MATHEMATICS DEPARTMENT

Black Oil Reservoir Simulation in 1-D via the Higher Order Godunov Method

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Abstract

In this report we review the Black-Oil model for petroleum reservoir simulation. The model includes compressibility and general mass transfer effects. The flow equation are formulated sequentially and split into a parabolic pressure equation and a hyperbolic system. We also review the Higher Order Godunov method which is used to discretise the hyperbolic part of the system. A possible technique of reducing the volume error discrepancy inherent in this sequential formulation is presented and numerical results given.

Contents

1 Introduction

The Black-Oil model is the industry standard phase behaviour model most often used in reservoir simulation and includes compressibility and mass transfer effects between the phases. The fluid flow exhibits both hyperbolic and parabolic behaviour, both of which can be treated by a fully implicit discretisation. However to effectively treat both types of behaviour present in the flow equations different types of numerical procedure are required for each and so a splitting of the flow equations into a system of hyperbolic conservation laws and a parabolic pressure equation is used. Also implicit discretisation of the unsplit equations typically use large amounts of numerical dispersion, providing further reason for use of the sequential strategy.

Sequential methods were first proposed by Sheldon et al in 1959 and by Stone and Garder in 1961. These early and some subsequent formulations contained some anomalies. Further developments were made, then in 1986 Bell, Shubin and Trangenstein [4] presented a new sequential method to treat two-phase, two-component fluid flow. This formulation is based on conservation of mass of the fluid components and does not have the consequences of a 'volume error discrepancy' which is present in later work. This later work by Trangenstein and Bell [2] treats three-phase, three-component fluid flow, and is the formulation we will work with. The consequence of the volume error discrepancy is essentially due to the complicated nature of the volume balance equation for three phase flow with general mass transfer.

Discontinuities are typically present in the solutions to the flow equations. Classical first order methods will produce smeared profiles whereas classical second order methods will reduce the smearing but will introduce oscillations at the discontinuities. Accurate resolution of fronts and control over oscillations at the discontinuities are needed, and so Trangenstein et al [1]-[5] advocate use of the Higher Order Godunov method to treat the hyperbolic part of the split equations. This method is particularly suitable for use with systems possessing eigenvector deficiencies and local linear degeneracies , as does the Black-Oil model . In [2] the hyperbolic part is first discretised via the Higher Order Godunov method which achieves second order accuracy in both space and time. The pressure equation is then solved using an implicit discretisation.

The aims of this work are to reproduce the 1-D results in [2] and to present techniques for reducing the volume-error-discrepancy encountered, both of these objectives are met with a fair degree of success.

In section 2 we give a review of the mathematical structure of the Black-Oil model . Section 3 contains a review of the Higher Order Godunov method including the modifications for systems possessing eigenvector deficiencies and local linear degeneracies . In section 4 we present the test problems from [2] and discuss some of the problems encountered in their solution. Section 5 contains a description of the technique which can be used to reduce the volume-error-discrepancy. In section 4 and 5 we also present graphical results to illustrate the discussions.

2 The Black-Oil Equations

2.1 Overview of model

The Black-Oil model [2] is the Industry standard phase behaviour model most often used in petroleum reservoir simulation. It includes compressibility and general mass transfer effects between the phases that are needed to model primary (pressure depletion) and secondary (water injection) recovery.

The flow equations are derived from four main principles:

i) Phase equilibrium - determines how the components combine to form phases,

ii) Equation of state - states that the fluid fills the pore volume,

iii) Darcy's law for the volumetric flow rates,

iv) Mass conservation equations for each component.

The flow equations are formulated sequentially, therefore they cannot be satisfied exactly at each step of the computation, and so some incompatibility is introduced. We satisfy phase equilibrium, Darcy's law and the component conservation laws exactly but the equation of state is linearised so that it is only satisfied approximately. This splitting of the flow equation is termed a 'volume-discrepancy splitting'. We now consider the equations in detail.

2.2 Thermodynamic equilibrium

2.2.1 Components and Phases

The reservoir is considered to be composed of three components oil, gas, and water. To reach phase equilibrium these components combine to form at most three phases, liquid, vapour and aqua. At each point in the reservoir the components associate into phases in order to attain thermodynamic equilibrium. The components flow in phases but their total mass is conserved, thus it is necessary to find how the mass of each component is apportioned into the phases. This phase equilibrium problem can be expressed as follows: given the pressure p and the vector $\mathbf{n} = [n_o, n_g, n_w]^T$ of mass component densities, find the matrix

$$N = \begin{bmatrix} n_{ol} & n_{ov} & 0 \\ n_{gl} & n_{gv} & n_{ga} \\ 0 & 0 & n_{wa} \end{bmatrix}$$

of component densities in each of the phases, subject to the mass balance condition

$$\mathbf{n} = N\mathbf{e},$$

where $\mathbf{e}$ is a vector of one's. Thus it can be seen from the definition of N that there is a restriction on the way the components are allowed to mix in the phases. i.e. oil may be allowed in the liquid and vapour phases, gas in all three phases and water in the aqueous phase only. Oil and water do not mix and steam is not treated.

