DEPARTMENT OF MATHEMATICS

Test Problems with Analytic Solutions for Steady Open Channel Flow

by

Ian MacDonald

Department of Mathematics
University of Reading
P O Box 220
Reading RG6 2AF
UK

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1. Introduction

The testing of numerical methods for computing the steady flow in an open channel has been hindered by the fact that, until now, there have been no readily available non-trivial test problems with known analytic solutions. Here we shall introduce a simple technique for constructing such test problems, including solutions with hydraulic jumps. To carry out the construction, we in effect invert the problem and ask the following question: Given a hypothetical depth profile throughout the channel, what must the slope of the channel be in order for this profile to be a steady solution of the Saint-Venant equations.

2. Smooth Solutions

In smooth regions of flow, steady solutions of the Saint-Venant equations satisfy a differential equation of the form

$$\frac{dy}{dx} = \frac{S_0(x) - \gamma_1(x,y)}{\gamma_2(x,y)}, \tag{2.1}$$

where $y(x)$ is the depth at distance x along the channel. If $z(x)$ is the height of the channel bottom above some horizontal datum, then $S_0(x)$ is the slope of the channel given by

$$S_0 = -\frac{dz}{dx}. \tag{2.2}$$

The particular form of γ_1 and γ_2, for a general channel, is not important; however, note that these functions are independent of z, and in general are well defined for all positive y.

The crux of our method is the following observation. Suppose we have some interval $[x_1, x_2]$ and we choose some shape of channel for this interval; this fixes γ_1 and γ_2. Now let the slope of the channel be given by

$$S_0(x) = \gamma_2(x, \hat{y}(x))\frac{d}{dx}\hat{y}(x) + \gamma_1(x, \hat{y}(x)), \tag{2.3}$$

where $\hat{y}$ is some positive function in $C^1[x_1, x_2]$. By definition an exact solution to equation (2.1) for the stretch of channel in this interval is $y(x) \equiv \hat{y}(x)$.

For simplicity we apply this method only to uniform rectangular channels here, although the method is not restricted to such simple channels. In our examples we use a channel in the interval $[0, L]$ where $L = 100m$. The channel has width $B = 10m$ and discharge $Q = 20m^3s^{-1}$ and we use Manning's friction law, with friction coefficient $n = 0.03$. Under these circumstances we have (see Chow[1])

$$\gamma_1(x,y) = S_f(x,y) = Q|Q|n^2\frac{(2y+B)^{4/3}}{(By)^{10/3}} \tag{2.4}$$

and

$$\gamma_2(x,y) = 1 - \left(\frac{y_c}{y}\right)^3, \tag{2.5}$$

where y_c is the critical depth, given by

$$y_c = \sqrt[3]{\frac{Q^2}{gB^2}}, \tag{2.6}$$

and g is the acceleration due to gravity. For our examples below, $y_c = 0.741617m$.

Example 1 *Subcritical Flow*
Here we take $\hat{y}$ of the form

$$\hat{y}(x) = y_c\left(1 + \alpha \exp\left[-\beta^2\left(\frac{x}{L} - \frac{1}{2}\right)^2\right]\right), \tag{2.7}$$

where $\alpha = \frac{1}{2}$ and $\beta = 2$. Figure 1a shows $\hat{y}$ as well as the corresponding channel slope, calculated from equation (2.3). Figure 1b shows the profile of the channel bottom, $z(x)$, obtained by numerically solving equation (2.2) with boundary condition, $z(L) = 0$. This figure also shows $\hat{y}(x) + z(x)$, the profile of the free surface.

Example 2 *Supercritical Flow*
This example is the same as example 1, except that the flow is rendered supercritical by taking $\alpha = -\frac{1}{4}$. Figure 2a shows $\hat{y}$ and the corresponding channel slope. Figure 2b shows the channel bottom and free surface profiles.

Example 3 *Transcritical Flow*
A critical point is a point along the channel where the channel slope changes from mild to steep, smoothly. Flow can change smoothly from subcritical to supercritical at such a point. At any point where $\hat{y}$ passes from subcritical to supercritical, the slope calculated from equation (2.3) will automatically give a critical point. We can also create solutions which change smoothly from supercritical to subcritical using this method, although these are not very useful. Here we put

$$\hat{y}(x) = y_c\left(1 - \frac{1}{2L}\left(x - \frac{L}{2}\right) + \frac{1}{3L^2}\left(x - \frac{L}{2}\right)^2\right). \tag{2.8}$$

Figure 3a shows $\hat{y}$ and the corresponding channel slope. Figure 3b shows the channel bottom and free surface profiles. Here the critical point is at $x = \frac{L}{2}$.

