that given by Glaister [3], [4] in the case of a single non-ideal fluid, by Roe and Pike [6] in the case of a single ideal fluid in one dimension and by Roe [5] in the ideal multifluid one dimensional case. We note that problem (*) is equivalent to seeking an approximation $\tilde{A}$ to the Jacobian A with eigenvalues $\tilde{\lambda}_i$ and eigenvectors $\tilde{\underline{e}}_i$, which is an alternative approach used in the single fluid case by Roe [1]. Furthermore, the Riemann solver that we construct must have the property that it reduces to the scheme of Roe [1] in the case of a single fluid, i.e. γ = constant throughout the flow, and to the scheme of Roe [5] in the one dimensional case.

The first step in the analysis of problem (*) is to write out equations (3.20) and (3.21) explicitly, namely,

$$\Delta\rho = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 \tag{3.25a}$$

$$\Delta(\rho u) = \tilde{\alpha}_1(\tilde{u} + \tilde{a}) + \tilde{\alpha}_2(\tilde{u} - \tilde{a}) + \tilde{\alpha}_3 \tag{3.25b}$$

$$\Delta(\rho v) = \tilde{\alpha}_1\tilde{v} + \tilde{\alpha}_2\tilde{v} + \tilde{\alpha}_3\tilde{v} + \tilde{\alpha}_4\tilde{v} \tag{3.25c}$$

$$\Delta(\rho w) = \tilde{\alpha}_1\tilde{w} + \tilde{\alpha}_2\tilde{w} + \tilde{\alpha}_3\tilde{w} + \tilde{\alpha}_5\tilde{w} \tag{3.25d}$$

$$\Delta e = \Delta\left(\frac{p}{\gamma-1}\right) + \Delta\left(\frac{\rho q^2}{2}\right) = \tilde{\alpha}_1\tilde{\delta} + \tilde{\alpha}_2\tilde{\varepsilon} + \tilde{\alpha}_3\tfrac{1}{2}\tilde{q}^2 + \tilde{\alpha}_4\tilde{v}^2 + \tilde{\alpha}_5\tilde{w}^2 + \tilde{\alpha}_6\tilde{\kappa} \tag{3.25e}$$

$$\Delta(\rho\gamma) = \tilde{\alpha}_1\tilde{\gamma} + \tilde{\alpha}_2\tilde{\gamma} + \tilde{\alpha}_3\tilde{\gamma} + \tilde{\alpha}_6\tilde{\gamma} \tag{3.25f}$$

$$\Delta(\rho u) = \tilde{\alpha}_1(\tilde{u} + \tilde{a}) + \tilde{\alpha}_2(\tilde{u} - \tilde{a}) + \tilde{\alpha}_3\tilde{u} \tag{3.25g}$$

$$\Delta(p + \rho u^2) = \Delta p + \Delta(\rho u^2) = \tilde{\alpha}_1(\tilde{u} + \tilde{a})^2 + \tilde{\alpha}_2(\tilde{u} - \tilde{a})^2 + \tilde{\alpha}_3\tilde{u}^2 \tag{3.25h}$$

$$\Delta(\rho uv) = \tilde{\alpha}_1(\tilde{u} + \tilde{a})\tilde{v} + \tilde{\alpha}_2(\tilde{u} - \tilde{a})\tilde{v} + \tilde{\alpha}_3\widetilde{uv} + \tilde{\alpha}_4\widetilde{uv} \tag{3.25i}$$

$$\Delta(\rho uw) = \tilde{\alpha}_1(\tilde{u} + \tilde{a})\tilde{w} + \tilde{\alpha}_2(\tilde{u} - \tilde{a})\tilde{w} + \tilde{\alpha}_3\widetilde{uw} + \tilde{\alpha}_5\widetilde{uw} \tag{3.25j}$$

$$\Delta\big(u(e + p)\big) = \Delta\left(\frac{\gamma up}{\gamma-1}\right) + \Delta\left(\frac{\rho uq^2}{2}\right)$$

$$= \tilde{\alpha}_1(\tilde{u} + \tilde{a})\tilde{\delta} + \tilde{\alpha}_2(\tilde{u} - \tilde{a})\tilde{\varepsilon} + \tilde{\alpha}_3\tilde{u}\tfrac{1}{2}\widetilde{q^2} + \tilde{\alpha}_4\widetilde{uv^2} + \tilde{\alpha}_5\widetilde{uw^2} + \tilde{\alpha}_6\widetilde{u\kappa} \tag{3.25k}$$

$$\Delta(\rho u\gamma) = \tilde{\alpha}_1(\tilde{u} + \tilde{a})\tilde{\gamma} + \tilde{\alpha}_2(\tilde{u} - \tilde{a})\tilde{\gamma} + \tilde{\alpha}_3\widetilde{u\gamma} + \tilde{\alpha}_6\widetilde{u\gamma} \tag{3.25l}$$

where

$$q^2 = u^2 + v^2 + w^2 \tag{3.26}$$

as before, and for convenience we have written

$$\widetilde{q^2} = \widetilde{u^2} + \widetilde{v^2} + \widetilde{w^2} \ . \tag{3.27}$$

Equation (3.25a) is satisfied by any average we care to define, while (3.25g) is the same as equation (3.25b); thus it remains to solve equations (3.25c-l). From equation (3.25g) we have

$$\Delta(\rho u) = \tilde{u}(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) + \tilde{a}(\tilde{\alpha}_1 - \tilde{\alpha}_2)$$

$$= \tilde{u}\Delta\rho + \tilde{\rho}\Delta u \ , \tag{3.28}$$

and from equation (3.25h) we obtain

$$\Delta(\rho u^2) = \widetilde{u^2}(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) + 2\widetilde{ua}(\tilde{\alpha}_1 - \tilde{\alpha}_2)$$

$$= \widetilde{u^2}\Delta\rho + 2\widetilde{ua}\Delta u \ . \tag{3.29}$$

Substituting for $\tilde{\rho}$ from equation (3.28) into equation (3.29) yields the following quadratic equation for $\tilde{u}$:

$$\widetilde{u^2} - 2\tilde{u}\Delta(\rho u) + \Delta(\rho u^2) = 0 \ . \tag{3.30}$$

Only one solution of equation (3.30) is productive, namely,

$$\tilde{u} = \frac{\Delta(\rho u) - \sqrt{\left(\Delta(\rho u)\right)^2 - \Delta\rho\Delta(\rho u^2)}}{\Delta\rho}$$

$$= \frac{\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad (3.31)$$

which, on substituting $\tilde{u}$ into equation (3.28), gives

$$\tilde{\rho} = \frac{\Delta(\rho u) - \tilde{u}\Delta\rho}{\Delta u} = \sqrt{\rho_L \rho_R} \ . \qquad (3.32)$$

From equations (3.25c-d) and (3.25f) we have

$$\Delta(\rho v) = \tilde{v}\Delta\rho + \tilde{\rho}\Delta v \qquad (3.33a)$$

$$\Delta(\rho w) = \tilde{w}\Delta\rho + \tilde{\rho}\Delta w \qquad (3.33b)$$

$$\Delta(\rho\gamma) = \tilde{\gamma}\Delta\rho + \tilde{\rho}\Delta\gamma \qquad (3.33c)$$

i.e.

$$\tilde{v} = \frac{\Delta(\rho v) - \tilde{\rho}\Delta v}{\Delta\rho} = \frac{\sqrt{\rho_L}\, v_L + \sqrt{\rho_R}\, v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \ , \qquad (3.34a)$$

$$\tilde{w} = \frac{\Delta(\rho w) - \tilde{\rho}\Delta w}{\Delta\rho} = \frac{\sqrt{\rho_L}\, w_L + \sqrt{\rho_R}\, w_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad (3.34b)$$

and

$$\tilde{\gamma} = \frac{\Delta(\rho\gamma) - \tilde{\rho}\Delta\gamma}{\Delta\rho} = \frac{\sqrt{\rho_L}\, \gamma_L + \sqrt{\rho_R}\, \gamma_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \ . \qquad (3.34c)$$

We have now determined $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{\gamma}$ and can now show that

$$\Delta(\rho U^2) - 2\widetilde{\rho U}\Delta U - \tilde{U}^2\Delta\rho = 0 \ , \qquad U = u,\ v \ \text{or} \ w \qquad (3.35a\text{-}c)$$

$$\Delta(\rho u V) - \widetilde{\rho u}\Delta V - \widetilde{Vu}\Delta\rho - \widetilde{\rho V}\Delta u = 0 \ , \quad V = v,\ w \ \text{or} \ \gamma \qquad (3.36a\text{-}c)$$

$$\Delta\left(\frac{\rho uU^2}{2}\right) - \frac{\widetilde{uU^2}}{2}\Delta\rho - \widetilde{\rho uU}\Delta U - \frac{\widetilde{\rho U^2}}{2}\Delta u = \frac{\tilde{\rho}^2(\Delta U)^2\Delta u}{2(\sqrt{\rho_L}+\sqrt{\rho_R})^2}\ , \quad U = u,\ v \text{ or } w \qquad (3.37a\text{-}c)$$

$$\Delta\left(\frac{\gamma up}{\gamma - 1}\right) - \tilde{u}\Delta\left(\frac{\gamma p}{\gamma - 1}\right) = \tilde{\rho}\Delta u\left(\frac{\sqrt{\rho_L}\dfrac{\gamma_L p_L}{\rho_L(\gamma_L - 1)} + \sqrt{\rho_R}\dfrac{\gamma_R p_R}{\rho_R(\gamma_R - 1)}}{\sqrt{\rho_L}+\sqrt{\rho_R}}\right)\ , \qquad (3.38)$$

and

$$\frac{\sqrt{\rho_L}U_L{}^2 + \sqrt{\rho_R}U_R{}^2}{\sqrt{\rho_L}+\sqrt{\rho_R}} - \tilde{U}^2 = \frac{\tilde{\rho}(\Delta U)^2}{(\sqrt{\rho_L}+\sqrt{\rho_R})^2}\ , \qquad U = u,\ v \text{ or } w \qquad (3.39a\text{-}c)$$

all of which will be used later. From equations (3.36a-c) and (3.24a-f) we can see that equations (3.25i-j) and (3.25l) are automatically satisfied.