2.2.2 Mass transfer and phase equilibrium

With each phase we associate a principal component, oil with the liquid phase, gas with the vapour phase and water with the aqueous phase. The amount of each component in each phase is related to the amount of the principal component in that phase by the ratio matrix R. When all three phases are formed R is given by

$$R = \begin{bmatrix} 1 & R_v & 0 \\ R_l & 1 & R_a \\ 0 & 0 & 1 \end{bmatrix} = ND_N^{-1},$$

where D_N is the diagonal part of N. The ratios R_l, R_v, and R_a must be non-negative functions of pressure and are supplied as part of the model.

Now we can determine how many phases are present. Firstly we define $T = R^{-1}$, then form the vector $T\mathbf{n}$. If all three entries of $T\mathbf{n}$ are positive then all three phases are formed and the fluid is termed saturated. Hence it can be shown that D_N is given by

$$D_N \mathbf{e} = T\mathbf{n},$$

and the solution to phase equilibrium by

$$N = RD_N.$$

2.2.3 Undersaturation

Although we have three components this does not necessarily mean that three phases are always formed. For example, if gas is allowed to dissolve in liquid ($R_l \neq 0$), then for sufficiently high pressures it is possible for all the gas to dissolve in the liquid phase. Hence the vapour phase is missing and the liquid phase is said to be undersaturated. Similarly if ($R_v \neq 0$), i.e. oil is allowed to volatilise into vapour, then it is possible that no liquid is formed and the vapour phase is undersaturated. In the present work neither R_l, R_v or R_a are identically equal to zero therefore both types of undersaturation are possible. Analysis shows that at most one phase can be missing due to undersaturation.

When the vapour phase is missing the physical meaning of the negative vapour component of $T\mathbf{n}$ is that the fluid pressure is higher than the bubble-point pressure p_b at which the vapour phase forms. The bubble point pressure is implicitly defined by the requirement that the vapour component of $T\mathbf{n}$ is zero, i.e.

$$-R_l(p_b)n_o + n_g - R_a(p_b)n_w = 0.$$

When the liquid phase is missing the volatile oil ratio is defined to make the liquid component of $T\mathbf{n}$ zero, i.e.

$$\overline{R}_v = \frac{n_o}{n_g - R_a(p)n_w}.$$

It is possible to derive a combined formulation of undersaturated phase behaviour. This is achieved by multiplying by a constant matrix Q which effectively removes the column corresponding to the missing phase, i.e.

$$\overline{R} = RQ \quad , \quad \overline{D}_N = Q^T D_N Q.$$

For example in the case of the vapour phase being undersaturated, Q is given by

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Throughout, quantities denoted with an overbar refer to the reduced matrix or vector in the undersaturated case. The solution to phase equilibrium is then given by

$$N = \overline{RD}_N Q^T.$$

2.2.4 Compressibility

Water and oil have small compressibilities, but the relatively high compressibility of gas, and the swelling effects caused by gas dissolving in liquid lead to important volume changes at reservoir pressures. The Black-Oil model incorporates these volume changes by relating the volumes of each of the phases to the amounts of the principal component in that phase. To quantify this relationship we first define D_u to be the diagonal matrix of phase volumes per pore volume. The entries of D_N and D_u are related by the formation volume factors,

$$B = \begin{bmatrix} B_l & 0 & 0 \\ 0 & B_v & 0 \\ 0 & 0 & B_a \end{bmatrix} = D_u D_N^{-1},$$

which are functions of pressure and the undersaturation variable ω in the case of undersaturated flow. In the case of undersaturated liquid ω is the bubble point pressure and in the case of vapour undersaturated ω is the volatile oil ratio. The formation volume factors actually relate the volume of the fluid at reservoir conditions to the volume occupied at standard conditions (pressure and temperature), and thus this is how compressibility is introduced. The vector $\mathbf{u}$ of phase volumes per pore volume is given by

$$\mathbf{u} = BT\mathbf{n}.$$

When the fluid is undersaturated we again use the constant matrix Q to define

$$\overline{B} = Q^T BQ,$$

and then $\mathbf{u}$ is given by

$$\overline{\mathbf{u}} = Q\overline{B}T\mathbf{n}.$$

We also wish to include rock compressibility, which means that the porosity is an increasing function of pressure,

$$\phi = \phi_0(1 + c_R(p - p_0)).$$

Where ϕ_0 is the porosity at the reference pressure p_0 and c_R is the rock compressibility.

