Direct Solution of Reservoir Flow Equations with Uncertain Parameters

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Abstract

This paper presents a direct method to determine the uncertainty in reservoir pressure, and other functions, using the time-dependent one phase 2- and 3-dimensional reservoir flow equations. The uncertainty in the solution is modelled as a probability distribution function. This is derived from probability distribution functions for input parameters such as permeability.

The method involves a perturbation expansion about a mean of the parameters. Coupled equations for second order approximations to the mean at each point and field covariance of the solution, are developed and solved numerically. This method involves only one (albeit complicated) solution of the equations, and contrasts with the more usual Monte-Carlo approach, where many such solutions are required.

The procedure is a development of earlier steady-state two-dimensional analyses and a transient mass-balance analysis using uncertain parameters.

These methods can be used to find the risked value of a field for a given development scenario.

1 Introduction

Difficulty in the mathematical and numerical modeling of physical systems, such as evaluation of the flow in underground oil reservoirs, may often arise when a precise knowledge of data is not available. Specifically, data that is crucial for describing the system, may only be known within certain limits of accuracy, or it may only be possible to specify certain statistical properties of

the data. This may be due to inaccuracy in measuring equipment, or inaccessibility, and a high level of heterogeneity, in materials whose parameters are involved in the model equations,

It is the effects of these latter sorts of uncertainty on the solutions of analytic and numerical systems which form the basis of this research project. The usual approach to problems of this type is to use Monte-Carlo methods. However, the number of realisations that need to be generated may, in some cases, be prohibitively large, and, for this reason, we have decided to restrict our study to the development of more direct methods of assessing the uncertainty in the solution.

We restrict our study to a fairly straightforward two-dimensional model equation (with the implicit assumption that most results obtained may be generalised to the three-dimensional case.) This is obtained by combining D'arcy's law for flow in a porous medium, [1], with the equation for single-phase flow in a fluid with a constant compressibility,

$$\gamma \frac{\partial p}{\partial t} - \nabla(k \nabla p) = f(\mathbf{r}, t), \tag{1}$$

where γ is the compressibility, p the pressure, k the permeability, and $f(\mathbf{r}, t)$ is some forcing function.

We are specifically considering the case where the uncertainties are in the permeability, and make the assumption that its statistical behaviour may be characterised by its mean value, $\langle k \rangle$, and the permeability autocorrelation function (P.A.F.) defined as a function of two spatial positions, $\mathbf{r_1}$ and $\mathbf{r_2}$,

$$\rho(\mathbf{r_1}, \mathbf{r_2}) = \frac{\langle (k(\mathbf{r_1}) - k_0(\mathbf{r_1}))(k(\mathbf{r_2}) - k_0(\mathbf{r_2})) \rangle}{\sigma_k(\mathbf{r_1}) \sigma_k(\mathbf{r_2})}, \tag{2}$$

and can be thought of as how strongly the statistical properties at points $\mathbf{r_1}$ and $\mathbf{r_2}$ are related. For practical purposes, the distribution is assumed to be of a lognormal form.

2 Hierarchical Equations

We begin by developing a set of hierarchical equations for a general admissible realisation. By developing these systems of equations as far as possible, before taking mean values on either side, we can obtain equations that allow us to solve for the statistical properties of the numerical solution.

2.1 Standard Form

For a permeability distribution function that is symmetric about the mean value, a simple linear perturbation about the mean can be considered. We

therefore treat the two-dimensional permeability field for a single realisation as a perturbation about some pre-defined mean value field,

$$k = k_0 + \alpha k_1. \tag{3}$$

We assume that $k_0 = \langle k \rangle$ is a deterministic mean, knowledge of which is available.

Equation (1) can then be written,

$$\gamma \frac{\partial p}{\partial t} - \nabla((k_0 + \alpha k_1)\nabla(p)) = f_0(\mathbf{r}, t) + \alpha f_1(\mathbf{r}, t), \tag{4}$$

where p is the pressure solution for the specific realisation under consideration.