We are now left with equations (3.25e) and 3.25k). Now, subtracting equation (3.25e) multiplied by $\tilde{u}$ from equation (3.25k) gives

$$\left(\frac{\gamma up}{\gamma - 1}\right) - \tilde{u}\Delta\left(\frac{\gamma p}{\gamma - 1}\right) + \Delta\left(\frac{\rho uq^2}{2}\right) - \tilde{u}\Delta\left(\frac{\rho q^2}{2}\right) = \tilde{\alpha}_1\widetilde{a\delta} - \tilde{\alpha}_2\widetilde{a\varepsilon}\ , \qquad (3.40)$$

and using equations (3.24a-b) together with $\frac{1}{\gamma-1} = \frac{\gamma}{\gamma-1} - 1$ equation (3.40) yields

$$\Delta\left(\frac{\gamma up}{\gamma - 1}\right) - \tilde{u}\Delta\left(\frac{\gamma p}{\gamma - 1}\right) + \Delta\left(\frac{\rho uq^2}{2}\right) - \tilde{u}\Delta\left(\frac{\rho q^2}{2}\right) = \left(\frac{(\tilde{\delta} - \tilde{\varepsilon})}{2\tilde{a}} - \tilde{u}\right)\Delta p + \tilde{\rho}\,\frac{(\tilde{\delta} + \tilde{\varepsilon})}{2}\Delta u\ . \qquad (3.41)$$

If we add equations (3.35a-c) each multiplied by $\frac{\tilde{u}}{2}$ and subtract from the result obtained by adding equations (3.37a-c) we find that

$$\Delta\left(\frac{\rho uq^2}{2}\right) - \tilde{u}\Delta\left(\frac{\rho q^2}{2}\right) = \frac{\widetilde{\rho q^2}}{2}\Delta u + \frac{\tilde{\rho}^2\Delta u}{2}\left((\Delta u)^2 + (\Delta v)^2 + (\Delta w)^2\right)\ . \qquad (3.42)$$

Finally, adding equations (3.39a-c), equation (3.42) can be written as

$$\Delta\left(\frac{\rho u q^2}{2}\right) - \tilde{u}\Delta\left(\frac{\rho q^2}{2}\right) = \frac{\tilde{\rho}\Delta u}{2} \frac{(\sqrt{\rho_L}\, q_L{}^2 + \sqrt{\rho_R}\, q_R{}^2)}{(\sqrt{\rho_L} + \sqrt{\rho_R})} , \tag{3.43}$$

where

$$q^2 = u^2 + v^2 + w^2 , \tag{3.44a}$$

$$\tilde{q}^2 = \tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2 \tag{3.44b}$$

and

$$q^2_{L(R)} = u^2_{L(R)} + v^2_{L(R)} + w^2_{L(R)} . \tag{3.44c}$$

Thus, using equations (3.38) and (3.43), equation (3.41) becomes

$$\left(\frac{\tilde{\delta} + \tilde{\varepsilon}}{2} - \frac{(\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R)}{\sqrt{\rho_L} + \sqrt{\rho_R}}\right)\tilde{\rho}\Delta u + \left(\frac{\tilde{\delta} - \tilde{\varepsilon}}{2\tilde{a}} - \tilde{u}\right)\Delta p = 0 \tag{3.45}$$

where

$$H_{L(R)} = \frac{\gamma_{L(R)} p_{L(R)}}{\rho_{L(R)}(\gamma_{L(R)} - 1)} + \tfrac{1}{2} q^2_{L(R)} . \tag{3.46}$$

Therefore, if we define a mean enthalpy $\tilde{H}$, by

$$\tilde{H} = \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{3.47}$$

where the enthalpy H is given by

$$H = \frac{e + p}{\rho} = \frac{\gamma p}{\rho(\gamma - 1)} + \tfrac{1}{2} q^2 . \tag{3.48}$$

equation (3.45) becomes

$$\left(\frac{\tilde{\delta} + \tilde{\varepsilon}}{2} - \tilde{H}\right)\tilde{\rho}\Delta u + \left(\frac{\tilde{\delta} - \tilde{\varepsilon}}{2\tilde{a}} - \tilde{u}\right)\Delta p = 0 . \tag{3.49}$$

We now focus attention on equation (3.25e) to be solved in conjunction with equation (3.49). Using equations (3.24a-f), (3.26), (3.27) and (3.35a-c) enables us to rewrite equation (3.25e) as

$$\Delta\left(\frac{p}{\gamma - 1}\right) - \frac{\Delta p}{\tilde{\gamma} - 1} - \frac{\widetilde{\rho\kappa}\Delta\gamma}{\tilde{\gamma}} = \left(\frac{\tilde{\delta} + \tilde{\varepsilon}}{2} - \frac{\tilde{q}^2}{2} - \frac{\tilde{a}^2}{\tilde{\gamma} - 1}\right)\frac{\Delta p}{\tilde{a}^2} + \left(\frac{\tilde{\delta} - \tilde{\varepsilon}}{2\tilde{a}} - \tilde{u}\right)\tilde{\rho}\Delta u \ . \qquad (3.50)$$

If we now introduce the internal energy $i = \dfrac{p}{\rho(\gamma-1)}$ then we see that

$$(\tilde{\gamma} - 1)\Delta\left(\frac{p}{\gamma - 1}\right) - \Delta p = (\tilde{\gamma} - 1)\Delta(\rho i) - \Delta\big((\gamma - 1)\rho i\big)$$

$$= \tilde{\gamma}\Delta(\rho i) - \Delta(\rho i \gamma)$$

$$= -\tilde{\rho}\,\frac{(\sqrt{\rho_L}\, i_L + \sqrt{\rho_R}\, i_R)}{\sqrt{\rho_L} + \sqrt{\rho_R}}\Delta\gamma \ . \qquad (3.51)$$

Thus, defining a mean internal energy $\tilde{i}$, by

$$\tilde{i} = \frac{\sqrt{\rho_L}\, i_L + \sqrt{\rho_R}\, i_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} = \frac{\sqrt{\rho_L}\dfrac{p_L}{\rho_L(\gamma_L-1)} + \sqrt{\rho_R}\dfrac{p_R}{\rho_R(\gamma_R-1)}}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad (3.52)$$

and using equation (3.51) enables us to rewrite equation (3.50) as

$$\left(\frac{\tilde{\delta} + \tilde{\varepsilon}}{2} - \frac{\tilde{q}^2}{2} - \frac{\tilde{a}^2}{\tilde{\gamma} - 1}\right)\frac{\Delta p}{\tilde{a}^2} + \left(\frac{\tilde{\delta} - \tilde{\varepsilon}}{2\tilde{a}} - \tilde{u}\right)\tilde{\rho}\Delta u + \left(\frac{\tilde{\kappa}}{\tilde{\gamma}} + \frac{\tilde{i}}{\tilde{\gamma} - 1}\right)\Delta\gamma = 0 \qquad (3.53)$$

Therefore, in order to complete our Riemann solver we need to solve equations (3.49) and (3.53).

There is only one consistent choice for the remaining averages $\tilde{a}$, $\tilde{\delta}$, $\tilde{\varepsilon}$ and $\tilde{\kappa}$ in order that equations (3.49) and (3.53) are satisfied for all variations Δu, Δp, $\Delta\gamma$ i.e.

$$\frac{\tilde{\delta} + \tilde{\varepsilon}}{2} - \tilde{H} = 0 \qquad (3.54a)$$

$$\frac{\tilde{\delta} + \tilde{\varepsilon}}{2} - \frac{\tilde{q}^2}{2} - \frac{\tilde{a}^2}{\tilde{\gamma} - 1} = 0 \qquad (3.54b)$$

$$\frac{\tilde{\delta} - \tilde{\varepsilon}}{2\tilde{a}} - \tilde{u} = 0 \qquad (3.54c)$$

and

$$\frac{\tilde{\kappa}}{\tilde{\gamma}} + \frac{\tilde{i}}{\tilde{\gamma} - 1} = 0 \ . \qquad (3.54d)$$

Therefore, as we have prescribed the averages $\tilde{\rho}$, $\tilde{\gamma}$, $\tilde{H}$, $\tilde{i}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and hence $\tilde{q}^2$ we solve equations (3.54a-d) to give the following averages for $\tilde{a}$, $\tilde{\delta}$, $\tilde{\varepsilon}$ and $\tilde{\kappa}$:

$$\tilde{a}^2 = (\tilde{\gamma} - 1)(\tilde{H} - \tfrac{1}{2}\tilde{q}^2) \qquad (3.55a)$$

$$\tilde{\delta} = \tilde{H} + \tilde{u}\tilde{a} = \frac{\tilde{a}^2}{\tilde{\gamma} - 1} + \tfrac{1}{2}\tilde{q}^2 + \tilde{u}\tilde{a} \qquad (3.55b)$$

$$\tilde{\varepsilon} = \tilde{H} - \tilde{u}\tilde{a} = \frac{\tilde{a}^2}{\tilde{\gamma} - 1} + \tfrac{1}{2}\tilde{q}^2 - \tilde{u}\tilde{a} \qquad (3.55c)$$

and

$$\tilde{\kappa} = - \frac{\tilde{\gamma}\tilde{i}}{(\tilde{\gamma} - 1)} \ . \qquad (3.55d)$$

By symmetry, similar results hold for the Jacobians $\frac{\partial G}{\partial \underline{w}}$, $\frac{\partial H}{\partial \underline{w}}$.

Summarising, we can now apply a three-dimensional Riemann solver for the Euler equations for multifluid (ideal) flows using the technique of operator splitting. We incorporate the results found here, together with the one dimensional scalar first order algorithm given in [6], and perform a sequence of one dimensional calculations along computational grid lines in the x, y and z-directions in turn. The algorithm along a line y = constant, z = constant can be described as follows.

Suppose at time level n we have data $\underline{w}_L$, $\underline{w}_R$ given at either end of the cell (x_L, x_R), (on a line $y = y_0$, $z = z_0$), then we update $\underline{w}$ to time level $n+1$ in an upwind manner. Schematically, we increment $\underline{w}$ as in Fig. 1.

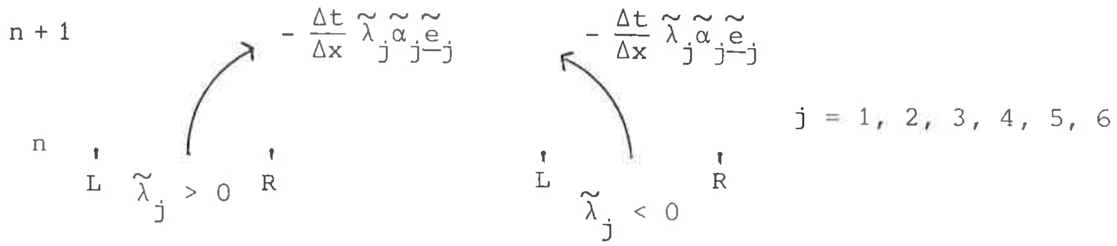


Figure 1

where $\Delta x = x_R - x_L$, Δt is the time interval from level n to $n+1$, and $\tilde{\lambda}_j$, $\tilde{\alpha}_j$, $\tilde{\underline{e}}_j$ are given by

$$\tilde{\lambda}_{1,2,3,4,5,6} = \tilde{u} + \tilde{a},\ \tilde{u} - \tilde{a},\ \tilde{u},\ \tilde{u},\ \tilde{u},\ \tilde{u}$$

$$\tilde{\underline{e}}_{1,2,3,4,5,6} = \begin{pmatrix} 1 \\ \tilde{u}+\tilde{a} \\ \tilde{v} \\ \tilde{w} \\ \tilde{H}+\tilde{u}\tilde{a} \\ \tilde{\gamma} \end{pmatrix},\ \begin{pmatrix} 1 \\ \tilde{u}-\tilde{a} \\ \tilde{v} \\ \tilde{w} \\ \tilde{H}-\tilde{u}\tilde{a} \\ \tilde{\gamma} \end{pmatrix},\ \begin{pmatrix} 1 \\ \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \frac{1}{2}\tilde{q}^2 \\ \tilde{\gamma} \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \tilde{v} \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \tilde{w} \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{\tilde{i}}{\tilde{\gamma}-1} \\ 1 \end{pmatrix}$$

$$\tilde{\alpha}_{1,2,3,4,5,6} = \frac{1}{2\tilde{a}^2}(\Delta p + \tilde{\rho}\tilde{a}\Delta u),\ \frac{1}{2\tilde{a}^2}(\Delta p - \tilde{\rho}\tilde{a}\Delta u),\ \Delta\rho - \frac{\Delta p}{\tilde{a}^2},\ \tilde{\rho}\Delta v,\ \tilde{\rho}\Delta w,\ \tilde{\rho}\Delta\gamma$$

$$\tilde{\rho} = \sqrt{\rho_L\rho_R}\ ,\qquad \tilde{U} = \frac{\sqrt{\rho_L}\,U_L + \sqrt{\rho_R}\,U_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}\qquad U = u,\ v,\ w,\ \gamma,\ H \text{ or } i$$

$$\tilde{q}^2 = \tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2\ ,\qquad \tilde{a}^2 = (\tilde{\gamma} - 1)(\tilde{H} - \tfrac{1}{2}\tilde{q}^2)\ ,$$