2.3 Darcy's Law

Darcy's law specifies how the phases flow through the reservoir and gives the vector of flow volumes per unit area per unit time, i.e. phase velocities, which in matrix-vector form is

$$\mathbf{v} = -L(\mathbf{e}\frac{\partial p}{\partial x} - \boldsymbol{\rho}g\frac{\partial d}{\partial x})\kappa. \tag{2.1}$$

i.e. Darcy's law states that the fluid flow is due to a pressure gradient and a gravitational potential. L is the diagonal matrix of phase mobilities which are phase relative permeabilities divided by phase viscosities (k_p/μ_p). Relative permeability describes how the prescence of each phase adversely affects the flow of the other phases and are functions of the saturations $\mathbf{s}$. Saturations are the fractions of the fluid volume occupied by each of the phases and are defined by

$$\mathbf{s} = \frac{\mathbf{u}}{\mathbf{e}^T\mathbf{u}}.$$

It should be noted that $\mathbf{s}$ must be considered distinct from $\mathbf{u}$ because of the volume error discrepancy of this formulation of the flow equations. In the case of no volume error discrepancy i.e. $\mathbf{e}^T\mathbf{u} = 1$ the vectors $\mathbf{s}$ and $\mathbf{u}$ are equivalent. The phase viscosity μ is the property of a phase that represents it's resistance to flow under the influence of a pressure gradient. κ is the total rock permeability, $\boldsymbol{\rho}$ is the vector of phase densities and $\frac{\partial d}{\partial x}$ is the depth gradient of the reservoir.

The total velocity is defined as the sum of phase velocities and obtained from (2.1),

$$v_T = \mathbf{e}^T\mathbf{v} = -\mathbf{e}^T L\mathbf{e}\frac{\partial p}{\partial x}\kappa + \mathbf{e}^T L\boldsymbol{\rho}g\frac{\partial d}{\partial x}\kappa. \tag{2.2}$$

We use (2.2) to solve for the pressure gradient and then substitution in (2.1) gives

$$\mathbf{v} = \frac{L\mathbf{e}v_T}{\mathbf{e}^T L\mathbf{e}} + [I - \frac{L\mathbf{e}\mathbf{e}^T}{\mathbf{e}^T L\mathbf{e}}]L\boldsymbol{\rho}g\frac{\partial d}{\partial x}\kappa.$$

This is the form of the Darcy velocities used in the model and is termed a total velocity splitting. The motivation for this step is that for incompressible flow a total velocity splitting correctly decouples the elliptic and hyperbolic character present in the flow equations.

2.4 Equation of state

When satisfied exactly the equation of state says that the fluid fills the pore volume, i.e.

$$\mathbf{e}^T\mathbf{u} = 1, \tag{2.3}$$

where $\mathbf{e}$ is the vector of one's. Equation (2.3) is linearised about time level t, to develop an equation for pressure,

$$\mathbf{e}^T\mathbf{u}\Big|_{t+\Delta t} \approx \mathbf{e}^T\mathbf{u}\Big|_t + \Delta t\mathbf{e}^T\frac{\partial \mathbf{U}}{\partial p}\frac{\partial p}{\partial t} + \Delta t\mathbf{e}^T\frac{\partial \mathbf{U}}{\partial \mathbf{n}}\frac{\partial \mathbf{n}}{\partial t}. \tag{2.4}$$

It is assumed that we have a volume error discrepancy at time level t i.e. $\mathbf{e}^T\mathbf{u} \neq 1$, and we wish to calculate pressure such that there is no volume error at time level $t + \Delta t$, i.e. $\mathbf{e}^T\mathbf{u} = 1$. Therefore equation (2.4) becomes

$$\frac{1 - \mathbf{e}^T\mathbf{u}}{\Delta t} \approx \mathbf{e}^T\frac{\partial \mathbf{u}}{\partial p}\frac{\partial p}{\partial t} + \mathbf{e}^T\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\frac{\partial \mathbf{n}}{\partial t},$$

and the quantity $\mathbf{e}^T\mathbf{u} - 1$ is termed the volume-error-discrepancy.