As in much work by Dagan, [2], and Dupuy and Schwydler, [3], we assume the pressure solution can be expressed in a series form

$$p = \sum_{m=0}^{N} \alpha^m p_m + R_{N+1}, \tag{5}$$

where R_{N+1} is the residue due to truncating the series for N^{th} order accuracy. Substituting equation (5) into (4), gives

$$\begin{aligned} \gamma \frac{\partial}{\partial t}(& \sum_{m=0}^{N} \alpha^m p_m + R_{N+1} \\ - \quad & \nabla((k_0 + \alpha k_1)\nabla \left(\sum_{m=0}^{N} \alpha^m p_m + R_{N+1} \right) = f_0(\mathbf{r}, t) + \alpha f_1(\mathbf{r}, t). \end{aligned} \tag{6}$$

If we define p_0 to be the solution of the mean value problem, also known as the deterministic problem,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla k_0 \nabla p_0 = f_0, \tag{7}$$

then, by equating successive powers of α, equation (6) can be split up into the $N + 1$ set of hierarchical equations,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla k_0 \nabla p_0 = f_0, \tag{8}$$

$$\gamma \frac{\partial p_1}{\partial t} - \nabla k_0 \nabla p_1 - \nabla k_1 \nabla p_0 = f_1, \tag{9}$$

$$\vdots$$

$$\gamma \frac{\partial p_m}{\partial t} - \nabla k_0 \nabla p_m - \nabla k_1 \nabla p_{m-1} = 0, \tag{10}$$

$$\vdots$$

$$\gamma \frac{\partial p_N}{\partial t} - \nabla k_0 \nabla p_N - \nabla k_1 \nabla p_{N-1} = 0, \tag{11}$$

$$\gamma \frac{\partial R_{N+1}}{\partial t} - \nabla (k_0 + k_1) \nabla \alpha R_{N+1} - \nabla k_1 \nabla p_N = 0. \tag{12}$$

This represents a set of coupled p.d.e.s for each admissible realisation. By truncating this series at the N^{th} term, we have imposed a level of accuracy on the possible solutions. In a statistical sense, we are not able to solve the $N+1^{th}$ equation (12), and so these equations are of N^{th} order accuracy. It may, of course, be possible to obtain bounds on the size of these residue terms over all admissible realisations. This would effectively give a measure of the accuracy of the hierarchical approximation.

2.2 Lognormal Distribution

If a Lognormal distribution function is assumed for the permeability, the expansion must be done about the geometric mean, [4]. This is equivalent to a linear expansion about the log of the permeability.

$$ln(k) = y = y_0 + \beta y_1,$$

where, $y_0 = \langle y \rangle$. So,

$$\begin{aligned} k &= e^{y_0} + \beta y_1 e^{y_0} + \frac{\beta^2 y_1^2}{2} e^{y_0} + \cdots \\ &= \kappa_g + \beta \kappa_1 + \beta^2 \kappa_2 + \cdots = \kappa_g + \sum_{m=1}^{\infty} \beta^m \kappa_m, \end{aligned}$$

where κ_g is the geometric mean.

Performing the same procedure, assuming the pressure has the form,

$$p = \sum_{m=0}^{N} \beta^m p_m + S_{N+1},$$

and substituting for pressure and permeability into equation (1), gives,

$$\gamma \frac{\partial}{\partial t} (\sum_{m=0}^{N} \beta^m p_m + S_{N+1}) - \nabla (\kappa_g + \sum_{m=1}^{\infty} \beta^m \kappa_m) \nabla (\sum_{m=0}^{N} \beta^m p_m + S_{N+1}) = f(\mathbf{r}, t). \tag{13}$$