$i = \frac{p}{\rho(\gamma - 1)}$ and $\Delta(\bullet) = (\bullet)_R - (\bullet)_L$. We note that factors $\tilde{v}$, $\tilde{w}$ have been taken out of $\underline{\tilde{e}}_4$, $\underline{\tilde{e}}_5$ so that $\tilde{\alpha}_4$, $\tilde{\alpha}_5$ will not become indeterminate, and the factor $\tilde{\gamma}$ has been taken out of $\underline{\tilde{e}}_6$. Similar results apply for updating in the y and z directions.

The Riemann solver we have constructed in this section is a conservative algorithm (also when incorporated with operator splitting) and has the important one-dimensional shock capturing property guaranteed by equations (3.20), (3.21) (see [1]). Problems may occur, as with all operator split schemes, when attempting to capture a shock that is oblique to the grid.

In the next section we describe the mechanism used to create a second order algorithm which is oscillation free, and maintains sharp shock and contact discontinuities.

4. SECOND ORDER ALGORITHM

In this section we describe the mechanism used to make the algorithm of §3 into a second order, oscillation-free scheme.

We begin by describing the scalar algorithm of Roe and Pike [6] written in the flux limiter notation given in Sweby [7]. The scheme incorporates the device of Sweby [8] needed to disperse entropy violating solutions, and can be extended to include irregular grids (see Glaister [9]). Consider the one dimensional scalar equation

$$u_t + f_x = 0 \qquad (x,\ t) \in (-\infty,\ \infty) \times [0,\ T]$$

where $u = u(x,\ t)$ and $f = f(u)$. A class of second order oscillation-free schemes for the solution of equation (4.1) can be written as

$$u^{j-1} = u_{j-1} + \phi^L_{j-\frac{1}{2}} - b^L_{j-\frac{1}{2}} + b^R_{j-\frac{1}{2}}$$
$$j = 1,2,\ldots \qquad \textbf{(4.1a-b)}$$
$$u^j = u_j + \phi^R_{j-\frac{1}{2}} - b^R_{j-\frac{1}{2}} + b^L_{j-\frac{1}{2}}$$

where u_j u^j denote the approximations to $u(j\Delta x,\ n\Delta t)$, $u(j\Delta x,\ (n+1)\Delta t)$, respectively. In addition

$$\phi^L_{j-\frac{1}{2}} = (-\nu^R_{j-\frac{1}{2}})^+(u_j - u_{j-\frac{1}{2}}) + (-\nu^L_{j-\frac{1}{2}})^+(u_{j-\frac{1}{2}} - u_j) \qquad (4.2a)$$

$$\phi^R_{j-\frac{1}{2}} = -\ (\nu^R_{j-\frac{1}{2}})^+(u_j - u_{j-\frac{1}{2}}) - (\nu^L_{j-\frac{1}{2}})^+(u_{j-\frac{1}{2}} - u_{j-1}) \qquad (4.2b)$$

$$\nu_{j-\frac{1}{2}} = \frac{\Delta t}{\Delta x} a_{j-\frac{1}{2}}\ , \qquad \nu^{L(R)}_{j-\frac{1}{2}} = \frac{\Delta t}{\Delta x} a^{L(R)}_{j-\frac{1}{2}} \qquad (4.3a\text{-}b)$$

$$u_{j-\frac{1}{2}} = \tfrac{1}{2}(u_j + u_{j-1})\ , \qquad c^+ = \tfrac{1}{2}(|c| + c) \qquad (4.4a\text{-}b)$$

$$a_{j-\frac{1}{2}} = \begin{cases} \dfrac{f(u_j) - f(u_{j-1})}{u_j - u_{j-1}} & u_j \neq u_{j-1} \\ f'(u_{j-\frac{1}{2}}) & u_j = u_{j-1} \end{cases} \tag{4.5a-b}$$

$$a^L_{j-\frac{1}{2}} = \min(a_{j-1}, a_{j-\frac{1}{2}}) \tag{4.6a}$$

$$a^R_{j-\frac{1}{2}} = \max(a_{j-\frac{1}{2}}, a_j) \tag{4.6b}$$

$$a_j = f'(u_j) \ , \qquad a_{j-1} = f'(u_{j-1}) \tag{4.7}$$

$$b^L_{j-\frac{1}{2}} = \beta_{j-\frac{1}{2}}\phi^L_{j-\frac{1}{2}}\psi(r^L_{j-\frac{1}{2}}) \tag{4.8a}$$

$$b^R_{j-\frac{1}{2}} = \beta_{j-\frac{1}{2}}\phi^R_{j-\frac{1}{2}}\psi(r^R_{j-\frac{1}{2}}) \tag{4.8b}$$

$$\beta_{j-\frac{1}{2}} = \tfrac{1}{2}(1 - |\nu_{j-\frac{1}{2}}|) \tag{4.9}$$

$$r^L_{j-\frac{1}{2}} = \frac{\beta_{j+\frac{1}{2}}\phi^L_{j+\frac{1}{2}}}{\beta_{j-\frac{1}{2}}\phi^L_{j-\frac{1}{2}}} \tag{4.10a}$$

$$r^R_{j-\frac{1}{2}} \quad \frac{\beta_{j-\frac{3}{2}}\phi^R_{j-\frac{3}{2}}}{\beta_{j-\frac{1}{2}}\phi^R_{j-\frac{1}{2}}} \tag{4.10b}$$

and $\psi(r)$ is a flux limiter, (see [7]) e.g. the 'Minmod' limiter

$$\psi(r) = \max\left(\min(1,r),\ 0\right) \tag{4.11}$$

or the 'Superbee' limiter

$$\psi(r) = \max\left(\min(2,r),\ \min(1,2r),\ 0\right) \ . \tag{4.12}$$

This scheme can be represented in schematic form as a first order increment stage together with a second order transfer stage as seen in Fig. 2.

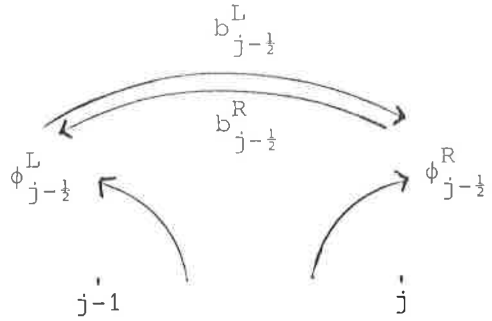


Figure 2

The algorithm given by equations (4.1a)-(4.10b) has the property that one or other of $\phi^L_{j-\frac{1}{2}}$, $\phi^R_{j-\frac{1}{2}}$ is zero, except at expansions.

We can now apply this algorithm to each of the plane waves in the approximate Riemann solver of §3. The schematic representation for updating $\underline{w}$ at time level n to time level $n+1$ using the scheme given by equations (4.1a)-(4.10b) is given in Fig. 3.

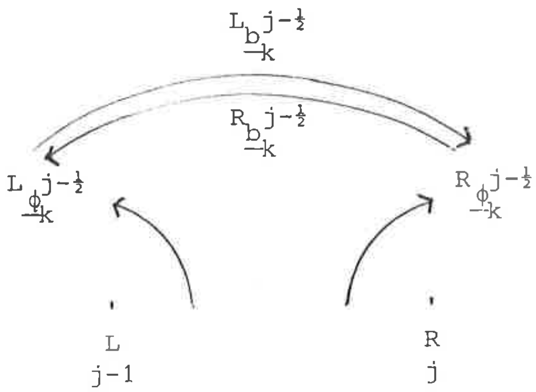


Figure 3

The algorithm consists of an increment stage and a transfer stage, and the quantities given in Fig. 3 are defined as follows:

$$ {}^{L}\underline{\phi}_k^{j-\frac{1}{2}} = {}^{L}\tilde{\sigma}_k^{j-\frac{1}{2}} \; \tilde{\alpha}_k^{j-\frac{1}{2}} \; \underline{\tilde{e}}_k^{j-\frac{1}{2}} \tag{4.13a} $$

$$ {}^{R}\underline{\phi}_k^{j-\frac{1}{2}} = {}^{R}\tilde{\sigma}_k^{j-\frac{1}{2}} \; \tilde{\alpha}_k^{j-\frac{1}{2}} \; \underline{\tilde{e}}_k^{j-\frac{1}{2}} \tag{4.13b} $$

$$ {}^{L}\tilde{\sigma}_k^{j-\frac{1}{2}} = \begin{cases} \left(-\tilde{\nu}_k^{j-\frac{1}{2}}\right)^{+} & {}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} = {}^{L}\tilde{\nu}_k^{j-\frac{1}{2}} \\[2ex] \dfrac{\left[\left(-{}^{R}\tilde{\nu}_k^{j-\frac{1}{2}}\right)^{+}\left(\tilde{\nu}_k^{j-\frac{1}{2}} - {}^{L}\tilde{\nu}_k^{j-\frac{1}{2}}\right) + \left(-{}^{L}\tilde{\nu}_k^{j-\frac{1}{2}}\right)^{+}\left({}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} - \tilde{\nu}_k^{j-\frac{1}{2}}\right)\right]}{\left({}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} - {}^{L}\tilde{\nu}_k^{j-\frac{1}{2}}\right)} & {}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} \neq {}^{L}\tilde{\nu}_k^{j-\frac{1}{2}} \end{cases} \tag{4.13a-b} $$