Multiplication by the porosity ϕ and use of the mass conservation laws to replace the time derivative of $\mathbf{n}$ results in

$$c\frac{\partial p}{\partial t} + \mathbf{w}^T\frac{\partial}{\partial x}\left(\mathbf{f}v_T + \mathbf{g}\tau\right) = q, \tag{2.5}$$

with the total velocity v_T being given by

$$v_T = [-\frac{\partial p}{\partial x} + \gamma]\tau, \tag{2.6}$$

where the coefficients c, $\mathbf{w}$, $\mathbf{f}$, $\mathbf{g}$, γ, q and τ depend on the pressure p and component density $\mathbf{n}$. Since we use c and τ explicitly we define them here

$$c = \phi\mathbf{e}^T\frac{\partial\mathbf{u}}{\partial p} - \mathbf{e}^T\mathbf{u}\frac{\partial\phi}{\partial p},$$

$$\tau = \mathbf{e}^T Le\kappa.$$

For more details and definitions of the other coefficients see [5].

To examine the character of (2.5) we need to examine the coefficient c of $\frac{\partial p}{\partial t}$ and the coefficient of $\frac{\partial^2 p}{\partial x^2}$ which is obtained after substitution of (2.6) into (2.5) and is given by

$$-\mathbf{w}^T\mathbf{f}\tau = -\mathbf{e}^T Le\kappa,$$

since $\mathbf{w}$ and $\mathbf{f}$ satisfy $\mathbf{w}^T\mathbf{f} = 1$. The transmissibility τ is positive, hence the coefficient of $\frac{\partial^2 p}{\partial x^2}$ is negative, and therefore for (2.5) to be parabolic we require c to be negative. As pressure increases the rock occupies a smaller volume hence porosity ϕ is a non-decreasing function of pressure ,$\frac{\partial\phi}{\partial p} \geq 0$. Therefore for c to be negative we require $\mathbf{e}^T\frac{\partial\mathbf{u}}{\partial p} < 0$ which is the condition of negative total fluid compressibility. This is guaranteed by placing restrictions on the formation volume factors B and ratios R.

Given the sequential formulation described, it is therefore accepted that mass is conserved at the expense of developing a volume error discrepancy .

2.5 Conservation of mass

We require that the mass of each component is conserved. The matrix ND_u^{-1} represents the density of each component in each phase, hence the conservation of mass equations are

$$\frac{\partial(\mathbf{n}\phi)}{\partial t} + \frac{\partial(ND_u^{-1}\mathbf{v})}{\partial x} = 0. \tag{2.7}$$

The flux function $\mathbf{h} = ND_u^{-1}\mathbf{v}$ can also be expressed as $\mathbf{h} = RB^{-1}\mathbf{v}$, which is the form that will be used in the characteristic analysis. The flux is a function of $\mathbf{n}$, p and v_T, hence in quasi-linear form we have

$$\phi\frac{\partial\mathbf{n}}{\partial t} + \frac{\partial\mathbf{h}}{\partial\mathbf{n}}\frac{\partial\mathbf{n}}{\partial x} = -\frac{\partial\mathbf{h}}{\partial p}\frac{\partial p}{\partial x} - \frac{\partial\mathbf{h}}{\partial v_T}\frac{\partial v_T}{\partial x}. \tag{2.8}$$

The system is hyperbolic if and only if

$$H = \frac{\partial\mathbf{h}}{\partial\mathbf{n}}$$

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has real eigenvalues for all values of **n**. The terms on the right hand side of (2.8) are source terms which must be included in the numerical scheme to obtain the desired accuracy. We also need to compute the right eigenvectors of $\frac{\partial \mathbf{h}}{\partial \mathbf{n}}$ to provide information about the structure of the wave fields, which will be used by the numerical method. p and v_T are considered to be independent of **n** for the purpose of the characteristic analysis. This is because the volume balance (2.3) is not enforced in the sequential formulation.