Again, by equating powers of β we obtain the system of hierarchical equations,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \kappa_g \nabla p_0 = f_0 \tag{14}$$

$$\gamma \frac{\partial p_1}{\partial t} - \nabla \kappa_g \nabla p_1 - \nabla \kappa_1 \nabla p_0 = 0 \tag{15}$$

$$\gamma\frac{\partial p_2}{\partial t} - \nabla\kappa_0\nabla p_2 - \nabla\kappa_1\nabla p_1 - \nabla\kappa_2\nabla p_0 = 0 \tag{16}$$

$$\vdots$$

$$\gamma\frac{\partial p_m}{\partial t} - \nabla\kappa_g\nabla p_m - \sum_{i=1}^{m}\kappa_i p_{m-i} = 0 \tag{17}$$

$$\vdots$$

$$\gamma\frac{\partial p_N}{\partial t} - \nabla\kappa_g\nabla p_N - \sum_{i=1}^{N}\kappa_i p_{N-i} = 0 \tag{18}$$

$$\gamma\frac{\partial S_{N+1}}{\partial t} - \nabla\kappa_g\nabla S_{N+1} - \nabla(\sum_{1}^{\infty}\beta^m\kappa_m)\nabla S_{N+1} - \sum_{i=1}^{N+1}\kappa_i p_{(N+1-i)} = 0 \tag{19}$$

3 Statistical Properties of Analytical Equations

To progress further, we must consider the statistical properties of the solutions to all of the above equations by taking mean values on either side.

3.1 Standard Form

For the purposes of this research, we restrict our consideration to the second order approximations for symmetric, or standard-form, permeability distribution functions.

Taking mean values on either sides of equations (8) to (11), for $N = 2$, and assuming k_1 is a perturbation about the absolute mean, so that $\langle k_1\rangle = 0$, we obtain,

$$\gamma\frac{\partial p_0}{\partial t} - \nabla k_0\nabla p_0 = f_0, \tag{20}$$

$$\gamma\frac{\partial\langle p_1\rangle}{\partial t} - \nabla k_0\nabla\langle p_1\rangle = \langle f_1\rangle, \tag{21}$$

$$\gamma\frac{\partial\langle p_2\rangle}{\partial t} - \nabla k_0\nabla\langle p_2\rangle - \nabla\langle k_1\nabla p_1\rangle = 0, \tag{22}$$

and,

$$\gamma\frac{\partial\langle R_3\rangle}{\partial t} - \nabla(k_0\nabla\langle R_3\rangle) + \nabla\langle\alpha k_1\nabla R_3\rangle - \nabla\langle k_1\nabla p_2\rangle = 0. \tag{23}$$

As they stand, these equations are not solvable, even just up to second order, due to the presence of the cross-correlation term, $\nabla\langle k_1\nabla p_1\rangle$. For this to be possible, a method to evaluate the correlation function, $\langle k_1\nabla p_1\rangle$ is needed.

Consider multiplying k_1 by the grad of equation (9) to give an extra p.d.e. The result of this is to give higher order cross-correlation terms, such as

$\langle k_1 \nabla^2 k_1 \nabla p_0 \rangle$ to evaluate which would involve introducing subsequently higher and higher order cross-correlation terms. This process, of course, is only feasible if a closure can be imposed on the system of equations, under consideration. As they stand, this is not possible.

3.2 Lognormal Distribution

The same procedure on the set of equations for the lognormal permeability distribution function, equations (14) to (16), gives the similar, but adapted equations,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \kappa_g \nabla p_0 = f_0, \tag{24}$$

$$\gamma \frac{\partial \langle p_1 \rangle}{\partial t} - \nabla \kappa_g \nabla \langle p_1 \rangle = \langle f_1 \rangle, \tag{25}$$

$$\gamma \frac{\partial \langle p_2 \rangle}{\partial t} - \nabla \kappa_g \nabla \langle p_2 \rangle - \nabla \langle \kappa_1 \nabla p_1 \rangle - \nabla \langle \kappa_2 \rangle \nabla p_0 = 0. \tag{26}$$

The difference here is the presence of the third term, $\nabla \langle \kappa_2 \rangle \nabla p_0$ in equation (26), but this term just links in the first equation in the series, with an extra moment of the distribution, $\langle \kappa_2 \rangle$ which is a known property of the distribution.