$$ {}^{L}\sigma_k^{j-\frac{1}{2}} = \begin{cases} -\left(\tilde{\nu}_k^{j-\frac{1}{2}}\right)^{+} & {}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} = {}^{L}\tilde{\nu}_k^{j-\frac{1}{2}} \\[2ex] -\dfrac{\left[\left({}^{R}\tilde{\nu}_k^{j-\frac{1}{2}}\right)^{+}\left(\tilde{\nu}_k^{j-\frac{1}{2}} - {}^{L}\tilde{\nu}_k^{j-\frac{1}{2}}\right) + \left({}^{L}\tilde{\nu}_k^{j-\frac{1}{2}}\right)^{+}\left({}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} - \tilde{\nu}_k^{j-\frac{1}{2}}\right)\right]}{\left({}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} - {}^{L}\tilde{\nu}_k^{j-\frac{1}{2}}\right)} & {}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} \neq \tilde{\nu}_k^{j-\frac{1}{2}} \end{cases} \tag{4.14c-d} $$

$$ \tilde{\nu}_k^{j-\frac{1}{2}} = \frac{\Delta t}{\Delta x} \tilde{\lambda}_k^{j-\frac{1}{2}} \tag{4.15a} $$

$$ \tilde{\nu}_k^{j-\frac{1}{2}} = \min\left(\tilde{\nu}_k^{j-\frac{1}{2}}, \; \tilde{\nu}_k^{j-\frac{3}{2}}\right) \tag{4.15b} $$

$$ {}^{R}\tilde{\nu}_k^{j-\frac{1}{2}} = \max\left(\tilde{\nu}_k^{j-\frac{1}{2}}, \; \tilde{\nu}_k^{j+\frac{1}{2}}\right) \tag{4.15c} $$

$$ c^{+} = \tfrac{1}{2}(|c| + c) \tag{4.16} $$

$$ {}^{L}\underline{b}_k^{j-\frac{1}{2}} = \tfrac{1}{2}\left(1 - \left|\tilde{\nu}_k^{j-\frac{1}{2}}\right|\right) {}^{L}\underline{\phi}_k^{j-\frac{1}{2}} \psi\left({}^{L}r_k^{j-\frac{1}{2}}\right) \tag{4.17a} $$

$$ {}^{R}\underline{b}_k^{j-\frac{1}{2}} = \tfrac{1}{2}\left(1 - \left|\tilde{\nu}_k^{j-\frac{1}{2}}\right|\right) {}^{R}\underline{\phi}_k^{j-\frac{1}{2}} \psi\left({}^{R}r_k^{j-\frac{1}{2}}\right) \tag{4.17b} $$

$$ {}^{L}r_k^{j-\frac{1}{2}} = \frac{\frac{1}{2}\left(1 - |\tilde{\nu}_k^{j+\frac{1}{2}}|\right)\left\{{}^{L}\underline{\phi}_k^{j+\frac{1}{2}}\right\}_q}{\frac{1}{2}\left(1 - |\tilde{\nu}_k^{j-\frac{1}{2}}|\right)\left\{{}^{L}\underline{\phi}_k^{j-\frac{1}{2}}\right\}_q} \tag{4.18a} $$

$$ {}^{R}r_k^{j-\frac{1}{2}} = \frac{\frac{1}{2}\left(1 - |\tilde{\nu}_k^{j-\frac{3}{2}}|\right)\left\{{}^{R}\underline{\phi}_k^{j-\frac{3}{2}}\right\}_q}{\frac{1}{2}\left(1 - |\tilde{\nu}_k^{j-\frac{1}{2}}|\right)\left\{{}^{R}\underline{\phi}_k^{j-\frac{1}{2}}\right\}_q} \tag{4.18b} $$

where Δt is the time increment from level n to $n + 1$, $\Delta x = x_R - x_L = x_j - x_{j-1}$, $\tilde{\lambda}_k^{j-\frac{1}{2}}$, $\tilde{\alpha}_k^{j-\frac{1}{2}}$, $\tilde{\underline{e}}_k^{j-\frac{1}{2}}$ for $k = 1 - 6$ are given in §3, ψ is one of the flux limiters given by equations (4.11), (4.12) and $\{\underline{v}\}_q$ denotes the qth component of the vector $\underline{v}$. (N.B. The superscript $j - \frac{1}{2}$ on $\tilde{\lambda}_k$, $\tilde{\alpha}_n$, $\tilde{\underline{e}}_k$ refers to the interval (x_L, x_R) i.e. (x_{j-1}, x_j) .)

It remains to choose q , i.e. the component of ${}^{L}\underline{\phi}_R^{j-\frac{1}{2}}\left({}^{R}\underline{\phi}_k^{j-\frac{1}{2}}\right)$ used in the limiter. We need to choose the most sensitive component of the conserved vector $\underline{w}$, and this is usually the density, ρ . The use 'Superbee' limiter (4.12) enables contact discontinuities to remain sharp, provided we choose $q = 1$. Unfortunately, this does not guarantee that the interface between fluids, represented by a jump in the variable γ , remains sharp. Following numerical experiments, we advocate using the sixth component $(q = 6)$ of ${}^{L}\underline{\phi}_k^{j-\frac{1}{2}}\left({}^{R}\underline{\phi}_k^{j-\frac{1}{2}}\right)$, corresponding to the variable $\rho\gamma$, to determine the part of the limiter chosen. In this way, we find that contact discontinuities remain sharp both in the density ρ , and in γ when the fluids either side of the discontinuity are different, i.e. at an interface.

In the next section we describe a test problem for the Euler equations for multifluid flows, and derive the exact solution.

5. A TEST PROBLEM

In this section we describe a test problem in multifluid gas dynamics and, using the Rankine-Hugoniot shock relations, we derive the exact solution to this problem.

The one dimensional test problem we look at is that of a shock wave passing from one gas to another. There are two cases to consider (see [10]); however, we shall deal only with the case where there is a transmitted and a reflected shock.

Consider a shock wave travelling in gas 1 with initial conditions given by Fig. 4. After the interaction of the shock at the interface of gas 1 and gas 2 the exact solution consists of a transmitted shock and a reflected shock together with a contact discontinuity whose characteristics are given by Fig. 5.

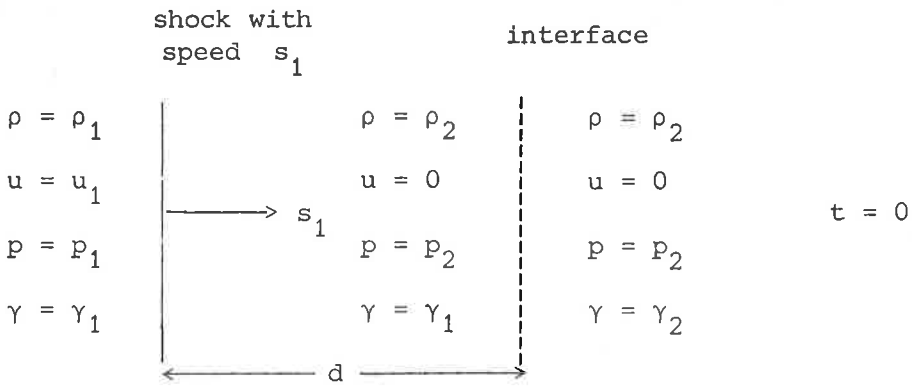


Figure 4

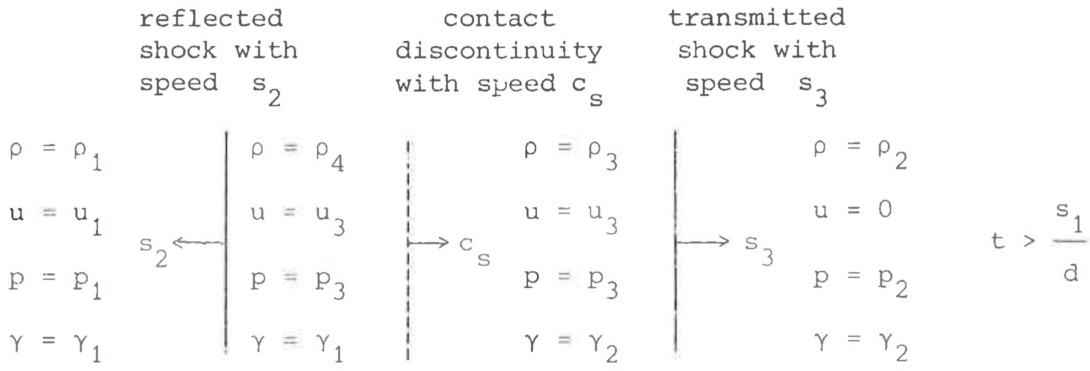


Figure 5

The Rankine-Hugoniot shock relations that hold across a shock wave are again given by

$$s = \frac{[\rho u]}{[\rho]} \tag{5.1a}$$

$$s = \frac{[p + \rho u^2]}{[\rho u]} \tag{5.1b}$$

and

$$s = \frac{[u(e + p)]}{[e]} \tag{5.1c}$$

where s is the speed of the shock, $[v] = v^+ - v^-$ denotes the jump in v across the shock, and v^+, v^- denote the values of v behind and in front of the shock, respectively. From equations (5.1a-b) we can show that

$$(p^+ - p^-)(\rho^+ - \rho^-) = \rho^+\rho^-(u^+ - u^-)^2 \tag{5.2a}$$

and from equations (5.1a), (5.1c)

$$\frac{(\gamma - 1)}{2}(u^+ - u^-)^2(u^+ + u^-)\rho^+\rho^- = \gamma(u^+p^+ - u^-p^-)(\rho^+ - \rho^-) - (p^+ - p^-)(\rho^+u^+ - \rho^-u^-) \tag{5.2b}$$

In particular, if $u^- = 0$ then equations (5.2a-b) yield

$$\frac{\rho^+}{\rho^-} = \frac{\dfrac{p^+}{p^-} + \mu}{\mu\dfrac{p^+}{p^-} + 1} \tag{5.3a}$$

and

$$(u^+)^2 = \frac{(p^+ - p^-)(\rho^+ - \rho^-)}{\rho^+\rho^-}$$

where

$$\mu = \frac{\gamma - 1}{\gamma + 1} . \tag{5.3c}$$

Therefore, for the initial shock travelling from left to right we have

$$\rho_1 = \left(\frac{\frac{p_1}{p_2} + \mu_1}{\mu_1 \frac{p_1}{p_2} + 1} \right) \rho_2 \tag{5.4a}$$

$$u_1 = \sqrt{\frac{(p_1 - p_2)(\rho_1 - \rho_2)}{\rho_1 \rho_2}} \tag{5.4b}$$

and

$$s_1 = \frac{\rho_1 u_1}{\rho_1 - \rho_2} \tag{5.4c}$$

where

$$\mu_1 = \frac{\gamma_1 - 1}{\gamma_1 + 1} \; . \tag{5.4d}$$

Thus, once we have chosen γ_1, ρ_2, p_2 and the shock strength $\frac{p_1}{p_2}$, the initial shock is specified.

