ON THE SOLUTION OF NONLINEAR DIFFUSION EQUATIONS

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Abstract

It is shown that a recent generalisation of the Cole-Hopf transformation has a physical interpretation in terms of the "mass" in the tail of a distribution. Solution procedures involving characteristics are discussed and are related to moving grid methods.

§1 Introduction

It is well-known [1] that the Cole-Hopf transformation

$$u - -2\epsilon \frac{\partial}{\partial x} \log w \qquad (1.1)$$

carries Burgers' equation

$$u_t + uu_x = \epsilon u_{xx} \qquad (\epsilon > 0) \qquad (1.2)$$

into the linear heat equation

$$w_t = \epsilon w_{xx} \qquad (1.3)$$

Conversely, the linear heat equation may be carried into Burgers' equation by the following.   Define

$$\phi = \int \frac{\epsilon}{w} \, dw = \epsilon \log w \qquad (1.4)$$

Then
$$\phi_t = \frac{\epsilon}{w} w_t \ , \qquad \phi_x = \frac{\epsilon}{w} w_x$$

and the linear heat equation (1.3) becomes

$$w\phi_t = \epsilon(w\phi_x)_x = \epsilon w_x \phi_x + \epsilon w \phi_{xx} \qquad (1.5)$$

or
$$\phi_t - \phi_x^2 = \epsilon \phi_{xx} \qquad (1.6)$$

Differentiating (1.6) with respect to x and setting

$$u = -2\phi_x \qquad (1.7)$$

retrieves Burgers' equation (1.2) (see [2]).

Solutions of the linear heat equation generally smooth out with time, but for the nonlinear heat equation

$$w_t = (D(w)w_x)_x \qquad (1.8)$$

or
$$w_t = D'(w)w_x^2 + D(w)w_{xx} \qquad (1.9)$$

where $D(w)$ is a solution-dependent diffusion coefficient, steep fronts may develop. In that case it is expedient to transform to a smoother variable. Following Please & Sweby [3] we define

$$\phi = \int \frac{D(w)}{w} \, dw \qquad (1.10)$$

from which

$$\phi_t = \frac{D(w)}{w} w_t \; , \qquad \phi_x = \frac{D(w)}{w} w_x \qquad (1.11)$$

Then the nonlinear heat equation (1.8) may be written

$$w_t = (w\phi_x)_x = w\phi_{xx} + w_x\phi_x$$

and multiplication of each side by $D(w)/w$ leads to

$$\phi_t - \phi_x^2 = D(w)\phi_{xx} \qquad (1.12)$$

Differentiation with respect to x and use of (1.7) then leads to

$$u_t + uu_x = \frac{\partial}{\partial x}(D(w)\frac{\partial u}{\partial x}) \qquad (1.13)$$

(c.f. (1.2)). In the case $D(w) = \epsilon$ this argument reduces to that for the linear heat equation.

The choice (1.10) is given a physical interpretation in [3]. Suppose that the form of the solution w of (1.8) exhibits a steep front whose foot lies near the x axis (see fig.1).
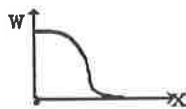

fig. 1

Then from conservation arguments the flux $D(w)w_x$ at a point within the front is balanced by an amount wS swept out by the front, where S is the speed of the front. Then

$$wS = -D(w)w_x . \qquad (1.14)$$

and, if ϕ is the velocity potential of S, then $\phi_x = -S$ and

$$w\phi_x = D(w)w_x \qquad (1.15)$$

which leads to (1.10).

In many cases ϕ is smoother than the original function w
and (1.12) is easier to solve numerically. The argument is good
for any number of dimensions. Note that the "hyperbolic" part

$$\phi_t - \phi_x^2 = 0 \tag{1.16}$$

of (1.12) is independent of the diffusion coefficient D(w) ,
unlike (1.10). Note also that the speeds of the front is one half
of the wavespeed u of (1.2).

§2 Alternative Physical Interpretation of ϕ

If there is no steep front in the solution the physical argument given in §1 does not hold up. However, we note that in , the case of the linear heat equation, the particular solution (Green's function)

$$w = \frac{e^{x^2/4\epsilon t}}{\sqrt{t}} \qquad (2.1)$$

leads to

$$\phi = \frac{-x^2}{4t} - \frac{\epsilon}{2} \log t \qquad (2.2)$$

so that the exponential behaviour of (2.1) is replaced by the smoother quadratic behaviour (in x) of ϕ .