However, the basic problem is the same, that is the presence of $\langle \kappa_1 \nabla p_1 \rangle$, for which, a method for solving simultaneously must be obtained, as for example, in [6], and [4].

3.3 Variance

A second order approximation to the covariance can be obtained in a similar way to [5], by considering $\gamma \frac{\partial p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t)}{\partial t}$, the values of the perturbation, at two distinct points,

$$\gamma \frac{\partial}{\partial t}(p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t)) = p_1(\mathbf{r_1},t) \gamma \frac{\partial p_1(\mathbf{r_2},t)}{\partial t} + p_1(\mathbf{r_2},t) \gamma \frac{\partial p_1(\mathbf{r_1},t)}{\partial t} \tag{27}$$

and substituting for $\gamma \frac{\partial p_1}{\partial t}$, etc. from (9),

$$\begin{aligned} &\gamma \frac{\partial}{\partial t}(p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t)) \\ &\quad - \nabla_2 k_0(\mathbf{r_2}) \nabla_2 p_1(\mathbf{r_1},t) p_1(\mathbf{r_2},t) - \nabla_2 k_1(\mathbf{r_2}) p_1(\mathbf{r_1},t) \nabla_2 p_0(\mathbf{r_2},t) \\ &\quad - \nabla_1 k_0(\mathbf{r_1}) \nabla_1 p_1(\mathbf{r_2},t) p_1(\mathbf{r_1},t) - \nabla_1 k_1(\mathbf{r_1}) p_1(\mathbf{r_2},t) \nabla_1 p_0(\mathbf{r_1},t) = 0. \end{aligned} \tag{28}$$

Taking the mean value on either side of this equation results in an equation for the behaviour of the covariance of the solution,

$$\gamma\frac{\partial}{\partial t}(\langle p_1(\mathbf{r_1},t)p_1(\mathbf{r_2},t)\rangle)$$
$$- \quad \nabla_2 k_0(\mathbf{r_2},t)\nabla_2\langle p_1(\mathbf{r_1},t)p_1(\mathbf{r_2},t)\rangle - \nabla_2\langle k_1(\mathbf{r_2})p_1(\mathbf{r_1},t)\rangle\nabla_2 p_0(\mathbf{r_2},t)$$
$$- \quad \nabla_1 k_0(\mathbf{r_1})\nabla_1\langle p_1(\mathbf{r_2},t)p_1(\mathbf{r_1},t)\rangle - \nabla_1\langle k_1(\mathbf{r_1})p_1(\mathbf{r_2},t)\rangle\nabla_1 p_0(\mathbf{r_1},t) = 0. \tag{29}$$

If the covariance, at time t, between pressure values at two points $\mathbf{r_1}$, and $\mathbf{r_2}$ is denoted by $C(\mathbf{r_1},\mathbf{r_2},t)$, then these equations are,

$$\gamma\frac{\partial}{\partial t}(C(\mathbf{r_1},\mathbf{r_2},t))$$
$$- \quad \nabla_2 k_0(\mathbf{r_2},t)\nabla_2 C(\mathbf{r_1},\mathbf{r_2},t) - \nabla_2\langle k_1(\mathbf{r_2})p_1(\mathbf{r_1},t)\rangle\nabla_2 p_0(\mathbf{r_2},t)$$
$$- \quad \nabla_1 k_0(\mathbf{r_1})\nabla_1 C(\mathbf{r_2},\mathbf{r_1},t) - \nabla_1\langle k_1(\mathbf{r_1})p_1(\mathbf{r_2},t)\rangle\nabla_1 p_0(\mathbf{r_1},t) = 0. \tag{30}$$

Evaluation of the terms in the expression is again rendered impossible, if no method for solving the cross-correlation term is available.