3. Discontinuous Solutions

We cannot substitute a discontinuous $\hat{y}$ into equation (2.3). However, if we wish to construct a solution containing one hydraulic jump, at $x = x^\star$, say, we can divide the solutions into two C^1 functions $\hat{y}_L$ and $\hat{y}_R$, the solutions on the left and right of the jump. We can use these two functions to calculate the channel slope on both sides of the jump. We must ensure that the hydraulic jump is valid, i.e. that the Specific Force is continuous across the jump (see Chow[1]).

In general, under the above construction, the resulting bed slope will be discontinuous at $x^\star$. However, by choosing the appropriate values for the derivatives of $\hat{y}_L$ and $\hat{y}_R$ at $x^\star$, we can ensure any required amount of smoothness in the channel slope. This assumes that we have enough smoothness from γ_1, γ_2, $\hat{y}_L$ and $\hat{y}_R$ to be able to differentiate equation (2.3) enough times. It is up to personal choice how much smoothness to require, if any. The problems here were created to test certain techniques and theory requiring the channel slope to be C^1 (see MacDonald[2]).

Here, for simplicity, we take $\hat{y}_L$ and $\hat{y}_R$ to be the polynomials

$$\hat{y}_L = y_c \sum_{r=0}^{n_L} \alpha_r (x - x^\star)^r \tag{3.9}$$

and

$$\hat{y}_R = y_c \sum_{r=0}^{n_R} \beta_r (x - x^\star)^r. \tag{3.10}$$

The first thing to note is that α_0 and β_0 are not independent. For a rectangular channel, the hydraulic jump is only valid if α_0, β_0 satisfy

$$\beta_0 = \frac{\alpha_0}{2}\left(\sqrt{1 + \frac{8}{\alpha_0^3}} - 1\right), \tag{3.11}$$

or the equivalent relation

$$\alpha_0 = \frac{\beta_0}{2}\left(\sqrt{1 + \frac{8}{\beta_0^3}} - 1\right). \tag{3.12}$$

For the channel slope to be continuous at $x^\star$, we require the following to hold:

$$\gamma_2(x^\star, \alpha_0 y_c)\alpha_1 y_c + \gamma_1(x^\star, \alpha_0 y_c) = \gamma_2(x^\star, \beta_0 y_c)\beta_1 y_c + \gamma_1(x^\star, \beta_0 y_c). \tag{3.13}$$

If α_0, β_0 are known, then this condition is just a linear relationship between α_1 and β_1. For $\frac{dS_0}{dx}$ to be continuous, differentiating equation (2.3) gives us the condition

$$\begin{aligned}
&\left(\frac{\partial \gamma_2}{\partial x}(x^\star, \alpha_0 y_c) + \alpha_1 y_c \frac{\partial \gamma_2}{\partial y}(x^\star, \alpha_0 y_c)\right)\alpha_1 y_c + \gamma_2(x^\star, \alpha_0 y_c) 2\alpha_2 y_c \\
&+ \frac{\partial \gamma_1}{\partial x}(x^\star, \alpha_0 y_c) + \alpha_1 y_c \frac{\partial \gamma_1}{\partial y}(x^\star, \alpha_0 y_c) = \\
&\left(\frac{\partial \gamma_2}{\partial x}(x^\star, \beta_0 y_c) + \beta_1 y_c \frac{\partial \gamma_2}{\partial y}(x^\star, \beta_0 y_c)\right)\beta_1 y_c + \gamma_2(x^\star, \beta_0 y_c) 2\beta_2 y_c \\
&+ \frac{\partial \gamma_1}{\partial x}(x^\star, \beta_0 y_c) + \beta_1 y_c \frac{\partial \gamma_1}{\partial y}(x^\star, \beta_0 y_c).
\end{aligned} \tag{3.14}$$

If α_0, β_0, α_1 and β_1 are known, then this condition is just a linear relationship between α_2 and β_2.

We now show, with some examples, how easy it is to satisfy the above conditions.

Example 4 In this example we specify $\hat{y}_L$ completely and then choose an appropriate $\hat{y}_R$ to satisfy our three conditions. We take $x^\star = \frac{2L}{3}$ and let

$$\hat{y}_L(x) = y_c\left(\frac{4}{3} - \frac{x}{L}\right) - \frac{9}{10L^2}(x - x^\star)x. \tag{3.15}$$

Equation (3.11) now gives us $\beta_0 = 1.4305$. Using equation (3.13) we get $\beta_1 = 0.14492$ and finally equation (3.14) gives us $\beta_2 = -0.00217112$. These coefficients define a quadratic, but if we just use this quadratic as $\hat{y}_R$, we have no control over what the right side of the solution looks like (it may become negative). To give us control we can arbitrarily add higher order terms; here we make $\hat{y}_R$ a quartic ($n_R = 4$). Now we must make a choice for β_3 and β_4. For this example we take $\beta_3 = 0.674202/L^3$ and $\beta_4 = 0.674202/L^4$. Figure 4a shows the resulting depth profile for the channel, as well as the corresponding channel slope. Figure 5 shows the channel bottom profile and the free surface profile. The resulting flow is subcritical at outflow, jumps to supercritical two thirds the way along the channel, and then returns to subcritical at $x = 45.13m$.