We now seek a physical interpretation of the transformation (1.10) in terms of the speed of the "tail" of the function w , which does not depend on there being a steep front and allows a more general interpretation of ϕ .   Following a suggestiion of R. LeVeque, define the tail of w   to be the region $x \geq X$ such that

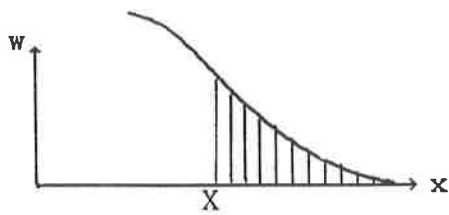$$\int_X^\infty w(x,t)dx = \text{constant}, \qquad (2.3)$$

small or not (see fig. 2).

fig. 2

then differentiation of (2.3) with respect to time yields

$$-w(X,t).\dot{X} + \int_X^\infty w_t dx = 0 \qquad (2.4)$$

and, using (1.8), we find that (assuming that $D(w)w_x \to 0$ as $x \to \infty$)

$$-w(X,t).\dot{X} - (D(w)w_x)_{x=X} = 0 , \qquad (2.5)$$

exactly as in (1.15) with $\dot{X} = S = -\phi_x$. So we have a physical interpretation of the transformation (1.10) for arbitrary $D(w)$ which does not depend on the existence of a steep front.

Note that the variable ϕ is the potential of the velocity of the "tail", i.e. the velocity of that point beyond which there is constant "mass".

A two-dimensional analogue may be constructed as follows: Replace (2.3) by

$$\int_{\Omega(t)} w(\underline{r},t)d\Omega = \text{constant} \qquad (2.6)$$

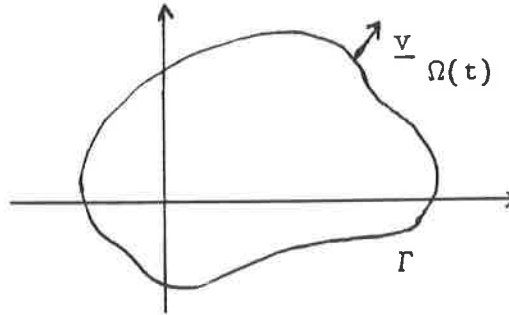where $\Omega(t)$ is the region exterior to a smooth closed curve Γ in the plane (see fig. 3).

fig. 3

Differentiation of (2.6) with respect to t gives

$$\int_{\Gamma} w\underline{V}.d\underline{\Gamma} + \int_{\Omega} w_t d\Omega = 0 \qquad (2.7)$$

where $\underline{V}$ is the velocity of a point on Γ. Then, from the two-dimensional heat equation

$$w_t = \underline{\nabla}(D(w)\underline{\nabla}w) \qquad (2.8)$$

with the assumption that $D(w)\underline{\nabla}w \to 0$ at infinity, we have

$$\int_{\Gamma} w\underline{V}.d\underline{\Gamma} + \int D(w)\underline{\nabla}w.d\underline{\Gamma} = 0 \qquad (2.9)$$

which is satisfied if we take

$$w \underline{\nabla} \phi = D(w) \underline{\nabla} w \qquad\qquad (2.10)$$

where $\underline{V} = -\nabla \phi$. So the interpretation of ϕ is that it is the potential of the velocity of a point of that surface Γ for which (2.6) holds. This velocity is most conveniently thought of as the normal velocity.

§3 Use of Characteristics and Moving Grids

The hyperbolic part (1.16) of (1.12) may be solved exactly by
the method of characteristics (see [4]) or approximately by the
moving finite element method (MFE) (see [5], [6]), which mimics the
characteristic solution (see [7]).

In the former case the solution is given by the ODEs

$$\frac{d\phi}{dt} = -\phi_x^2 \quad \text{on} \quad \frac{dx}{dt} = -2\phi_x \tag{3.1}$$

Alternatively, using $u = -2\phi_x$ we obtain the equation

$$u_t + uu_x = 0 \tag{3.2}$$

with characteristics solution

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u \tag{3.3}$$

from which ϕ is obtained by

$$\phi = -\frac{1}{2} \int^x u\,dx \ . \tag{3.4}$$

Since u is identified with twice the value of S , the velocity
of the front of the constant "mass" tail, the characteristics move
with twice the speed of the tail: ϕ_x is preserved on these
characteristics.

In the approximate case, if the solution ϕ and position x are expanded as piecewise linear functions

$$\phi = \sum_j \phi_j \alpha_j \qquad x = \sum_j x_j \alpha_j \qquad (3.5)$$

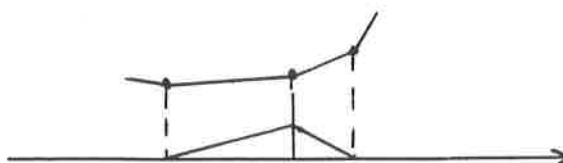with basis functions α_j and moving nodes as in fig. 4,

fig. 4

the MFE solution is given by

$$\frac{d\phi_j}{dt} = -m_L m_R \quad \text{on} \quad \frac{dx_j}{dt} = -(m_L + m_R) \qquad (3.6)$$

where m_L, m_R are the <u>slopes</u> of the approximation each side of node j (fig. 4).