To determine the exact solution after interaction we need to find $\rho_3, \rho_4, u_3, p_3, s_2, s_3$ and c_s subject to the shock relations (5.1a-c). From equations (5.3a-c) for the transmitted shock

$$\rho_3 = \rho_2 \left[\frac{\mu_2 + \frac{p_3}{p_2}}{\mu_2 \frac{p_3}{p_2} + 1} \right] \tag{5.5a}$$

and

$$u_3 = \sqrt{\frac{(p_3 - p_2)(\rho_3 - \rho_2)}{\rho_3 \rho_2}} \tag{5.5b}$$

where

$$\mu_2 = \frac{\gamma_2 - 1}{\gamma_2 + 1} \; , \tag{5.5c}$$

and from equation (5.2a) for the reflected shock

$$\rho_4 = \frac{\rho_1(p_3 - p_1)}{\left(p_3 - p_1 - \rho_1(u_3 - u_1)^2\right)} . \tag{5.5d}$$

If we write down equation (5.2b) for the reflected shock and substitute for ρ_4 from equation (5.5d), the following non-linear equation for p_3 is obtained

$$\gamma_1(u_3p_3 - u_1p_1)\rho_1(p_3 - p_1) - \rho_1\gamma_1(u_3p_3 - u_1p_1)\left(p_3 - p_1 - \rho_1(u_3 - u_1)^2\right)$$
$$- (p_3 - p_1)^2\rho_1u_3 + \rho_1u_1(p_3 - p_1)\left(p_3 - p_1 - \rho_1(u_3 - u_1)^2\right)$$
$$- \tfrac{1}{2}(\gamma_1 - 1)(u_3 - u_1)^2(u_3 + u_1)\rho_1^{\,2}(p_3 - p_1) = 0 \tag{5.6}$$

where $u_3 = u_3(p_3)$ is given by equations (5.5a-b). As the reflected shock is usually a weak shock, we solve equation (5.6) by the method of bisection for p_3 close to p_1 where $p_3 > p_1$. The exact solution is now completely specified since ρ_3, u_3, ρ_4 are given by equations (5.5a), (5.5b) and (5.5d), respectively. In addition, the shock and contact speeds s_2, s_3 and c_s are found to be

$$s_2 = \frac{\rho_4u_3 - \rho_1u_1}{\rho_4 - \rho_1} \tag{5.7a}$$

$$s_3 = \frac{\rho_3u_3}{\rho_3 - \rho_2} \tag{5.7b}$$

and

$$c_s = u_3 . \tag{5.7c}$$

In the next section we give the numerical results obtained for the test problem described here.

6. NUMERICAL RESULTS

In this section we show the numerical results obtained for the test problem given in §5 using the Riemann solver given in §3 and the scalar algorithm given in §4.

Each of the figures refers to the one dimensional test problem described in §5. The equations governing the flow are

$$\underline{w}_t + \underline{F}_x = \underline{0} \tag{6.1a}$$

where

$$\underline{w} = (\rho,\ \rho u,\ e,\ \rho\gamma)^T \tag{6.1b}$$

$$F(\underline{w}) = (\rho u,\ p+\rho u^2,\ u(e+p),\ \rho u\gamma)^T \tag{6.1c}$$

and

$$e = \frac{p}{\gamma - 1} + \tfrac{1}{2}\rho u^2 . \tag{6.1d}$$

The particular case we test is the refraction of a shock wave at an air-helium interface, i.e. $\gamma_1 = \frac{7}{5}$ and $\gamma_2 = \frac{5}{3}$. The parameters of the initial shock are $p_1 = 1{\cdot}0$, $\rho_2 = 0{\cdot}2$, with p_2 chosen so that the shock strength $\frac{p_1}{p_2}$ takes the three values ∞, 100 and 10, and ρ_1, u_1, s_1 are given by equations (5.4a-d). In each case we take 100 mesh points in $0 \leq x \leq 1$, and choose the output times to be before and after the shock has been refracted at the interface. All computations have been done using the 'Superbee' limiter (4.12). The shock is initially at $x = 0{\cdot}1$ and the interface at $x = 0{\cdot}4$.

Figures 6, 7, 10, 11, 14 and 15 refer to the time just before the shock is refracted after 55 time steps with $\Delta t = 0{\cdot}002$. Figures 8, 9, 12, 13, 16 and 17 refer to the time after the shock has been refracted after a further 75 time steps with $\Delta t = 0{\cdot}002$. The flow variables ρ, u, p, γ are plotted and Figures 6-9, 10-13 and 14-17 refer to the cases $\frac{p_1}{p_2} = \infty$, 100 and 10, respectively. The solid line represents the exact solution.

We see that in each case the approximate solution gives a good representation of the exact solution, in particular the correct shock speeds have been attained, and the contact discontinuity remains quite sharp.