Example 5 In this example we specify $\hat{y}_R$ first and then choose an appropriate $\hat{y}_L$. We take $x^\star = \frac{L}{3}$ and let

$$\hat{y}_R(x) = y_c\left(\frac{5}{6} + \frac{L - x}{2L}\right) + \frac{4}{10L^2}(x - L)(x - x^\star). \tag{3.16}$$

In this case our three conditions give us $\alpha_0 = 0.850042$, $\alpha_1 = 0.031725$ and $\alpha_2 = 0.00179329$. We make $\hat{y}_L$ a quartic, and choose $\alpha_3 = 18.8777/L^3$ and $\alpha_4 = -10.7872/L^4$. Figure 5a shows the depth profile, as well as the channel slope. Figure 5b shows the channel bottom profile and the free surface profile. The flow in this example behaves oppositely to that in example 4. The flow is supercritical at inflow, jumps to subcritical and the returns to supercritical at $x = 55.93m$.

4. Concluding Remarks

Although the examples chosen here are very simple, they should demonstrate to the modeller how to construct test problems with the characteristics they require. With careful choice of the free parameters, it is possible to achieve such goals as keeping the channel slope small or keeping it positive The method is even more useful in the case of non-rectangular and non-prismatic channels. Here the behaviour of the solutions and numerical schemes is less well understood and there are many methods for approximating the channel geometry. We now have a concrete way to evaluate these different methods. The method could also be useful for networks of channels, where a test network can be constructed with a known solution.

Altogether, we now have a method of constructing test problems that can be used to measure and compare, exactly, the performance of different numerical methods. The test problems given here are summarised in the Appendix.

References

[1] **Chow,V.T.** Open Channel Hydraulics, MaGraw-Hill Book Company, 1959.

[2] **MacDonald,I.** Analysis and Computation of Steady Open Channel Flow using a Singular Perturbation Problem. Numerical Analysis Report, University of Reading. To appear.

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Appendix

For convenience we summarise the details of the examples given in the main text. For each problem, we state the appropriate boundary conditions.

Example 1 *Subcritical Flow*
A rectangular channel, $0 \leq x \leq 100m$, has width $10m$ and a discharge of $20m^3s^{-1}$. The slope of the channel is given by

$$S_0(x) = \left(1 - \frac{4}{g[\hat{y}(x)]^3}\right)\hat{y}'(x) + \frac{9}{2500[\hat{y}(x)]^2}\left(\frac{1}{5} + \frac{1}{\hat{y}(x)}\right)^{4/3},$$

where

$$\hat{y}(x) = \left(\frac{4}{g}\right)^{1/3}\left(1 + \frac{1}{2}\exp\left[-4\left(\frac{x}{100} - \frac{1}{2}\right)^2\right]\right),$$

and

$$\hat{y}'(x) = -\left(\frac{4}{g}\right)^{1/3}\frac{1}{25}\left(\frac{x}{100} - \frac{1}{2}\right)\exp\left[-4\left(\frac{x}{100} - \frac{1}{2}\right)^2\right].$$

Manning's friction coefficient for the channel is 0.03. The flow is subcritical at outflow, with depth $\hat{y}(100)$, and subcritical at inflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 1.

Example 2 *Supercritical Flow*
A rectangular channel, $0 \leq x \leq 100m$, has width $10m$ and a discharge of $20m^3s^{-1}$. The slope of the channel is given by

$$S_0(x) = \left(1 - \frac{4}{g[\hat{y}(x)]^3}\right)\hat{y}'(x) + \frac{9}{2500[\hat{y}(x)]^2}\left(\frac{1}{5} + \frac{1}{\hat{y}(x)}\right)^{4/3},$$

where

$$\hat{y}(x) = \left(\frac{4}{g}\right)^{1/3}\left(1 - \frac{1}{4}\exp\left[-4\left(\frac{x}{100} - \frac{1}{2}\right)^2\right]\right),$$

and

$$\hat{y}'(x) = \left(\frac{4}{g}\right)^{1/3} \frac{1}{50}\left(\frac{x}{100} - \frac{1}{2}\right) \exp\left[-4\left(\frac{x}{100} - \frac{1}{2}\right)^2\right].$$

Manning's friction coefficient for the channel is 0.03. The flow is supercritical at inflow, with depth $\hat{y}(0)$ and supercritical at outflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 2.