So, we have found that developing a method to solve equations for the lowest moments of the distribution function of the solution to equation (1), in this case, second order accurate approximations to mean and variance, requires some method of solving, or evaluating the cross-correlation terms $\langle k_1\nabla p_1\rangle$, for values of spatial separation and time. Finding a solvable equation for these terms has proved problematic, but it has been found that we are able to obtain closure if we consider the discretised equations.

4 Discretisation

We now show that the problem of providing a solution for $\langle k_1\nabla p_1\rangle$, or $\langle \kappa_1\nabla p_1\rangle$, may be overcome by consideration of the discretised versions of these equations.

4.1 Standard Form

We consider a discretisation of the equations (7) to (9), with a simple explicit time scheme, and a general (unspecified) spatial discretisation,

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h(k_{ij}^0\nabla_h p_{0\ ij}^{n}) = f_{0\ ij}^{n}, \tag{31}$$

$$\frac{\gamma p_{1\ ij}^{n+1} - \gamma p_{1\ ij}^{n}}{\Delta t} - \nabla_h(k_{ij}^0\nabla_h p_{1\ ij}^{n}) - \nabla_h(k_{ij}^1\nabla_h p_{0\ ij}^{n}) = f_{1\ ij}^{n}, \tag{32}$$

and,

$$\frac{\gamma p_{2\ ij}^{n+1} - \gamma p_{2\ ij}^{n}}{\Delta t} - \nabla_h(k_{ij}^0 \nabla_h p_{2\ ij}^{n}) - \nabla_h(k_{ij}^1 \nabla_h p_{1\ ij}^{n}) = 0, \tag{33}$$

where the (i,j) indices refer to spatial points $(i\Delta x, j\Delta y)$ in cartesian co-ordinates, and $p_{z\ ij}^{n}$ refers to the numerical solution for $p_z(\mathbf{r}, n\Delta t)$, where $\mathbf{r}$ is also in Cartesian co-ordinates.

Now let us denote a general value of the perturbation k_1 at a discrete point $(i\Delta x, j\Delta y)$ by $k_{i'j'}^1$, and consider the value at a second reference point, (i', j'). Multiplying this into equation (32), and taking the mean values throughout the resultant, together with equations (31) and (33)), gives,

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h(k_{ij}^0 \nabla_h p_{0\ ij}^{n}) = f_{0\ ij}^{n}, \tag{34}$$

$$\frac{\gamma\langle k_{i'j'}^1 p_{1\ ij}^{n+1}\rangle - \gamma\langle k_{i'j'}^1 p_{1\ ij}^{n}\rangle}{\Delta t} - \langle k_{i'j'}^1 \nabla_h(k_{ij}^0 \nabla_h p_{1\ ij}^{n})\rangle - \langle k_{i'j'}^1 \nabla_h(k_{ij}^1 \nabla_h p_{0\ ij}^{n})\rangle = \langle k_{i'j'}^1 f_{1\ ij}^{n}\rangle, \tag{35}$$

$$\frac{\gamma\langle p_{2\ ij}^{n+1}\rangle - \gamma\langle p_{2\ ij}^{n}\rangle}{\Delta t} - \nabla_h(k_{ij}^0 \nabla_h\langle p_{2\ ij}^{n}\rangle) - \langle\nabla_h(k_{ij}^1 \nabla_h p_{1\ ij}^{n})\rangle = 0. \tag{36}$$

This is now a complete set of coupled (numerical) p.d.e.s that can be solved. When these equations are being solved, simultaneously, the cross-correlation function is found, from equation (35), and then substituted into equation (36). In this form, it is a function of two (discretised) spatial points. The discretised autocorrelation function of the permeability field occurs in the $\langle k_{i'j'}^1 \nabla_h(k_{ij}^1 \nabla_h p_{0\ ij}^{n})\rangle$ terms. These are basically just linear combinations of the autocorrelation parameters, with coefficients specifically dependent on the particular spatially-discretised scheme under consideration. The boundary conditions have been incorporated into the right hand side terms of the equations.