For saturated flow a similarity transformation and eigenvector deflation is used to derive the eigenvectors X, of H as

$$X = RB^{-1} \begin{bmatrix} 1 & -\mathbf{e}^T \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & X_c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mathbf{a} & I \end{bmatrix}$$

where X_c is the matrix of eigenvectors of the matrix

$$C = [\, 0 \quad I \,] \frac{\partial \mathbf{v}}{\partial \mathbf{s}} \begin{bmatrix} -\mathbf{e}^T \\ I \end{bmatrix},$$

and Λ_c is the matrix of eigenvalues, i.e. $CX_c = X_c \Lambda_c$. **a** is given by the vector which solves

$$X_c \Lambda_c \mathbf{a} = \mathbf{c} = -C[\, 0 \quad I \,]\, \mathbf{s}.$$

The corresponding full matrix of eigenvalues is then

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & (\frac{1}{\mathbf{e}^T \mathbf{u}})\Lambda_c \end{bmatrix}.$$

In the case of undersaturated flow a similarity transformation of H is also used to find the eigenvectors and eigenvalues ,which are given by

$$X = [\, \overline{RB}^{-1} \quad \mathbf{q} \,] \begin{bmatrix} X_{\bar{v}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & -\mathbf{a} \\ 0 & 1 \end{bmatrix}.$$

Here **q** is a constant vector dependent on which phase is missing, and the vector **a** solves

$$X_{\bar{v}} \left((\frac{1}{\mathbf{e}^T \mathbf{u}})\Lambda_{\bar{v}}\mathbf{a} - \mathbf{a}\lambda \right) = \overline{BT} H \mathbf{q}.$$

$X_{\bar{v}}$ and $\Lambda_{\bar{v}}$ are given by

$$X_{\bar{v}} = \begin{bmatrix} \mathbf{s}_* & -1 \\ \mathbf{s}_a & 1 \end{bmatrix} \quad, \quad \Lambda_{\bar{v}} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial v_a}{\partial s_a} - \frac{\partial v_a}{\partial s_*} \end{bmatrix},$$

where s_* is the saturation of the other existing phase. The full matrix of eigenvalues is then given by

$$\Lambda = \begin{bmatrix} (\frac{1}{\mathbf{e}^T \mathbf{u}})\Lambda_{\bar{v}} & 0 \\ 0 & \lambda \end{bmatrix},$$

the eigenvalue λ corresponding to a weighted sum of particle velocities and is given in the liquid undersaturated and the vapour undersaturated cases respectively by

$$\frac{R_l n_o \frac{v_l}{u_l} + R_a n_w \frac{v_a}{u_a}}{R_l n_o + R_a n_w} \quad \text{and} \quad \frac{v_v}{u_v}.$$

Further details of this characteristic analysis can be found in [2].

3   The Higher Order Godunov Scheme

To solve the system of flow equations arising from the Black-Oil model we need to use a numerical approximation. To provide accurate resolution of the sharp fronts typically found in oil reservoir simulations, requires a high order method. Hence we use Higher Order Godunov discretisations of the conservation laws. The pressure equation is solved via an implicit discretisation.

The Higher Order Godunov method [1] achieves second order accuracy in both space and time. The scheme's framework includes modifications for both eigenvector deficiencies and local linear degeneracies and hence gives better performance on systems of conservation laws which posses these properties, than do other second order schemes.

We now give a review of the second-order Godunov scheme, which can be considered as a 5-step procedure.

1. Beginning with the piecewise constant approximation $\mathbf{U}_j^n$, compute 'centred', 'left' and 'right' slopes, whilst maintaining conservation.

2. 'Limit' the slopes using monotonicity criteria, again whilst maintaining conservation. This provides a piecewise discontinuous linear approximation to the solution at time t.

3. Trace along characteristics, using a Taylor series extrapolation, to derive left and right states at grid block interfaces at time $t + \frac{1}{2}\Delta t$. It is possible that the traced states are unphysical, if this is the case the physical cell centred value is used instead.

4. Solve the Riemann problem with these left and right states.

5. Use a conservative difference approximation to (2.7) to produce a piecewise constant approximation to $\mathbf{U}_j^{n+1}$.

We now describe the main steps of the method in more detail.

3.1   Monotonised slope computation

We begin with a piecewise constant approximation $\mathbf{U}_j^n$ to the solution in each grid block. We wish to calculate an approximation to the slope $\frac{\partial \mathbf{U}}{\partial x}$ so that we may express the data as a linear profile, which will be piecewise discontinuous. For example in the scalar case a piecewise constant approximation would look something like that in Figure 1. We require a discontinuous piecewise linear approximation which would look something like that in Figure 2.

We define a monotonised centred-difference approximation,

$$\left.\frac{\partial \mathbf{U}}{\partial x}\right|_{x=x_j} \approx \frac{\Delta \mathbf{U}_j}{\Delta x}, \tag{3.1}$$

where in order to calculate $\Delta \mathbf{U}_j$ we compute undivided centered, right and left differences and expand these differences in the right eigenvectors $\mathbf{r}_k$ of the system