Example 3 *Transcritical Flow*
A rectangular channel, $0 \leq x \leq 100m$, has width $10m$ and a discharge of $20m^3s^{-1}$. The slope of the channel is given by

$$S_0(x) = \left(1 - \frac{4}{g[\hat{y}(x)]^3}\right)\hat{y}'(x) + \frac{9}{2500[\hat{y}(x)]^2}\left(\frac{1}{5} + \frac{1}{\hat{y}(x)}\right)^{4/3},$$

where

$$\hat{y}(x) = \left(\frac{4}{g}\right)^{1/3}\left(1 - \frac{(x-50)}{200} + \frac{(x-50)^2}{30000}\right),$$

and

$$\hat{y}'(x) = \left(\frac{4}{g}\right)^{1/3}\left(-\frac{1}{200} + \frac{(x-50)}{15000}\right).$$

Manning's friction coefficient for the channel is 0.03. The flow is subcritical at inflow and supercritical at outflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 3.

Example 4 *Sub-Super-Subcritical Flow with Hydraulic Jump*
A rectangular channel, $0 \leq x \leq 100m$, has width $10m$ and a discharge of $20m^3s^{-1}$. The slope of the channel is given by

$$S_0(x) = \left(1 - \frac{4}{g[\hat{y}(x)]^3}\right)\hat{y}'(x) + \frac{9}{2500[\hat{y}(x)]^2}\left(\frac{1}{5} + \frac{1}{\hat{y}(x)}\right)^{4/3},$$

where

$$\hat{y}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\left(\frac{4}{3} - \frac{x}{100}\right) - \frac{9x}{1000}\left(\frac{x}{100} - \frac{2}{3}\right) & x \leq \frac{200}{3} \\ \left(\frac{4}{g}\right)^{1/3}\Big(0.674202\left(\frac{x}{100} - \frac{2}{3}\right)^4 + 0.674202\left(\frac{x}{100} - \frac{2}{3}\right)^3 & x > \frac{200}{3} \\ -21.7112\left(\frac{x}{100} - \frac{2}{3}\right)^2 + 14.492\left(\frac{x}{100} - \frac{2}{3}\right) + 1.4305\Big) & \end{cases},$$

and

$$\hat{y}'(x) = \begin{cases} \frac{-1}{100}\left(\frac{4}{g}\right)^{1/3} - \frac{9}{500}\left(\frac{x}{100} - \frac{1}{3}\right) & x \leq \frac{200}{3} \\ \left(\frac{4}{g}\right)^{1/3}\Big(0.02696808\left(\frac{x}{100} - \frac{2}{3}\right)^3 + 0.02022606\left(\frac{x}{100} - \frac{2}{3}\right)^2 & x > \frac{200}{3} \\ -0.434224\left(\frac{x}{100} - \frac{2}{3}\right) + 0.14492\Big) & \end{cases}.$$

Manning's friction coefficient for the channel is 0.03. The flow is subcritical at outflow, with depth $\hat{y}(100)$, and subcritical at inflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 4.

Example 5 *Super-Sub-Supercritical Flow with Hydraulic Jump*
A rectangular channel, $0 \le x \le 100m$, has width $10m$ and a discharge of $20m^3s^{-1}$. The slope of the channel is given by

$$S_0(x) = \left(1 - \frac{4}{g[\hat{y}(x)]^3}\right)\hat{y}'(x) + \frac{9}{2500[\hat{y}(x)]^2}\left(\frac{1}{5} + \frac{1}{\hat{y}(x)}\right)^{4/3},$$

where

$$\hat{y}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\left(-10.7872\left(\frac{x}{100} - \frac{1}{3}\right)^4 + 18.8777\left(\frac{x}{100} - \frac{1}{3}\right)^3\right. & x \le \frac{100}{3} \\ \left. +17.9329\left(\frac{x}{100} - \frac{1}{3}\right)^2 + 3.1725\left(\frac{x}{100} - \frac{1}{3}\right) + 0.850042\right) & \\ & \\ \left(\frac{4}{g}\right)^{1/3}\left(\frac{5}{6} + \frac{(100-x)}{200}\right) + \frac{4}{10}\left(\frac{x}{100} - \frac{1}{3}\right)\left(\frac{x}{100} - 1\right) & x > \frac{100}{3} \end{cases},$$

and

$$\hat{y}'(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3}\left(-0.431488\left(\frac{x}{100} - \frac{1}{3}\right)^3 + 0.566331\left(\frac{x}{100} - \frac{1}{3}\right)^2\right. & x \le \frac{100}{3} \\ \left. +0.358658\left(\frac{x}{100} - \frac{1}{3}\right) + 0.031725\right) & \\ & \\ \frac{-1}{200}\left(\frac{4}{g}\right)^{1/3} + \frac{4}{500}\left(\frac{x}{100} - \frac{2}{3}\right) & x > \frac{100}{3} \end{cases}.$$

Manning's friction coefficient for the channel is 0.03. The flow is supercritical at inflow, with depth $\hat{y}(0)$ and supercritical at outflow.