4.2 Lognormal Form

Performing the expansion for a lognormal distribution function, about the geometric mean, results in an extra term in the second order equation, as seen in equation (12). In discretised form, the set of coupled numerical equations becomes,

$$\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} - \nabla_h(k_{ij}^0 \nabla_h p_{0\ ij}^{n}) = f_{0\ ij}^{n}, \tag{37}$$

$$\frac{\gamma\langle k_{i'j'}^1 p_{1\ ij}^{n+1}\rangle - \gamma\langle k_{i'j'}^1 p_{1\ ij}^{n}\rangle}{\Delta t} - \langle k_{i'j'}^1 \nabla_h(k_{ij}^0 \nabla_h p_{1\ ij}^{n})\rangle - \langle k_{i'j'}^1 \nabla_h(k_{ij}^1 \nabla_h p_{0\ ij}^{n})\rangle = \langle k_{i'j'}^1 f_{1\ ij}^{n}\rangle, \tag{38}$$

$$\frac{\gamma\langle p_{2\ ij}^{n+1}\rangle - \gamma\langle p_{2\ ij}^{n}\rangle}{\Delta t} - \nabla_h(k_{ij}^0\nabla_h\langle p_{2\ ij}^{n}\rangle) - \langle\nabla_h(k_{ij}^1\nabla_h p_{1\ ij}^{n})\rangle - \nabla_h\langle k_{ij}^2\rangle\nabla_h p_{0\ ij}^{n} = 0. \tag{39}$$

4.3 Variance Equations

The same discretisation performed on the covariance equations (30) (which have the same form in the linear case and lognormal distributions) results in the following equations,

$$\begin{aligned}
&\frac{\gamma C_{i'j'ij}^{n+1} - \gamma C_{i'j'ij}^{n}}{\Delta t} \\
&- \quad \nabla_h k_{ij}^0\nabla_h C_{i'j'ij}^{n} - \nabla_h\langle k^1 p_1\rangle_{i'j'ij}^{n}\nabla_h p_{0\ ij}^{n} \\
&- \quad \nabla_h k_{i'j'}^0\nabla_h C_{iji'j'}^{n} - \nabla_h\langle k^1 p_1\rangle_{iji'j'}^{n}\nabla_h p_{0\ i'j'}^{n} = 0.
\end{aligned} \tag{40}$$

The quantity of particular interest is the variance of the pressure distribution, an important characterisation of the complete distribution function. In discretised form, the variance for time level $n\Delta t$, at spatial position $(i\Delta x, j\Delta y)$ is the value of C_{ijij}^n. Unfortunately, in the process of solving for this value, the correlation values for distinct points, $C_{i'j'ij}^n$ must also be solved and stored for each time-level. These can be considered as a bonus to the required information, having an academic, rather than practical point of interest. Although, an idea for the correlation length of the solution variable is now clearly available through this technique.

4.4 Summary

The result of the manipulation of the hierarchical equations (8) to (12) and (14) to (16), gives us a set of coupled numerical p.d.e.s for the first two moments that characterise the probability distribution function of the pressure solution. These are equations (34), (35), (36), and (40) for the standard form, and (37) to (40) for the lognormal form. They can be solved at each successive time-level to follow their progression in time. This results in an approximate idea of the time development of the distribution function.

5 Application

We now apply this technique to a specific example of a discretisation.