Before interaction

Pressure ratio $p_1/p_2 = \infty$

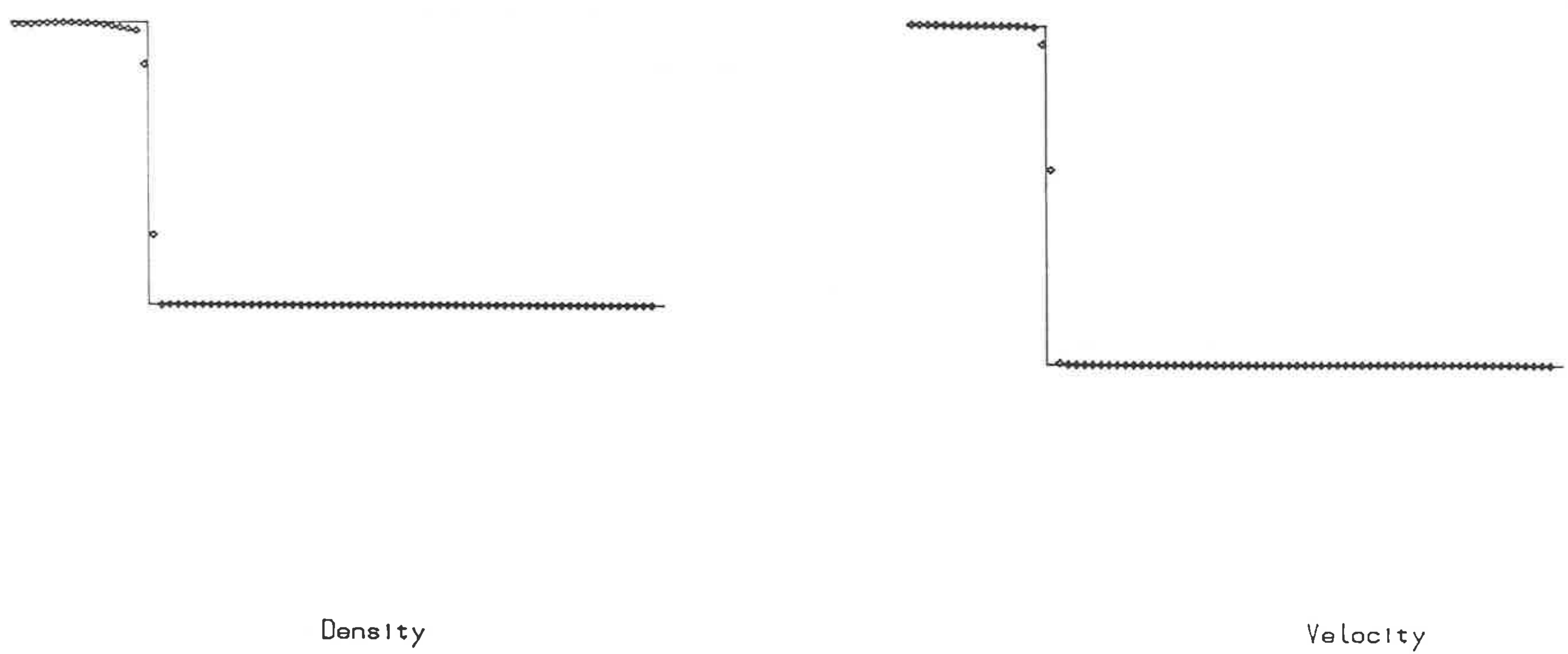


Figure 6

Before interaction

Pressure ratio $p_1/p_2 = \infty$

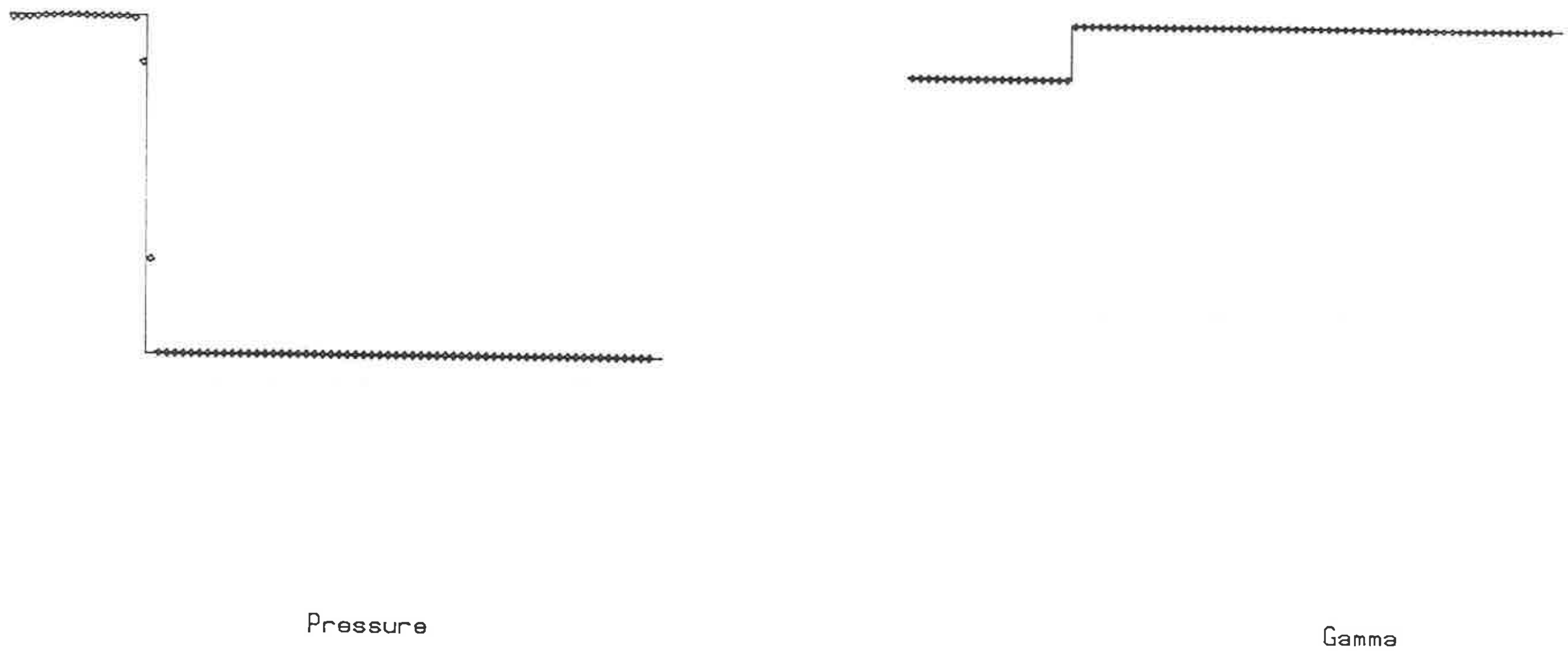


Figure 7

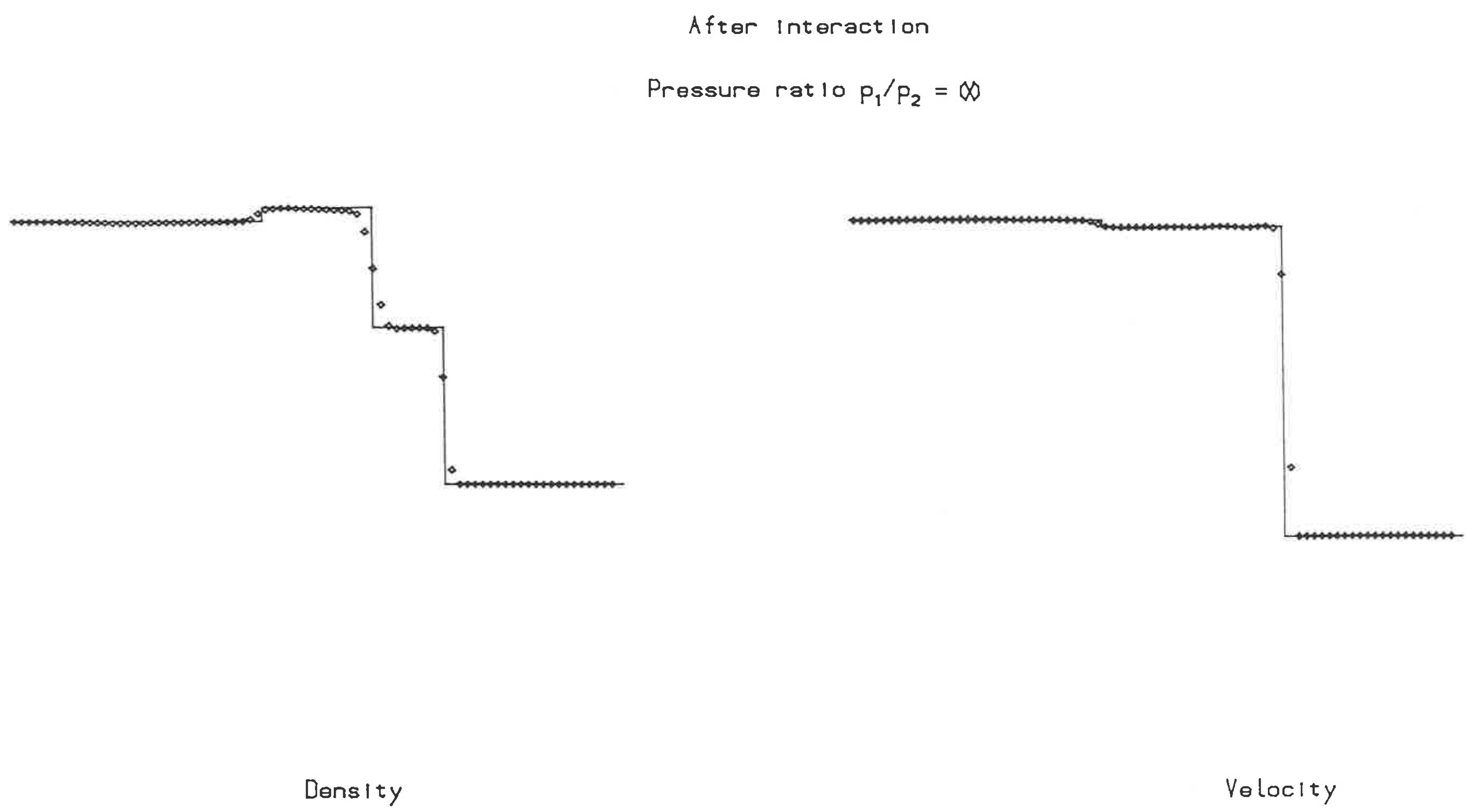

After interaction
Pressure ratio $p_1/p_2 = \infty$
Density
Velocity

Figure 8

After interaction

Pressure ratio $p_1/p_2 = \infty$

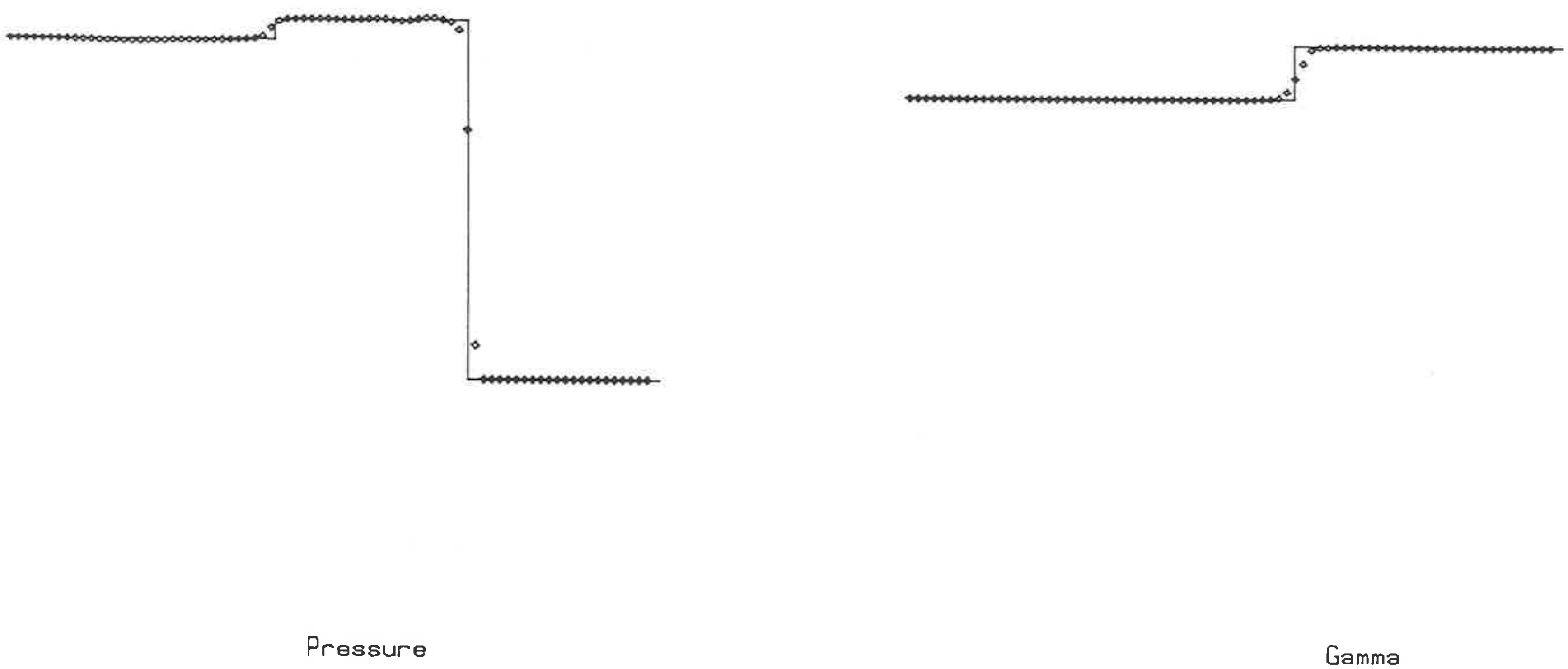


Figure 9

Before Interaction

Pressure ratio $p_1/p_2 = 100$

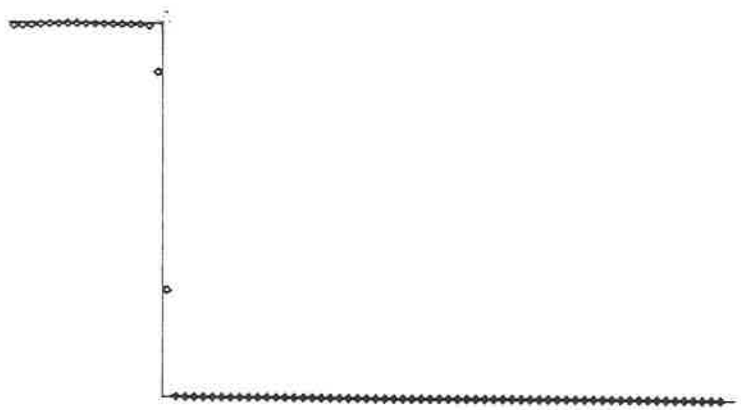
Density

Velocity

Figure 10

Before Interaction

Pressure ratio $p_1/p_2 = 100$



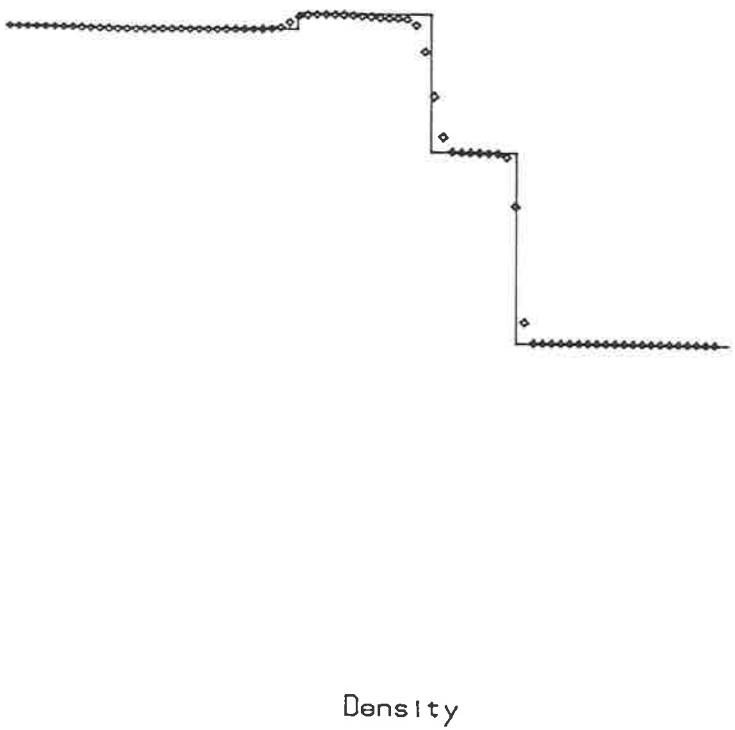
Pressure

Gamma

Figure 11

After Interaction

Pressure ratio $p_1/p_2 = 100$



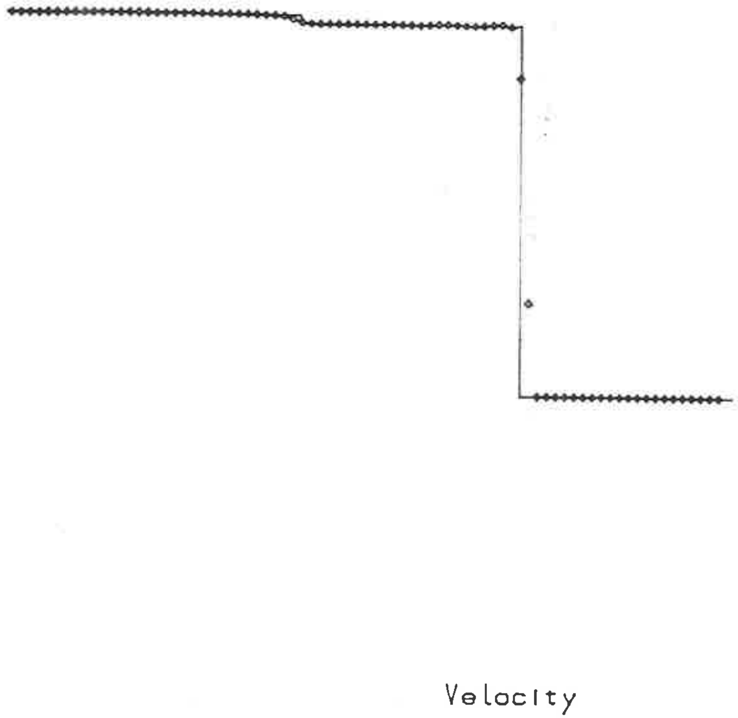


Figure 12

After interaction

Pressure ratio $p_1/p_2 = 100$

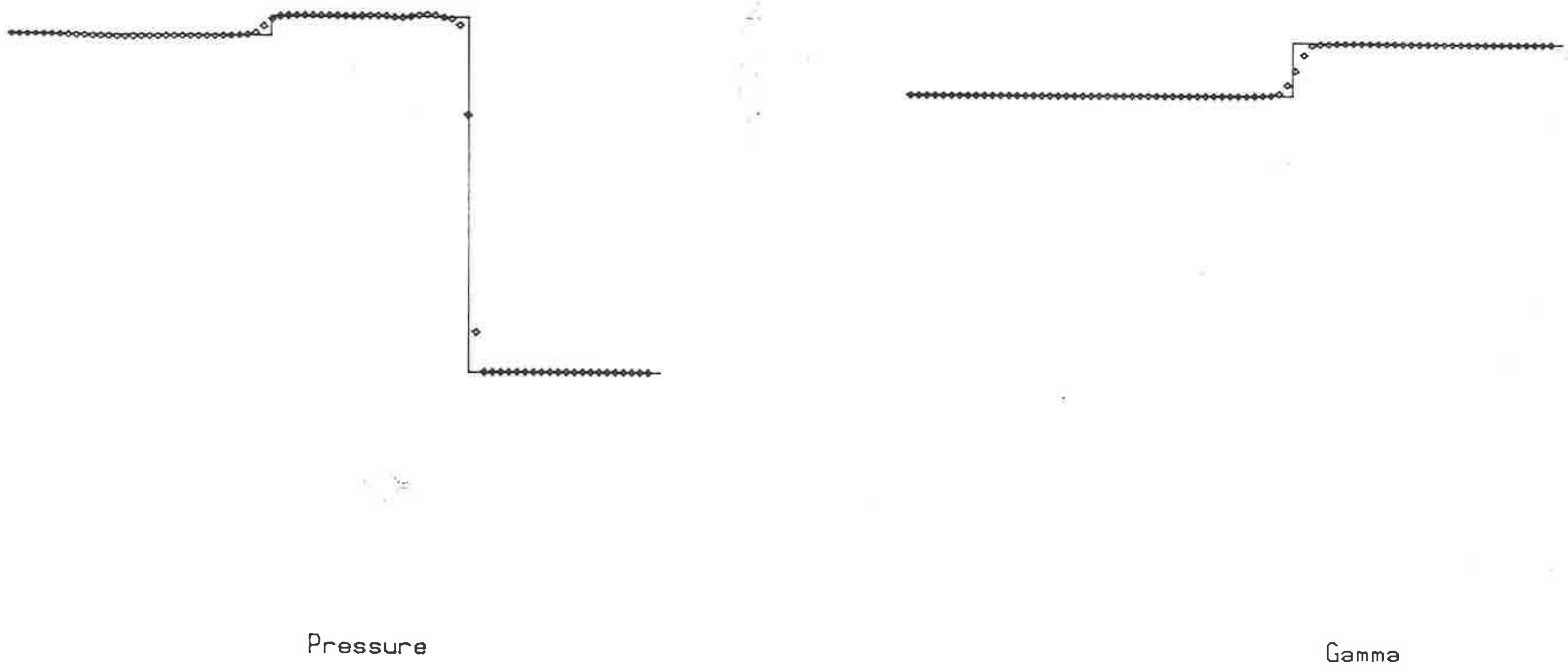


Figure 13

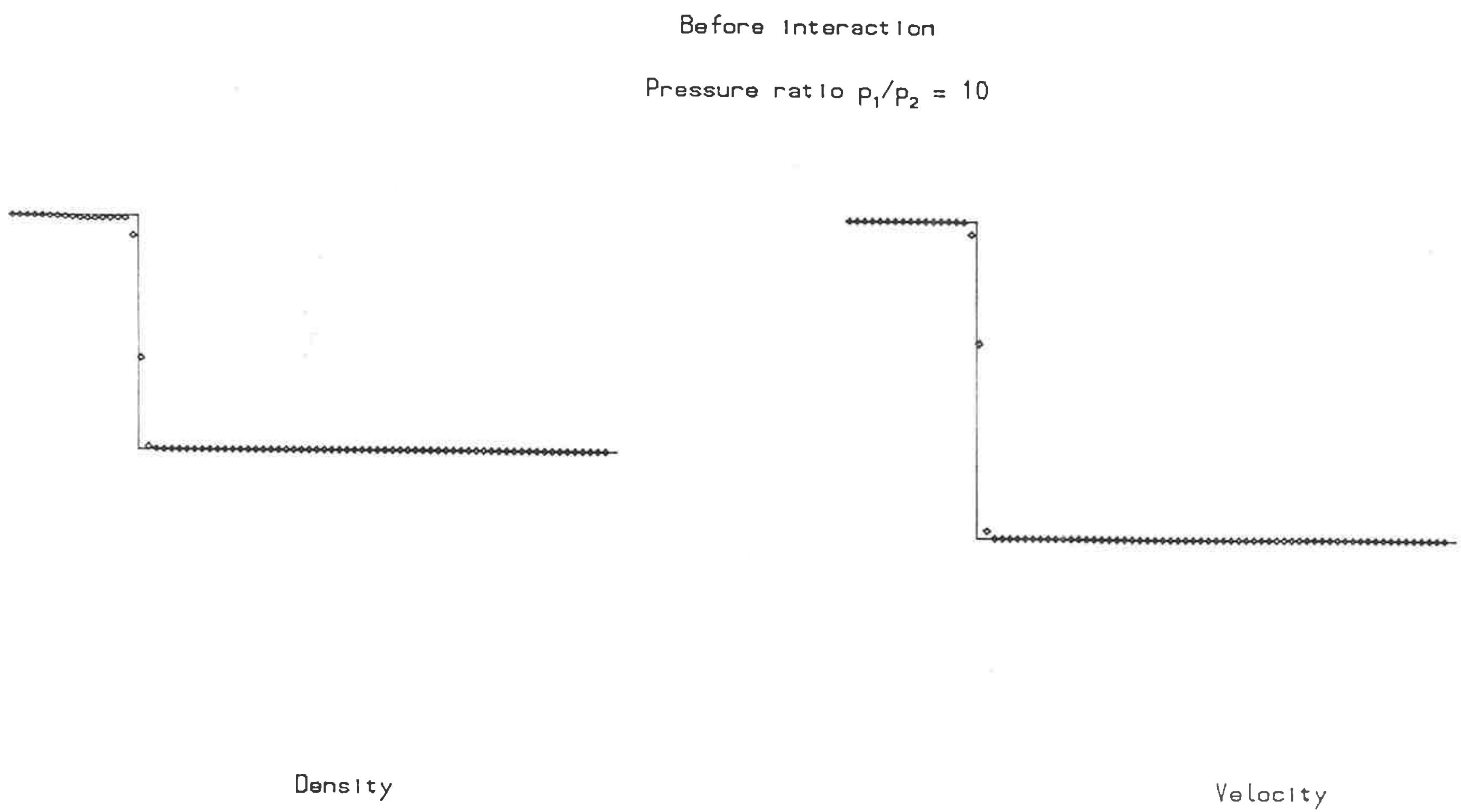

Before interaction
Pressure ratio $p_1/p_2 = 10$
Density
Velocity

Figure 14

Before interaction

Pressure ratio $p_1/p_2 = 10$

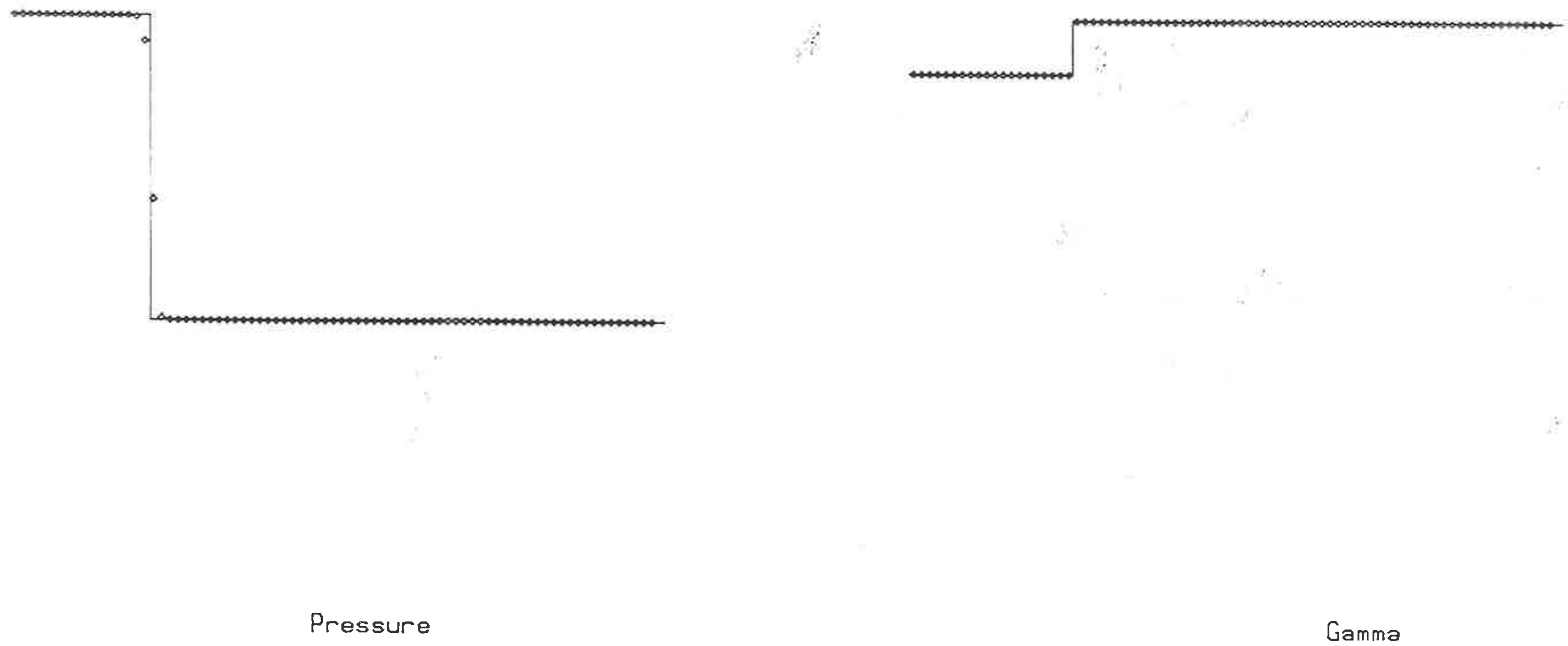


Figure 15

After Interaction

Pressure ratio $p_1/p_2 = 10$

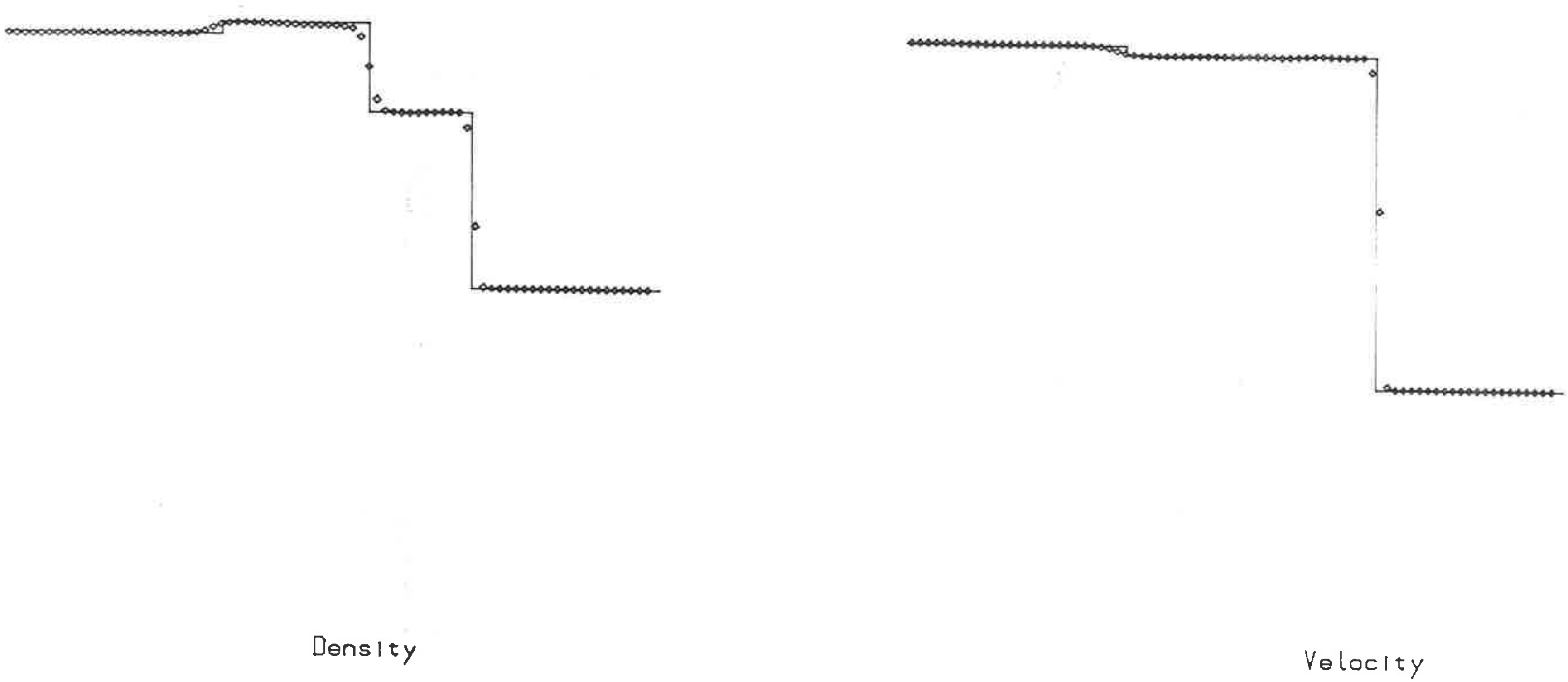


Figure 16

After Interaction

Pressure ratio $p_1/p_2 = 10$

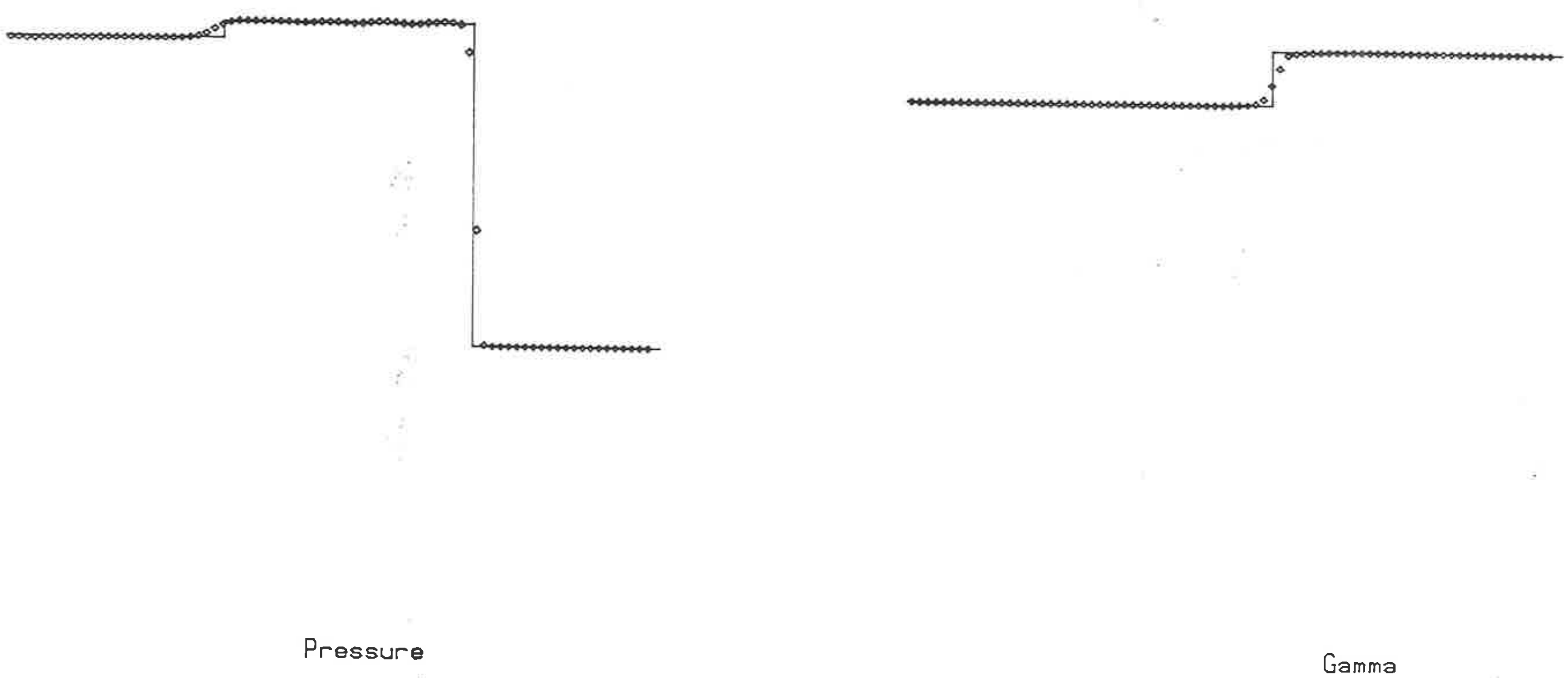


Figure 17

7. CONCLUSIONS