The exact solution for this problem is $y(x) \equiv \hat{y}(x)$ and is shown in figure 5.

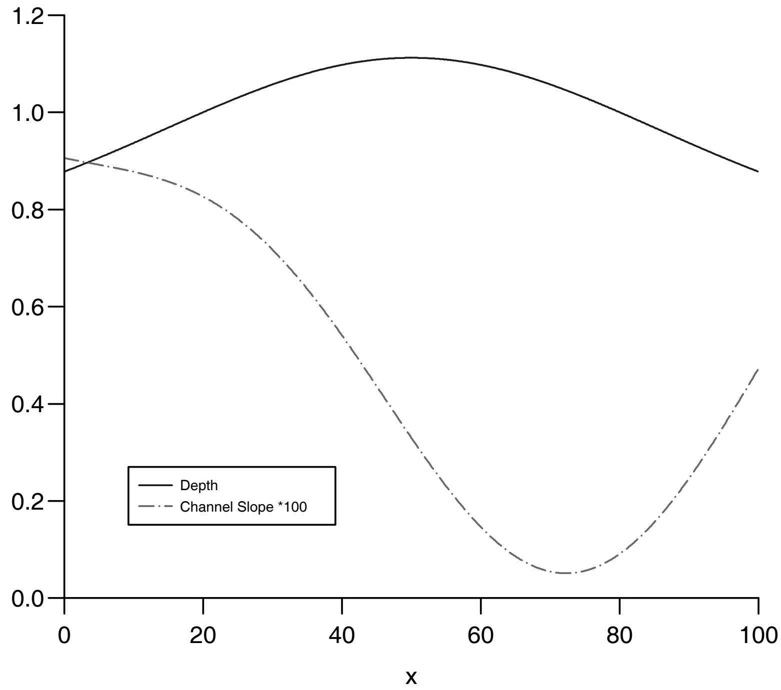

Figure 1a: Depth, $\hat{y}$, and bed slope, S_0, for example 1

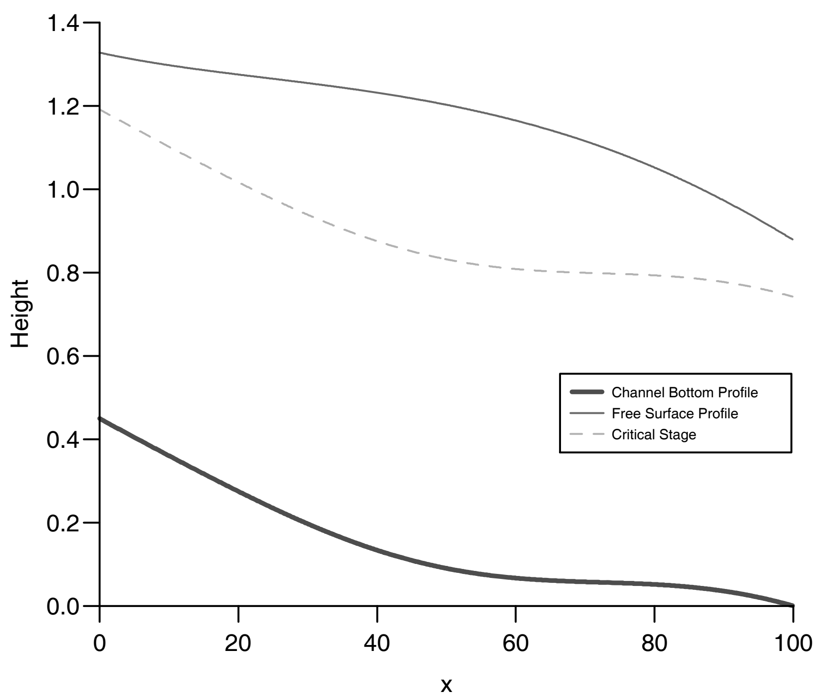

Figure 1b: Channel bottom, z, and free surface, $\hat{y} + z$, for example 1