Consider a simple five-point difference scheme, where the value of the permeability at points halfway between adjacent gridpoints (i, j) and $(i \pm 1, j)$ or $(i, j \pm 1)$ is always approximated by an average of the two values at the

grid-points. Equation (31) in this case becomes,

$$\begin{aligned}
&\frac{\gamma p_{0\ ij}^{n+1} - \gamma p_{0\ ij}^{n}}{\Delta t} \\
&+ \frac{(k_{i+1j}^0 + k_{ij}^0)}{2\Delta x^2} p_{0\ i+1j} + \frac{(k_{i-1j}^0 + k_{ij}^0)}{2\Delta x^2} p_{0\ i-1j} \\
&+ \frac{(k_{ij+1}^0 + k_{ij}^0)}{2\Delta y^2} p_{0\ ij+1} + \frac{(k_{ij-1}^0 + k_{ij}^0)}{2\Delta y^2} p_{0\ ij-1} \\
&- \left\{ \frac{(k_{i+1j}^0 + k_{i-1j}^0 + 2k_{ij}^0)}{2\Delta x^2} + \frac{(k_{ij+1}^0 + k_{ij-1}^0 + 2k_{ij}^0)}{2\Delta y^2} \right\} p_{0\ ij} = f_{0\ ij}. \quad (41)
\end{aligned}$$

The stability condition for this deterministic scheme is

$$\frac{4\Delta t k}{\gamma h^2} < 1. \quad (42)$$

6 Results

In this section we present some illustrative samples of the type of results that we have obtained using this method to solve the full statistical problem.

In each case we consider a single Fourier mode as the initial condition, with no flow conditions around the boundary, and zero forcing function. The region under investigation is square with unit length. All lengths and times are normalised for the purposes of this research.

Using a single Fourier mode as the initial condition means that in the case of a homogeneous mean value for the permeability, the solution to the p.d.e. under consideration, equation (1), may be expressed as the Fourier mode with an exponentially decaying amplitude,

$$p(x, y, t) = e^{-\pi^2 \frac{\langle k \rangle}{\gamma} t} cos(\pi x). \quad (43)$$

It is fairly trivial to show by substitution that this is a solution to the model equation, satisfying the zero boundary conditions. We choose this test function as it is a straightforward solution whose deterministic behaviour is well-known.

The experiments performed have included using different values of $\langle k \rangle$, with both constant, and spatially-varying function forms. We also tried different sizes of variance, σ_k^2, and different correlation lengths, λ_x and λ_y, for the P. A. F. Both the isotropic case, where $\lambda_x = \lambda_y$, and the anisotropic case, $\lambda_x \neq \lambda_y$ were considered.

Typical results of evolution can be seen in Figures 6.1 to 6.3. In this case, we have the case where the homogeneous mean value is 0.2, and the variance $\sigma^2 = 0.05$. Correlation lengths in both the x- and y- directions are the same, equal to 1.0, the size of the region under investigation. In Figure 6.1, we show

the initial condition for the deterministic solution, a one-dimensional Fourier mode, given by equation (43), at $t = 0$,

$$p(x, y) = cos(\pi x). \tag{44}$$

The numerical amplitude at time $t = 1.0$ is 0.140 compared to the analytic value of $e^{-\pi^2 \times 0.2} = 0.139$.

In Figures 6.2(a), 6.2(b), 6.2(c), and 6.2(d) we show three dimensional plots of the variances throughout the region. The initial value of the variance is taken to be zero throughout the region (equivalent to a deterministic initial condition), and the figures show how the variance function changes over time intervals of 0.3, starting at $t = 0.1$, and then at 0.4, 0.7, and 1.0 for respective figures 8.5.1(b) to 8.5.1(e). Numerically, the covariances have maximum values 5.637×10^{-5} at $t = 0.1$, 2.515×10^{-4} at $t = 0.4$, 2.381×10^{-4} at $t = 0.7$, and 1.629×10^{-4} at $t = 1.0$.

Figures 6.3(a), 6.3(b), 6.3(c), and 6.3(d) show the second order correction to the mean value with respect to the deterministic solution, $\langle p_2 \rangle$, at time intervals $t = 0.1$, 0.4, 0.7, and 1.0 respectively. The values have the following maxima, 4.640×10^{-4} for 6.3(a), 2.253×10^{-3} 6.3(b), 4.010×10^{-3} 6.3(c), and 5.434×10^{-3} for figure 6.3(d).