We have extended the one-dimensional results of Roe [5] to give a three-dimensional Riemann solver incorporating the technique of operator splitting. In particular, we have shown that the particular averages sought are unique. In addition, we have achieved satisfactory results for a problem of shock refraction.

ACKNOWLEDGEMENTS

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APPENDIX

Consider a general orthogonal curvilinear co-ordinate system (x_1, x_2, x_3) where a line element $\underline{ds}$ is given by

$$\underline{ds} = h_1 dx_1 \hat{\underline{x}}_1 + h_2 dx_2 \hat{\underline{x}}_2 + h_3 dx_3 \hat{\underline{x}}_3$$

and $\hat{\underline{x}}_1, \hat{\underline{x}}_2, \hat{\underline{x}}_3$ are orthogonal. The vector $\hat{\underline{x}}_i$ is of unit length and parallel to the co-ordinate lines with x_i increasing. Consider also a scalar field $\alpha=\alpha(x_1, x_2, x_3)$ and a vector field $\underline{v} = \underline{v}(x_1, x_2, x_3) = v_1\hat{\underline{x}}_1 + v_2\hat{\underline{x}}_2 + v_3\hat{\underline{x}}_3$. Then the definitions of $\text{div}\,\underline{v}$ and $\underline{\nabla}$ are as follows

$$\text{div}\,\underline{v} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x_1}(h_2h_3v_1) + \frac{\partial}{\partial x_2}(h_1h_3v_2) + \frac{\partial}{\partial x_3}(h_1h_2v_3)\right)$$

and