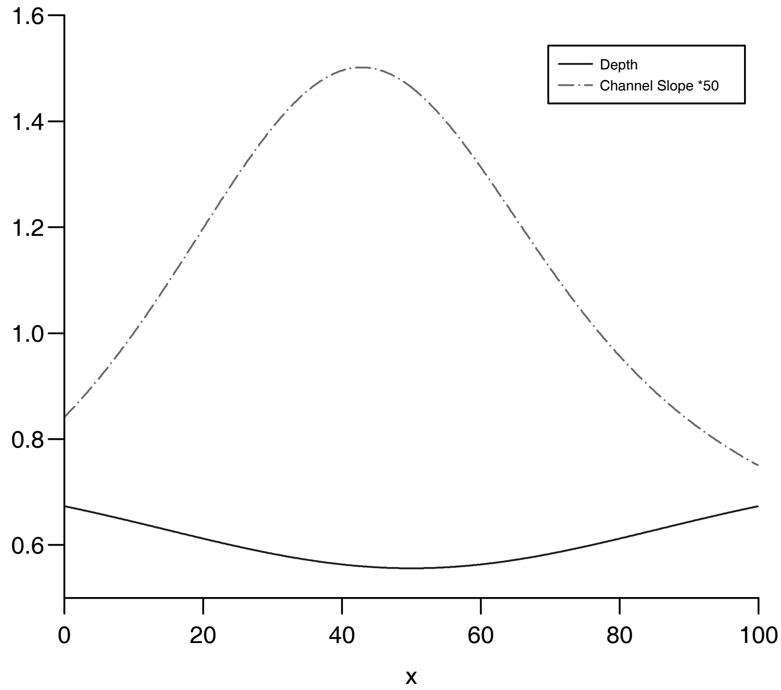

Figure 2a: Depth, $\hat{y}$, and bed slope, S_0, for example 2

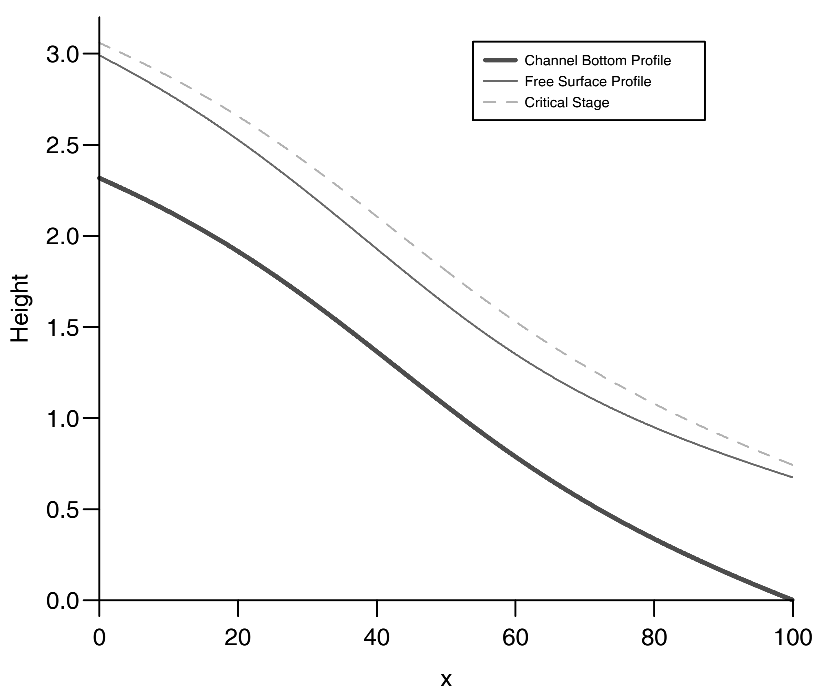

Figure 2b: Channel bottom, z, and free surface, $\hat{y}+z$, for example 2

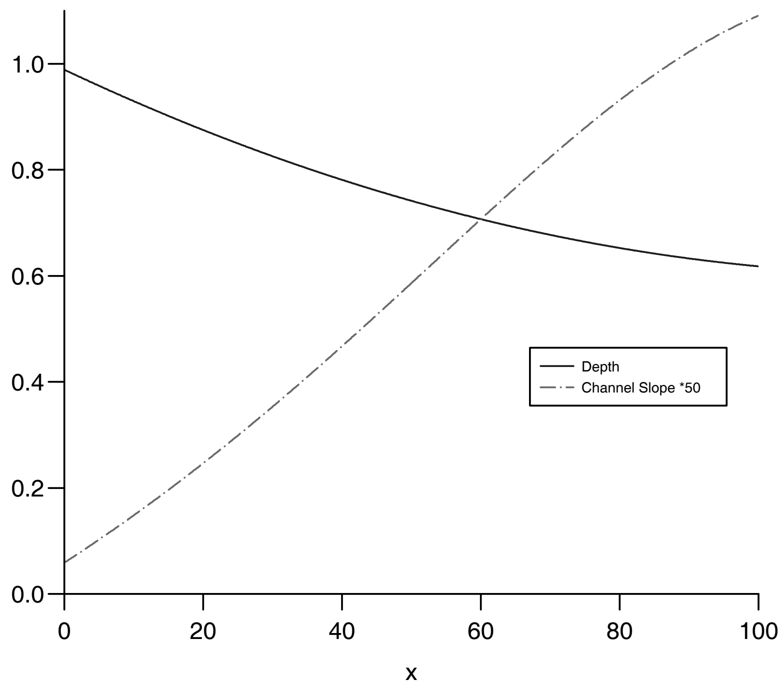