Figure 6.1

Figure 6.2(a)

Figure 6.2(b)

Figure 6.2(c)

Figure 6.2(d)

Figure 6.3(a)

Figure 6.3(b)

Figure 6.3(c)

Figure 6.3(d)

The deterministic solution, shown only at one time value, in Figure 6.1, behaves as expected, decaying exponentially, whilst retaining the basic shape of the (one-dimensional) mode. The basic shape of the three-dimensional plot of the variance, in Figures 6.2(a) to 6.2(d), remains the same throughout the time region under investigation, with maxima at the two edges of the region

given by $x = 0.0$, and $x = 1.0$. The maximum variance was seen to reach a maximum at around $t = 0.5$, thereafter gradually decreasing, with the maximum variance concentrating in the corners whilst it decays. The second order correction to the mean, in Figures 6.3(a) to 6.3(d), begins by taking a similar shape to the deterministic solution, on a much smaller scale, of course. This value is much more subject to instabilities than the variance and deterministic approximations, and we see large increases for large time values.

Compared to experiments done assuming a higher mean value, we naturally see a correspondingly slower decay rate, for example, when $\langle k \rangle = 0.1$, the numerical decay rate is halved. The general shape assumed by the variance and second order approximations after one time unit are the same. The numerical value of the variance is, however, higher due to a greater relative spread in admissible realisations. There is a lower numerical value for p_2 after the time interval. This may be due to the fact that p_2 is related to the decay of the Fourier mode.

When we do experiments with a larger variance, $\sigma^2 = 0.1$, compared to Figures 6.2(a) to 6.3(d), as expected, we see both the variance and the correction term with larger numerical values, 6.517×10^{-4} as a maximum for the variance, and 2.173×10^{-2} for the correction term, after a time interval of one unit. The general shape assumed, though, is similar.

Some further experiments were performed assuming anisotropic correlation lengths. In the case of strong correlation in the y-direction, and much less correlation in the x-direction, we saw that the statistical properties throughout the region are more homogeneous in themselves than in the case where the strong correlation is in the x-direction, and there are much higher variance figures concentrated in the corners. This seems to be partly due to the numerical process in solution of the stochastic p.d.e. which seems to favour correlated properties in the y-direction. In the case where we considered small isotropic correlation lengths in both directions we saw a similar concentration of variance in the corners, with numerical values of one order of magnitude lower, which is the sort of behaviour we would expect if the statistical properties are weakly correlated.

Other experiments were performed on cases where the mean function value of the permeability was heterogeneous. We have not yet obtained useful results for these types of models yet, due in particular to the rather limiting stability condition of the deterministic scheme we chose.

The examples we discuss in this chapter are basically a selection of illustrative examples of the general type of behaviour we have seen using this method of evaluation. We employed a very simple explicit numerical discretisation scheme, which turned out to be severely limiting on the examples we were able to effectively use. We found that the scheme would generally become unstable in cases where there was a significant probability that admissible realisations would lie outside the general stability range of the scheme. Experiments on

the specific point at which instabilities start to occur have yet to be done, but it has been observed that they can certainly be shown to occur when $\langle k \rangle + 3\sigma$ can be shown to lie outside the stability range for our scheme.

7 Conclusions

The results presented here should only be considered as an introduction to this approach of studying uncertain p.d.e.s in oil reservoir modeling. We have shown some of the early results that this method provides us with, but feel that much more research can be done in this specific area. Some further areas of potential research may include,

- Investigation of the differing effect of other schemes on this method, especially implicit methods;
- A full investigation of the effect of grid-size, in relation to correlation length, and, in particular, how their relative size effects results;
- Improving the efficiency of the method by a severe reduction in the number of cross-correlation terms, such as those in equation (37), that are computed;
- Investigation of the numerical correlation length of the results obtained, using the numerical values of the solution covariance function, that have already been computed in equation (40);
- Investigation of the convergence of the schemes, using results obtained from earlier work;
- Comparison of results with those obtained by Monte-Carlo simulations.

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