$$\underline{\nabla} = \frac{1}{h_1}\hat{\underline{x}}_1\frac{\partial}{\partial x_1} + \frac{1}{h_2}\hat{\underline{x}}_2\frac{\partial}{\partial x_2} + \frac{1}{h_3}\hat{\underline{x}}_3\frac{\partial}{\partial x_3}.$$

Thus,

$$\text{div}(\alpha\beta\underline{v}) = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x_1}(h_2h_3\alpha\beta v_1) + \frac{\partial}{\partial x_2}(h_1h_3\alpha\beta v_2) + \frac{\partial}{\partial x_3}(h_1h_2\alpha\beta v_3)\right)$$

$$= \frac{1}{h_1h_2h_3}\left(\alpha\frac{\partial}{\partial x_1}(h_2h_3\beta v_1) + \alpha\frac{\partial}{\partial x_2}(h_1h_3\beta v_2) + \alpha\frac{\partial}{\partial x_3}(h_1h_2\beta v_3)\right.$$

$$\left. + h_2h_3\beta v_1\frac{\partial\alpha}{\partial x_1} + h_1h_3\beta v_2\frac{\partial\alpha}{\partial x_2} + h_1h_2\beta v_3\frac{\partial\alpha}{\partial x_3}\right)$$

$$= \alpha\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x_1}(h_2h_3\beta v_1) + \frac{\partial}{\partial x_2}(h_1h_3\beta v_2) + \frac{\partial}{\partial x_3}(h_1h_2\beta v_3)\right)$$

$$+ \beta\left(v_1\frac{\partial\alpha}{\partial x_1} + v_2\frac{\partial\alpha}{\partial x_2} + v_3\frac{\partial\alpha}{\partial x_3}\right)$$

i.e.

$$\text{div}(\alpha\beta\underline{v}) = \alpha\,\text{div}(\beta\underline{v}) + \beta(\underline{v}.\underline{\nabla})\alpha$$

since $\hat{\underline{x}}_1, \hat{\underline{x}}_2, \hat{\underline{x}}_3$ are orthonormal.