Figure 3a: Depth, $\hat{y}$, and bed slope, S_0, for example 3

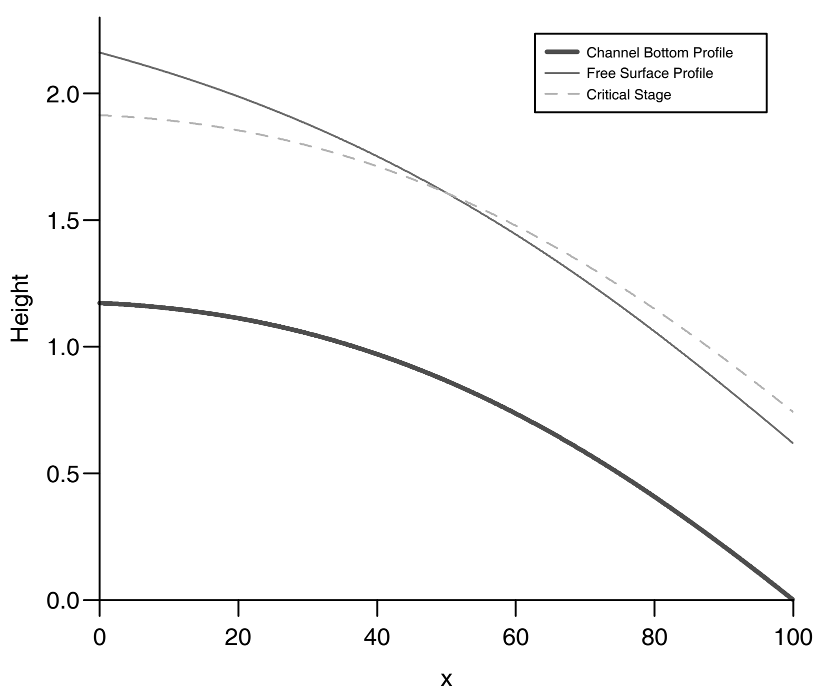

Figure 3b: Channel bottom, z, and free surface, $\hat{y} + z$, for example 3

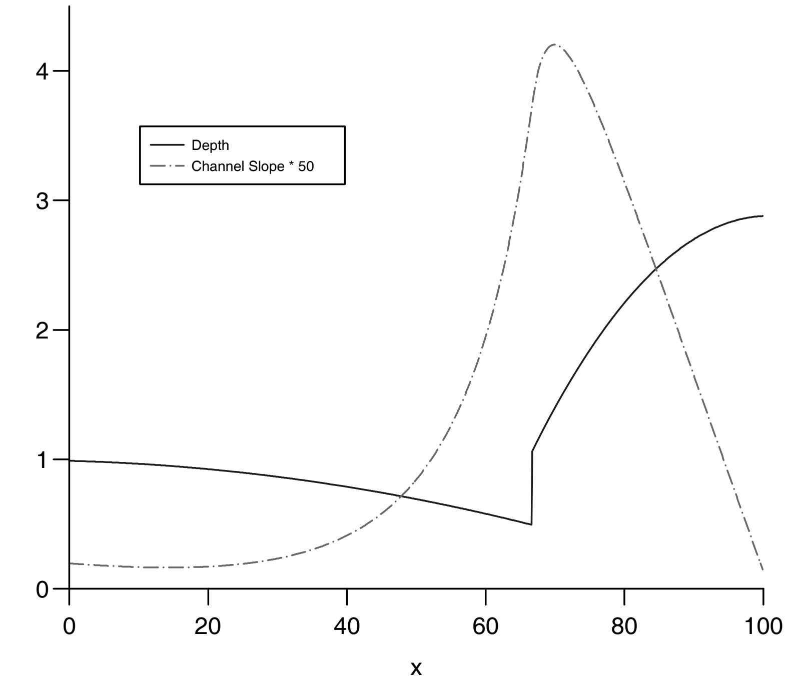

Figure 4a: Depth, $\hat{y}$, and bed slope, S_0, for example 4

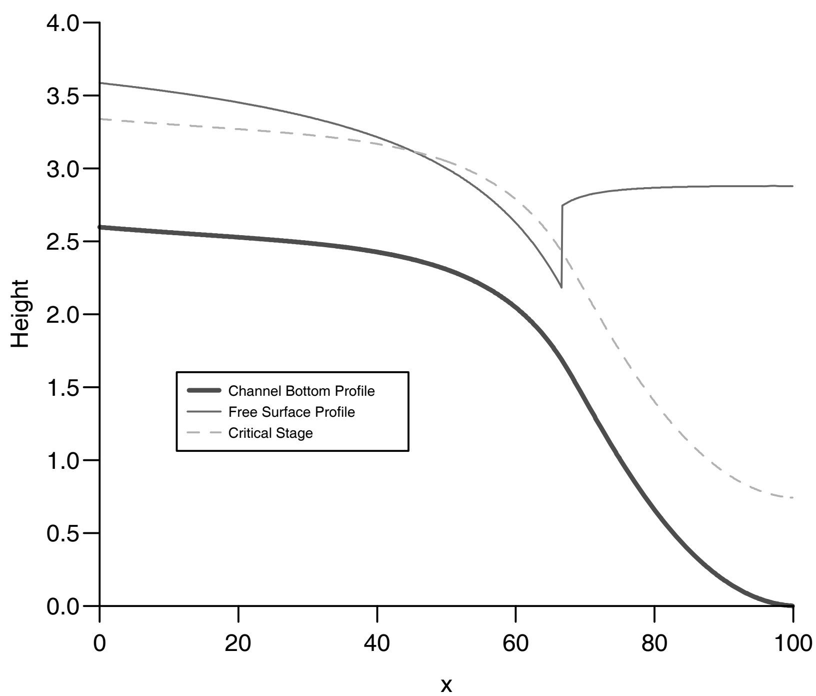