These solutions give underlying convective solutions into which the effect of the diffusion terms must be incorporated. One approach to the solution of (1.8) is, therefore, use of the transformation (1.10), yielding (1.12), together with the solution of (c.f. (3.1))

$$\frac{d\phi}{dt} = D(w)\phi_{xx} - \phi_x^2 \quad \text{on} \quad \frac{dx}{dt} = -2\phi_x \qquad (3.7)$$

or its MFE counterpart, where w is given by (1.10).

Another possibility for an approximate solution is to solve (3.3) numerically and obtain ϕ from (3.4). The MFE method gives correspondingly

$$\dot{u}_j = 0 \quad \text{on} \quad \dot{x}_j = -\frac{1}{2}(\mu_L + \mu_R) \qquad (3.8)$$

where μ_L, μ_R are the slopes of the piecewise linear approximation of u either side of node j . With this speed it is u and hence ϕ (rather than ϕ_x) that is convected along the characteristic, unchanged.

From (1.12), differentiation with respect to x and substitution of $u = -\frac{1}{2}\phi_x$ gives

$$u_t + uu_x = D(w)u_{xx} - \frac{1}{2}w\frac{D'(w)}{D(w)}u^2 \qquad (3.9)$$

requiring the solution (c.f. (3.3)) of

$$\frac{du}{dt} = D(w)u_{xx} - \frac{1}{2}w\frac{D'(w)}{D(w)}u^2 \quad \text{on} \quad \frac{dx}{dt} = u$$

(3.10)

where w is given in terms of u by (1.10) and (1.7). Note the additional source term.

If the MFE method with linear elements is used for u then ϕ is represented by piecewise quadratics, which means that for the linear heat equation case (2.2) will be represented exactly in space (although there will still of course be time discretisation errors).

A halfway house is obtained by sticking to ϕ as the main variable with the speed given by (3.3). Then the solution required is of

$$\frac{d\phi}{dt} = D(w)\phi_{xx} \quad \text{on} \quad \frac{dx}{dt} = \phi_x \qquad (3.11)$$

and the first of these may readily be solved by any convenient implicit method available for the linear heat equation, with w calculated from (1.10).

From a numerical point of view, the first of (3.7) has exactly the same difficulties encountered in the solution of convection diffusion equations as the original form (1.12) with the convective speed reversed, and so gives no advantage (unless the property that ϕ_x is preserved is particularly valuable). Equation (3.10) isolates the diffusion from the convection at the expense of a source term. The form (3.11) appears to be the best compromise, isolating the diffusion but keeping the ϕ equation simple.

To illustrate the points made in this section consider the nonlinear diffusion equation

$$w_t = ((\epsilon + w)w_x)_x \qquad (3.12)$$

in which $D(w) = \epsilon + w$. This particular equation is of interest in semiconductor process modelling [8].

The corresponding "potential" ϕ is, from (1.10),

$$\phi = w + \epsilon \log w \qquad (3.13)$$

leading to

$$\phi_t - \phi_x^2 = (\epsilon + w)\phi_{xx} \qquad (3.14)$$

Equation (3.7) is then

$$\frac{d\phi}{dt} + \phi_x^2 = (\epsilon + w)\phi_{xx} \quad \text{on} \quad \frac{dx}{dt} = -2\phi_x \qquad (3.15)$$

while (3.10) is

$$\frac{du}{dt} = (\phi + w)u_{xx} - \frac{1}{2} \frac{w}{\epsilon + w} u^2 \quad \text{on} \quad \frac{dx}{dt} = -\phi_x \qquad (3.16)$$

The best form is that of (3.11) with

$$\frac{d\phi}{dt} = (\epsilon + w)\phi_{xx} \quad \text{on} \quad \frac{dx}{dt} = \phi_x \qquad (3.17)$$

Recall that w is given in terms of ϕ by inverting (3.13) and in terms of u by solving

$$w + \epsilon \log w = -\frac{1}{2} \int u\,dx \ . \qquad (3.18)$$

§4 Generalised Cole-Hopf

Returning finally to Cole-Hopf one may conjecture that there exists a transformation which carries Burgers' equation with a nonlinear diffusion term into a single nonlinear diffusion equation.   Writing the Burgers' equation as

$$u_t + uu_x = (E(u)u_x)_x \qquad\qquad (4.1)$$

from (1.13) the appropriate transformation is of the form

$$u = -2\frac{\partial}{\partial x}\{\int \frac{D(w)}{w}dw\} \qquad\qquad (4.2)$$

where $\qquad\qquad\qquad D(w) = E(u) \ . \qquad\qquad (4.3)$

Although generating (1.8), since (4.2) and (4.3) are coupled the transformation is implicit.

Acknowledgement

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