Figure 4b: Channel bottom, z, and free surface, $\hat{y} + z$, for example 4

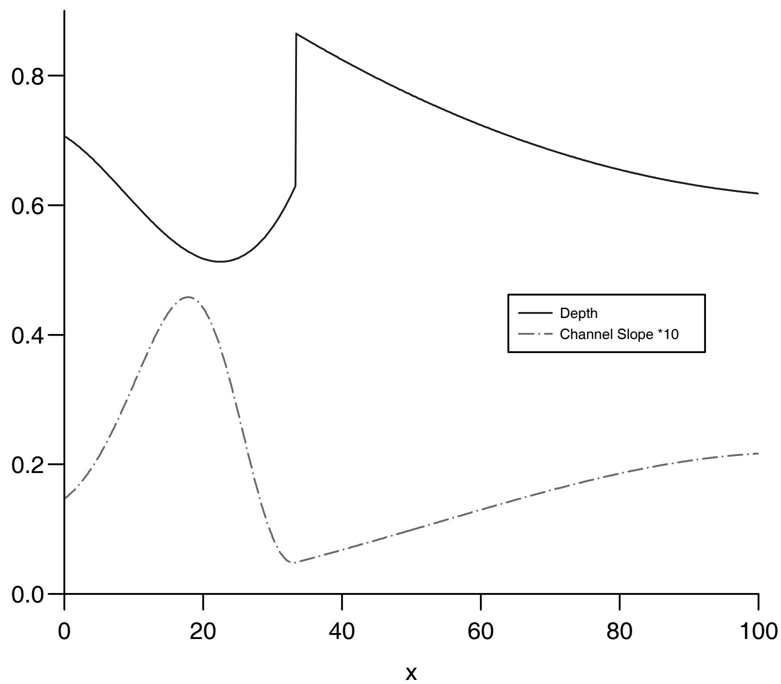

Figure 5a: Depth, $\hat{y}$, and bed slope, S_0, for example 5

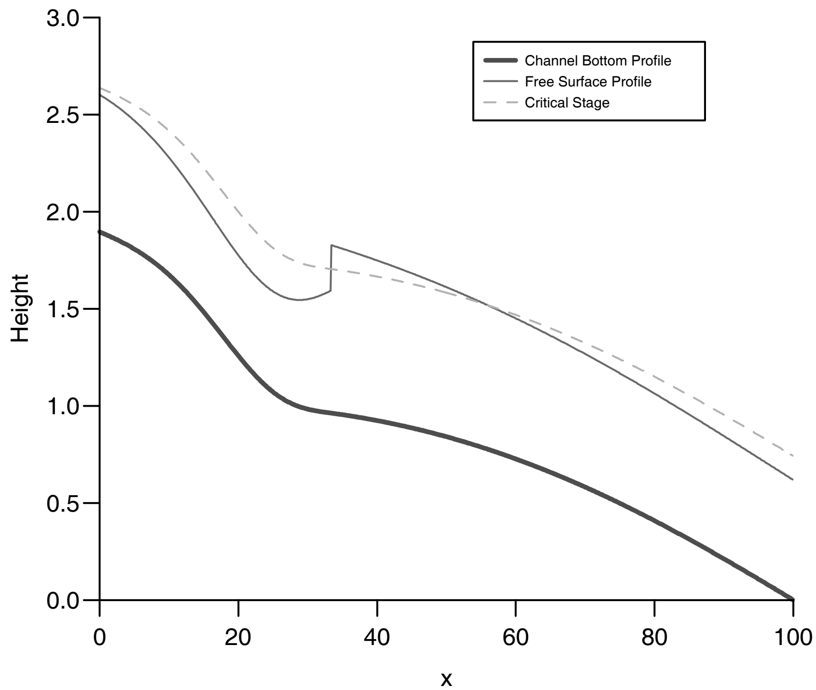

Figure 5b: Channel bottom, z, and free surface, $\hat{y} + z$, for example 5